TRANSPORT PROCESSES IN PLASMAS

2. Neoclassical transport theory

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Preface

The problem of the transport of matter, momentum and energy in a plasma submitted to temperature, pressure and velocity gradients combined with external electric and magnetic fields is, without any doubt, one of the most crucial aspects of plasma physics. In a nutshell, it can be said that *the object of transport theory is the response of a system (here: a plasma) to a sustained external constraint*. Let us make this statement more explicit.

Consider a material system which is in some kind of stationary state (it may be a true thermodynamic equilibrium state, or some more general kind of quasi-steady state, examples of which will be met later). At some time, a "*thermodynamic force*" is applied, which disturbs the initial equilibrium. This force may be an external force field, or it may be a spatial inhomogeneity (i.e., a gradient) of density, velocity or temperature. The system *responds* to this stimulus: various kinds of motion set in. There may be an overall material motion of the system, but also more subtle flows of momentum, energy or electric charge, which are not necessarily accompanied by a global motion of the matter.

The general tendency of the system is to react in such a way as to recover its initial steady state (*if* the latter is *stable*) when the driving forces are removed. If, however, the forces are sustained, many differrent things can happen: the system evolves now under an *external constraint*, opposing its return to equilibrium.

If the forces are maintained at a steady level and are not too strong, the system *may* (or may not!) reach, after some time, a new, *non-equilibrium steady state*, in which its state variables (such as the density and the temperature) retain constant values in time, but in which there exist non-vanishing, *constant fluxes* of matter, momentum, energy and/or electric charge. If, however, the constraint is too strong, or if it varies in time, the response of the system will be much more complex.

In any case, the result of the application of a thermodynamic force will be a redistribution of matter, momentum, energy and charge throughout the system, hence the appearance of more or less violent fluxes of these quantities. We entered the realm of *non-equilibrium physics*, a field that is in full expansion in the present days.

The importance of transport theory to plasma physics is enormous. It can be briefly illustrated by the following examples.

The problem of *Controlled Thermonuclear Fusion* is based on the possibility of confining a plasma inside a magnetic field configuration for a time long enough to permit the start-up of the nuclear fusion reactions and the collection of the energy produced by these reactions. But such "magnetic bottles" are

never tight: there are unavoidable "leaks" which may be fatal to the confinement. The control of these leaks implies a perfect knowledge and mastership of the mechanisms of transport of matter and energy from the region of production or deposition (i.e., in the core of the plasma) towards the walls, in the presence of a strong confining magnetic field of complex toroidal geometry.

Astrophysics and Geophysics is another important field of applications of plasma physics, as more than 90% of the matter in the universe is in the plasma state. Here again, the problem of transport is crucial in understanding a variety of complex phenomena. To quote just a few examples, the problem of the solar phenomena (flares, protuberances, spots,...) requires a deep study of magnetohydrodynamics in conditions where dissipation, hence transport, is quite important. The fascinating processes of magnetic reconnection are basic in the understanding of the phenomena (such as magnetic storms) occurring in the magnetospheres surrounding the earth as well as other planets, each with its own specificity. A fundamental question in the theory of reconnection is the mechanism of "anomalous transport" which produces the necessary dissipation. The latter, in absence of sufficiently frequent interparticle collisions, must arise from subtle collective electromagnetic phenomena, including plasma instabilities and turbulent processes.

Finally, we may briefly mention the more daily applications of plasma physics to such phenomena as electrical discharges in gases, arcs,..., where the problems of electrical conductivity and thermal conductivity are again fundamental.

Besides these motivations arising from the applications, we may stress the interest of transport theory for the theoretical physicist in the general framework of *non-equilibrium statistical physics, kinetic theory* and *non-equilibrium thermodynamics*. This problem was already of great historical importance since the times of the founders of the kinetic theory of gases: Maxwell, Boltzmann, Chapman, Enskog, Cowling. Since the relatively recent development of plasma physics, an entirely new dimension appeared in transport theory. Unlike neutral gases, plasmas consist of electrically charged particles interacting through long-range forces: this in itself induces a vast variety of new phenomena (especially collective behaviour of the particles). Moreover, because of their electric charge, these particles are affected by external magnetic and electric fields. This feature opens up a new level of interaction between the system and the experimentalist. The latter may control the transport not only by imposing temperature and pressure gradients, but also electromagnetic fields of arbitrarily complex and subtle shape and time dependence. In turn, the system may react to these external fields in ways which are often not predicted by straightforward intuition: it may develop peculiar drift motions, or instabilities that may lead to the breaking of the initial topology of the

magnetic fields, and in the last stages, the motion (as well as the internal electromagnetic fields) may become completely chaotic and turbulent. All these features strongly influence the transport mechanisms. As a result, the transport theory of plasmas is an incommensurably richer field than the classical transport theory of neutral gases.

I have been "in love" with the statistical physics of plasmas over my whole scientific life. Having studied for years the problems related to the "kinetic stage", i.e. the derivation of irreversible kinetic equations from the reversible equations of classical and quantum mechanics. I wanted to complete the programme of statistical physics by going over to the "hydrodynamic stage", in which the microscopic, molecular information is transferred to the observable, macroscopic level. Curiously enough, this stage has been somewhat neglected by statistical physicists in recent years. This is the more regrettable because, particularly in the field of plasma physics, the "hydrodynamic stage" leads to the plasma transport theory with all its fascinating aspects.

For many years I was surprised by the absence of a specific monograph on this subject. At present, a physicist wanting to enter this field has to begin his study with five "classical" review papers:

– S.I. Braginskii, 1965, in: Reviews of Plasma Physics, Vol. 1, ed. M.N. Lenotovich (Consultants Bureau, New York) [for the classical theory].

– A.A. Galeev and R.Z. Sagdeev, 1979, Reviews of Plasma Physics, vol. 7.

– F.L. Hinton and R.D. Hazeltine, 1976, Reviews of Modern Physics.

– S.P. Hirshman and D.J. Sigmar, 1981, Nuclear Fusion [the latter three references concern the neoclassical theory].

– P.C. Liewer, 1985, Nuclear Fusion [for the anomalous transport].

He then must start the search for specific research papers, scattered in a large number of journals, conference proceedings or laboratory reports. He will have difficulties in the fact that these papers (including the review papers!) do not try to present a global formulation within a general framework of plasma transport, but rather present a special chapter from a specific point of view.

The existing general textbooks or monographs on plasma physics usually contain only a very small chapter devoted to plasma transport: it is presented in a very elementary and non-rigorous way.

I therefore decided to try to fill this gap and started working on the project at the beginning of 1981. I soon began to understand the reason of this gap in the literature. I realized that, if the subject were to be treated in the spirit I had chosen, it would be absolutely impossible to cover the whole matter in a single volume, even after the elimination of less essential aspects.

On the other hand, I realized that this matter could be divided in two volumes, which could be conceived as self-contained entities. The first one would include the so-called *classical and neoclassical transport theories* and the

second one the *anomalous transport theories*. In the first volume, one studies the transport mechanisms explained by the properties of *individual particles*, which are the actors of the interactions among themselves and with the external electromagnetic fields. In the second volume one would deal with the *collective transport mechanisms* mentioned above. At the end of the work on the first volume, it appeared that even this would give rise to a too thick book, whose use would be unhandy: it was therefore decided to split it into two parts. The numbering of the chapter remained, however, continuous, in order to stress the intrinsic unity of the work. Thus, Part I is devoted to the *classical transport theory* and contains chapters 1–7, and Part II devoted to the *neoclassical transport theory*, contains chapters 8–19. The work on the second volume, devoted to the anomalous transport is, at present, in a preliminary stage and will not be published before several years.

I now wish to comment on the general spirit of this work. My objective was to write a self-contained book, which the reader could study without having to resort to side-references at every step. I therefore chose to present plasma transport theory as a discipline strongly embedded in the framework of theoretical physics. By the latter statement, I mean not only "plasma physics" in a restricted sense, but more generally: Hamiltonian mechanics, statistical physics, thermodynamics, hydrodynamics, electrodynamics. Plasma transport theory precisely realizes a synthesis of all these fields: it can be characterized as a truly *interdisciplinary activity*. This explains the presence of several chapters in which the important concepts of these "peripheral" fields are introduced, briefly but completely.

In making the unavoidable selections of the material, I decided to focus on relatively simple situations. Thus, the plasma model used throughout the book is a simple, fully ionized plasma consisting of electrons and a single species of ions. Within this framework, the presentation is as exhaustive as possible: the calculations are given in great detail) much greater than in the published papers) in order to make its study as easy as possible. The assumptions are discussed and every crucial step in the argument is commented both in its physical and its mathematical aspects.

I developed a formalism which allows the presentation of all the features of classical and neoclassical transport in a unified way, by using the same tools, concepts, methods and notations throughout the book. The basic tool of this formalism is the Hermitian moment method of solution of the kinetic equations (chs. 4, 11, 12). Although inspired from the classic work of GRAD, it goes well beyond its initial formulation, making it applicable to the whole range of problems under consideration here.

The final results are given in the form of completely explicit and simple *analytical formulae* (a new feature!). These formulae enable one to calculate numerical values of the transport coefficients with excellent precision on a

pocket calculator. A large number of tables, figures and graphs are presented, a feature usually absent in the transport literature. It is extremely helpful (in my opinion) for the general intuition of the problem to have a pictorial view of the phenomena. Moreover, many of these graphs allow a simple and rapid reading of the numerical values of the transport coefficients.

A monograph like this one implies a large amount of compilation of a vast amount of literature. I should like to stress, however, that the presentation is never left at the primitive stage of a compilation, or even of a critical review paper. I endeavoured to rethink all the known results and reformulate them in the framework of a unified theory. Every single calculation in the book was done and redone several times. As a result, the text went through several successive versions before its finalization in its present form.

The level of the mathematics is rather unsophisticated. Most of the problems treated here appeal only to linear algebra and to elementary aspects of differential equations. This should make the book accessible to a wide audience of experimental physicists, as well as to good undergraduate students. The only less usual mathematical tools appearing here are those related to the toroidal geometry: all the necessary concepts are discussed in detail in the text and in the appendices.

The prerequisites are just the basic principles of statistical physics, kinetic theory, electromagnetism and a feeling of plasma physics, as can be expected from any physicist or engineer.

In conclusion, my main purpose in writing this book was a comprehensive and unified presentation of the transport theory in plasmas. The reader will judge the degree of success in the achievement of this endeavour.

Contents

Introduction

The general philosophy and the structure of this work were discussed at length in the general preface of this work. I shall therefore be very brief here, and discuss only the characteristics of the second Part of the Volume.

This Part is devoted to the *neoclassical transport theory*. In this field, the interest focusses on the strange behaviour of a plasma in the presence of an *inhomogeneous and curved magnetic field* (such as those designed for the toroidal confinement devices in fusion physics). It was one of the great discoveries at the end of the 1960's, due to Galeev and Sagdeev, that the *global geometry* of the magnetic field (as opposed to its *local value*) has a very strong influence on the transport processes in a plasma. This was, indeed, a very surprising discovery. It is not presumptuous to invoke here an analogy with general relativity. Just as gravitation introduces a curvature of space which is felt at each point as a force acting on the bodies, the curvature of the magnetic field produces a peculiar drift of the charged particles at each point, which modifies their motion.

It may be interesting to explain the name of "neoclassical" given to this theory. It is fully justified in the sense that the basic mechanisms at work in this situation are the same as in the classical theory: it represents an interplay between the action of the external magnetic (and electric) fields on the particle motion and the effect of the interparticle collisions. In other words, no collective effects (such as instabilities or turbulence) are taken into account in this theory. The only really new feature, as said above, is the curvature of the magnetic field.

This characteristic of the magnetic field yields, however, to the existence of a regime that has no non-trivial counterpart in the classical theory. Indeed, in such a situation, even when the collisions are extremely rare, the particles move in a confined region of space. Hence, we may imagine a regime in which the mean free path of the particles is very long, of the order of or larger than the typical hydrodynamic length, without the particles' bouncing into the walls of the box that contains them. It so happens that, at the very high temperatures required for controlled fusion, the plasmas are precisely in such a long mean free path regime. The classical transport theory (which is typically a short mean free path theory) needs, therefore, a highly non-trivial generalization, in order to cover such situations. I insist, however, again that the difference between these two theories lies in the mathematical treatment of the

problem rather than in the physics. In particular, the collisions, even though very rare, play an essential role in the neoclassical transport theory.

The neoclassical transport theory was the object of an enormous amount of work by many physicists throughout the world. Looking through the literature on this theory (which has taken a "new look" since the recent important contributions of Hirshman and Sigmar at the end of the 1970's), one may note that the development of the field was, perhaps, too fast under the pressure of the technological constraints of the Fusion programme. As a result – in my opinion – the underlying structural beauty of the theory was largely overlooked. In working on this subject, I was surprised to discover that some of the known results have a universality which could not be suspected from the existing presentations. Even more surprising was the discovery of a set of relations between the neoclassical transport coefficients, which apparently went completely unnoticed before. As a result, the neoclassical theory appears now as a "corpus" as complete and self-consistent as the classical theory, and thus worthy of a presentation in a textbook.

The structure of this second Part closely parallels that of the first. In chapter 8 the stage is set for the forthcoming treatment: the drama is played within a *torus* (where "...the beginning and the end coincide..."). The subsequent chapters reproduce the structure of Part I, by showing, at each stage, the effect of the toroidal geometry. Thus, chapter 9 studies the motion of an individual particle in a toroidal field. Chapters 10 and 11 are devoted to the study of the kinetic equation appropriate to the situation that prevails in the neoclassical theory. Chapter 12 is devoted to the general study of the macroscopic moment equations in toroidal geometry. In chapter 13, the first new transport equations are derived: they include the strange Pfirsch–Schlüter effect. In chapter 14, the method of solution of the kinetic equation in the long mean free path regime is developed. Chapter 15 is again a central chapter: the typical long mean free path neoclassical transport equations are obtained and discussed: their very peculiar differences with the classical ones are emphasized. Chapter 16 introduces an intermediate mean free path regime, as well as a method of interpolation of the results over the whole range of collisionalities. Chapter 17 is also crucial in our opinion: it provides the connection of the transport theory with non-equilibrium thermodynamics, in a regime (long mean free path) where the applicability of the latter seems, at first sight, to be questionable. Nevertheless, a complete and consistent thermodynamic theory can be set up, even in this regime. Finally, Chapter 18 goes back to the hydrodynamical equations and treats the problem of their closure (in toroidal geometry): the objective of transport theory is thus realized.

The neoclassical theory has a prerequisite: the existence and stability of the toroidal geometric structure (technically speaking: the existence of toroidal magnetic surfaces). Whenever this structure is destroyed, as a result of an

external action or of the growth of an internal instability, the theory no longer is valid, and must be continued into a theory of the so-called anomalous transport. In chapter 19, we have illustrated on a simple (but important) example how this transition can appear.

Summarizing this discussion, I believe to have shown in this work that a unified and complete presentation of the classical and neoclassical transport theories is now possible: this was my endeavour. I now feel ready to continue an analogous work on the anomalous transport problem (a much harder objective!) for the next part of this project.

8

Toroidal magnetic confinement

8.1. Magnetic surfaces

In this section we study a few elementary properties of an important class of magnetic fields used in most of the present-day thermonuclear fusion devices. This study will set the framework for the forthcoming treatment of the mechanism of transport of matter, momentum and energy in these devices.

In all magnetic fusion experiments, the first goal to be achieved is the *confinement of a hot plasma of sufficiently high density for a sufficiently long time*. If these conditions are met, the fusion reactions can be sustained in the plasma and produce more energy than the amount fed into the system. Recent discussions of the status of magnetic fusion research are found in the work by Ribe (1975) Freidberg (1982) Kadomtsev et al. (1983), and Kadomtsev and Shafranov (1983).

In magnetic fusion, the confinement is achieved by designing a magnetic field which causes the motion of the charged particles of the plasma to remain confined in a bounded region.

We already know that a homogeneous field produces good confinement in two dimensions (perpendicular to its direction), but no confinement at all in the third direction (section **1**.5, fig. **1**.5.1 *). Hence, a confining field must be necessarily *inhomogeneous* and *curved*.

Let us concentrate on the field lines associated with the magnetic field (see section **1**.3). The fact that $\nabla \cdot \boldsymbol{B} = 0$ imposes conditions on the topology of the field lines; these must either be closed, or extend to infinity. Fields of the latter type are used in so-called *open configuration* fusion devices, the best known of which is the *magnetic mirror*. In such configurations there always exists a "loss-cone", which must be "plugged" by various means. We do not discuss these systems here.

* Whenever reference is made to a section, equation or figure in another chapter, the number is preceded by the chapter number in boldface type

Closed magnetic lines are necessarily confined in a bounded region of space *. But even in this case, the global topology of these lines is not always simple, except when the total length of the lines is finite. But this does not need to be the case. The field line may wander indefinitely through the bounded region in which it is confined before closing on itself after an infinite journey (which is tantamount to saying that such a line actually never closes!). An inescapable consequence of this topology is the fact that such a field line *densely fills some region of the domain* in which it is confined. This means that it passes arbitrarily close to any point of this region. The latter may be the whole domain, or some three-dimensional subdomain; finally, and most interesting to us, the region densely covered by the field line may be a *two-dimensional manifold*. We now note that if such a surface exists and has no edges (i.e. it does not intersect its domain of definition) and if the field $\boldsymbol{B}$ vanishes nowhere on it, a non-trivial theorem of topology tells us that the manifold must be a *toroid* ** (i.e. it must be topologically equivalent to a torus) Alexandroff and Hopf 1935, ch. 14.4, theorem III, Kruskal and Kulsrud 1958). This type of topology is used in most of the important fusion machines, such as the *tokamak* or the *stellarator*. From here on, we limit our discussion to the case of a toroidal topology. We now formulate these questions more quantitatively.

Let $\psi(\boldsymbol{x})$ be a real, sufficiently regular function of the position $\boldsymbol{x}$ in space, and ψ be a real number. Then

$$\psi(\boldsymbol{x}) = \psi \tag{1.1}$$

defines a surface; we take function $\psi(\boldsymbol{x})$ in such a way that this surface is a toroid. Moreover, by varying the value of the number ψ, we find that (1.1) represents a family of *nested toroids*, as shown in fig. 1.1.

We furthermore assume that the magnetic field of interest is such that each one of its field lines lies on some surface of the family (1.1). In other words, the magnetic field $\boldsymbol{B}(\boldsymbol{x})$ is, at each point $\boldsymbol{x}$, tangent to the toroid passing through $\boldsymbol{x}$. This condition is expressed by

$$\boldsymbol{B}(\boldsymbol{x}) \cdot \nabla\psi(\boldsymbol{x}) = 0. \tag{1.2}$$

The surfaces $\psi(\boldsymbol{x}) = \psi$ are then called *magnetic surfaces*. Within the family, there exists one innermost degenerate surface, which reduces to a closed curve, called the *magnetic axis*. We assume that the magnetic axis closes on itself after a single excursion around the toroid. More complicated topologies (such

* This, obviously, by no means implies that such fields are able to confine a plasma in a bounded region.

** Or a Klein bottle; but the latter cannot be realized in physical space.

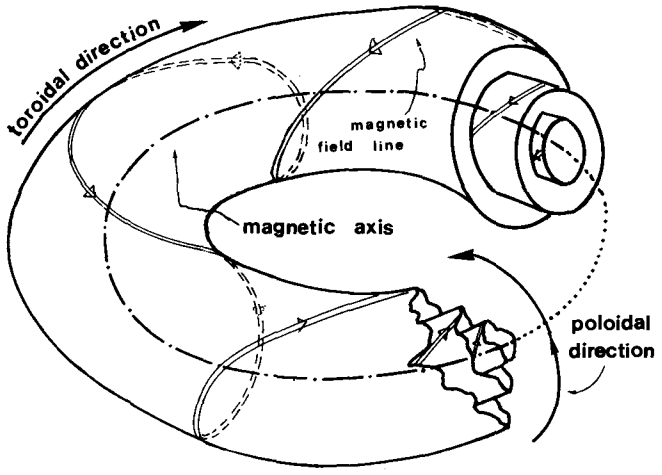

Fig 1.1 Family of nested toroidal magnetic surfaces

as *magnetic islands* formed around magnetic axes circling the toroid several times before closing on themselves) will be discussed in the future volume on anomalous transport processes.

An important concept will be introduced now. Let γ be a closed loop obtained by intersecting a magnetic surface with some plane perpendicular to the magnetic axis (fig. 1.2). Consider the field line passing through some point a_0 on this loop. We follow this line until it completes a turn, the long way (i.e. in the so-called toroidal direction) around the torus and intersects again the

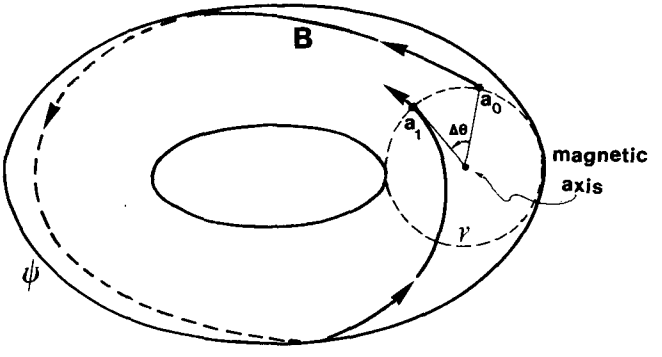

Fig. 1.2. Definition of the rotational transform. For clarity, we represented a case where the field line turns only once the short way around the torus.

loop γ. (Note that during this journey the line may turn a number of times the short way around the torus, i.e. in the poloidal direction). In general, the new intersection a_1 does not coincide with the starting point a_0. The points $a_0 a_1$ define an arc $\Delta\theta_1$ as measured from the magnetic axis. Next, we continue following the field line for another complete toroidal circuit, until it meets again the loop at a_2; the arc $a_1 a_2$ defines a new angle $\Delta\theta_2$ which may (or may not) be different from $\Delta\theta_1$. But it can be shown that the average value of $\Delta\theta_j$, after a sufficiently large number of circuits, approaches a finite limiting value. This allows us to define the *rotational transform* ι as

$$\iota = \langle \Delta\theta \rangle = \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \Delta\theta_j . \tag{1.3}$$

This number depends only on the particular surface chosen. This important fact will be proven in section 8.4. Let us also define the frequently used *safety factor* q;

$$q = \frac{2\pi}{\iota} . \tag{1.4}$$

We now note that two possibilities exist. If the ratio

$$\frac{\iota}{2\pi} = \frac{n}{m} , \tag{1.5}$$

where n and m are integers, i.e. if the ratio $\iota/2\pi$ is a rational number, the field line closes on itself after a finite number of turns (because after m turns the intersection a_m has turned by an angle $m\iota = 2\pi n$ and has recovered the starting point). As ι is a characteristic of each magnetic surface, it follows that any surface with rational $\iota/2\pi$ is covered by magnetic field lines forming closed loops of finite length. These are called *rational surfaces*.

If, on the contrary, the number $\iota/2\pi$ is irrational, the field line will always "miss" its starting point and will continue indefinitely, covering densely the toroidal surface. On such an *ergodic surface, each single field line* passes arbitrarily close to any point of the surface; all field lines have necessarily infinite length. Thus, the concept of rotational transform allows us to make the discussion of the beginning of this section more precise. If the rotational transform $\iota = \iota(\psi)$ varies continuously as we go from the magnetic axis outward, we see that "the majority" of the magnetic surfaces are ergodic. More precisely, the rational surfaces form a set of measure zero. Nevertheless, in every toroidal geometry, the ergodic surfaces are densely interspersed with infinitely many rational surfaces.

8.2. Magnetohydrodynamic equilibrium. Toroidal coordinate systems

In the problem of plasma confinement, it is clear that an arbitrarily specified magnetic field cannot be imposed by external means. If we tried to do so, the presence of charged particles creating their own field would interfere with the external one and modify it until a self-consistent balance is achieved, in which the actual magnetic field is determined both by the external sources and by the plasma.

A particularly simple situation arises when the plasma, imbedded in the magnetic field, is in a state of *static mechanical equilibrium.* Its special importance stems from several facts. If such a configuration could be realized, and if moreover it were stable, it would be the ideal situation for a thermonuclear reactor. But such a reactor would not work in a steady state: in its functioning regime there are flows of electricity, matter, energy and momentum traversing it in complex patterns. In the study of these transport phenomena, which are the main purpose of this book, the equilibrium state determines a reference: the departures from this state provide the generalized forces driving the transport phenomena.

In a static plasma equilibrium, all hydrodynamic quantities [the density $\rho(\boldsymbol{x})$, the temperatures $T_e(\boldsymbol{x})$, $T_i(\boldsymbol{x})$, the pressure $P(\boldsymbol{x})$, the electromagnetic fields $\boldsymbol{E}(\boldsymbol{x})$ and $\boldsymbol{B}(\boldsymbol{x})$] are independent of time. Moreover, the plasma is everywhere electrically neutral [$\sigma(\boldsymbol{x}) = 0$] and its local velocity $\boldsymbol{u}(\boldsymbol{x})$ is zero. In such a state there is no dissipation, hence the dissipative pressure tensors $\boldsymbol{\pi}^\alpha$ and the heat flows $\boldsymbol{q}^\alpha$ vanish identically. According to the discussion in section 7.4, the reference equilibrium state is (traditionally) determined in the framework of ideal MHD.

In such conditions the macroscopic state of the plasma is determined by the momentum balance equation (7.4.1) which expresses the equilibrium between the pressure gradient and the Lorentz force,

$$\nabla P = c^{-1}\boldsymbol{j} \wedge \boldsymbol{B}. \tag{2.1}$$

This equation is coupled to the magnetostatic Maxwell equations (I.5), (I.7) of table 7.3.1,

$$\nabla \wedge \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{j}, \tag{2.2}$$

$$\nabla \cdot \boldsymbol{B} = 0. \tag{2.3}$$

The solution of these three equations constitutes the problem of *plasma equilibrium* which has been extensively studied in the literature. We shall only

treat here those aspects which will be needed subsequently, and refer the reader to the following review papers and books for further details: Kruskal and Kulsrud (1958), Greene and Johnson (1961) Morozov and Soloviev (1966) Grad (1967), Soloviev and Shafranov (1970), Greene et al. (1971), Krall and Trivelpiece (1986), Bateman (1978), Freidberg (1982), Golant et al. (1980), Hazeltine and Meiss (1985), Gross (1984), Raeder et al. (1986).

An immediate consequence of (2.1) is

$$\boldsymbol{B}(\boldsymbol{x}) \cdot \nabla P(\boldsymbol{x}) = 0. \tag{2.4}$$

Comparing with (1.2) we arrive at the following important statement: *In equilibrium the surfaces of equal pressure (isobars) $P(\boldsymbol{x}) = P$ are magnetic surfaces.* Thus, the picture of section 8.1 becomes more concrete. In a toroidally confined plasma, the isobars form nested toroids as in fig. 1.1; the pressure increases as we follow a radial path from the outside towards the magnetic axis; we shall assume here that this increase is monotonic, hence $\nabla P \neq 0$ everywhere, except at the magnetic axis.

We now define a coordinate system (or rather, a class of coordinate systems) which are well adapted to the geometrical situation discussed here. As a first coordinate, the pressure is a good candidate: in equilibrium, the value of the pressure labels a magnetic surface, and the vector $-\nabla P$ defines a direction pointing outward from the magnetic axis in each transverse section of the toroid (see fig. 2.1). We further define two coordinates which are angular in nature. The *toroidal angle* ζ is defined as a continuous multivalued function $\zeta(\boldsymbol{x})$ which increases by 2π upon a single circuit the long way around the torus. The *poloidal angle* θ is defined as a continuous (except at the axis) multivalued function $\theta(\boldsymbol{x})$ increasing by 2π upon a single circuit the short way

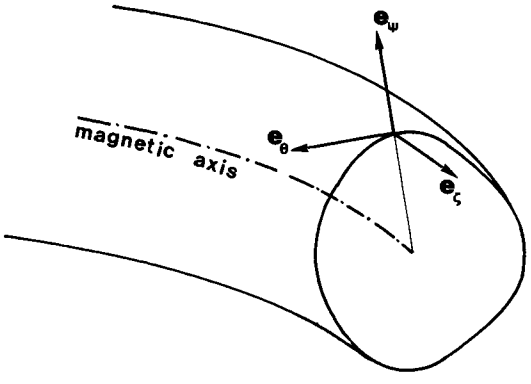

Fig. 2.1. Coordinate system adapted to toroidal geometry. e_ψ is the radial direction, e_θ is the poloidal direction, e_ζ is the toroidal direction.

around the magnetic axis. We must still say something about the directions of increasing θ and ζ. This is done in relation with the increasing radial direction $(-\nabla P)$. Concretely, we define three (dimensionless) unit vectors $\boldsymbol{e}_\psi$, $\boldsymbol{e}_\theta$, $\boldsymbol{e}_\zeta$,

$$\boldsymbol{e}_\psi = -\frac{\nabla P(\boldsymbol{x})}{|\nabla P(\boldsymbol{x})|}, \qquad \boldsymbol{e}_\theta = \frac{\nabla \theta(\boldsymbol{x})}{|\nabla \theta(\boldsymbol{x})|}, \qquad \boldsymbol{e}_\zeta = \frac{\nabla \zeta(\boldsymbol{x})}{|\nabla \zeta(\boldsymbol{x})|}. \tag{2.5}$$

Functions $\theta(\boldsymbol{x})$ and $\zeta(\boldsymbol{x})$ are defined in such a way that the vectors $\boldsymbol{e}_\psi$, $\boldsymbol{e}_\theta$, $\boldsymbol{e}_\zeta$ form a right-handed triad at each point (see General Appendix **G2**.1),

$$\boldsymbol{e}_\psi \cdot (\boldsymbol{e}_\theta \wedge \boldsymbol{e}_\zeta) > 0. \tag{2.6}$$

There is still a considerable freedom in the choice of a specific coordinate system. One may use this freedom in order to construct an orthogonal coordinate system. In this case, the unit vectors $\boldsymbol{e}_\psi$, $\boldsymbol{e}_\theta$, $\boldsymbol{e}_\zeta$ are mutually perpendicular and

$$\boldsymbol{e}_\psi \wedge \boldsymbol{e}_\theta = \boldsymbol{e}_\zeta, \tag{2.7}$$

and relation (2.6) simply becomes

$$\boldsymbol{e}_\psi \cdot (\boldsymbol{e}_\theta \wedge \boldsymbol{e}_\zeta) = +1 \tag{2.8}$$

(see General Appendix **G2**.1, **G2**.2).

The way in which this freedom of choice is used is largely a matter of convenience. Orthogonal systems are usually the simplest. But sometimes it is worthwhile sacrificing this feature in order to achieve more simplicity in other aspects of the problem.

A set of numbers ψ, θ, ζ completely specifies any point in the confinement domain. This domain is necessarily bounded. * The coordinate system (ψ, θ, ζ) is clearly much better adapted to the geometry of the plasma than the arbitrary Cartesian coordinates (x, y, z). In particular, ψ characterizes a magnetic surface as a whole, on which θ, ζ form a curvilinear coordinate grid.

8.3. Surface quantities

A toroidal confining field configuration was defined in section 8.1 by a set of surfaces $\psi(\boldsymbol{x}) = \psi$ having the topology of nested toroids and satisfying eq. (1.2). We have seen in section 8.2 that the isobars of a plasma in mechanical equilibrium form a set of such magnetic surfaces. But the surfaces can also be characterized by other intrinsic quantities, some of which have a clear and

* This can be clearly understood if we think of the particular example of concentric circular tori. If ψ is taken as the radial distance counted from the magnetic axis, it is clear that ψ cannot exceed the major radius of the torus, i.e. the radius of the magnetic axis.

important physical meaning. We define a *surface quantity* as any function $s(\boldsymbol{x})$ which is constant on a magnetic surface defined by (1.1) and (1.2). The value of any surface quantity may therefore be used to characterize globally (or label) the surface.

An obvious, but important surface quantity is the *volume* enclosed by a magnetic surface,

$$V = \int_{\psi} \mathrm{d}^3\boldsymbol{x}, \tag{3.1}$$

where the domain of integration (here and in the following formulae) is the interior of the magnetic surface $\psi(\boldsymbol{x}) = \psi$. A little reflexion will convince one that the reason why V can be used for labelling a unique surface lies in the nested structure of the surfaces.

The following two functions are particularly important in applications. We define the *toroidal flux* as

$$\Phi_{\mathrm{T}} = \frac{1}{2\pi} \int_{\psi} \mathrm{d}^3\boldsymbol{x}\, \boldsymbol{B} \cdot \nabla\zeta, \tag{3.2}$$

where ζ is the toroidal coordinate function defined in section 8.2 *.

Note that (3.2) can also be written, because of (2.3), as

$$\Phi_{\mathrm{T}} = \frac{1}{2\pi} \int_{\psi} \mathrm{d}^3\boldsymbol{x}\, \nabla \cdot (\boldsymbol{B}\zeta). \tag{3.3}$$

We may transform this volume integral into a surface integral by using the Gauss theorem; in order to do so, we must first make a cut through the surface at $\zeta = 0$ (say) and separate slightly the two resulting edges (fig. 3.1A). The Gauss theorem now tells us that the integral in (3.3) equals the flux going out of the toroidal surface ψ, together with its two "lids" at $\zeta = 0$ and $\zeta = 2\pi - \varepsilon$, where ε is an infinitesimally small positive number. However, the flux through the lateral surface is zero, because the magnetic field is everywhere tangent to it (see eq. 1.2). Hence, we are left with

$$\begin{aligned} \Phi_{\mathrm{T}} &= \lim_{\epsilon \to 0} \frac{1}{2\pi} \left(\int_{\zeta = 2\pi - \epsilon} \mathrm{d}\boldsymbol{S} \cdot (\boldsymbol{B}\zeta) - \int_{\zeta = 0} \mathrm{d}\boldsymbol{S} \cdot (\boldsymbol{B}\zeta) \right) \\ &= \frac{2\pi}{2\pi} \int_{\zeta = 2\pi} \mathrm{d}\boldsymbol{S} \cdot \boldsymbol{B} = \int_{\Sigma_{\mathrm{T}}} \mathrm{d}\boldsymbol{S}^{\zeta} \cdot \boldsymbol{B}. \end{aligned} \tag{3.4}$$

* Note that $\boldsymbol{B}\cdot\nabla\zeta$ is simply the *contravariant toroidal component*, $\hat{B}^{\zeta}$, of the magnetic field (see eqs. **G2** 1.15, **G2** 1.21)

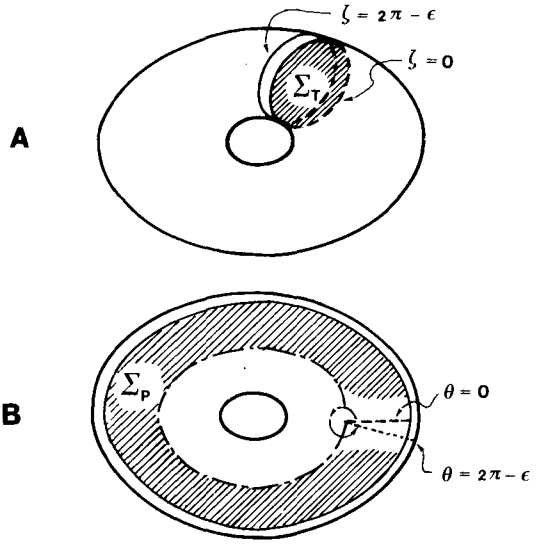

Fig 3 1 Surfaces Σ_T and Σ_P defining the toroidal flux (A) and the poloidal flux (B)

In the last integral we introduced the surface element $d\boldsymbol{S}^{\zeta}$ normal to the "lid" $\zeta = 2\pi$ defined in (**G2**.1.37) or (**G2**.2.13); we renamed the "lid" $\zeta = 2\pi$: Σ_T as in fig. 3.1. From (3.4) it is now clear that Φ_T is equivalent to the usual definition of the flux of the toroidal component of the magnetic field through a section of the magnetic surface perpendicular to the magnetic axis.

The *poloidal flux* Φ_P is another surface quantity, very similar to the toroidal flux,

$$\Phi_P = \frac{1}{2\pi} \int_{\psi} d^3\boldsymbol{x}\, \boldsymbol{B} \cdot \nabla\theta, \tag{3.5}$$

where θ is the poloidal angle. By a calculation similar to the previous one we show that

$$\Phi_P = \int_{\Sigma_P} d\boldsymbol{S}^{\theta} \cdot \boldsymbol{B}. \tag{3.6}$$

The surface Σ_P is defined as follows (fig. 3.1B): One cuts the toroid radially along the surface $\theta = 0$ (say), from the outer surface to the magnetic axis, and separates slightly the two edges; one then applies Gauss' theorem to the surface formed by the torus and the two "ribbons" $\theta = 0$ and $\theta = 2\pi - \varepsilon$.

Two more quantities are of physical interest. The *total toroidal current* I_T is defined as

$$I_T = \frac{1}{2\pi} \int_\psi d^3x\, \boldsymbol{j} \cdot \nabla\zeta \tag{3.7}$$

and the *total poloidal current* I_P is given by

$$I_P = \frac{1}{2\pi} \int_\psi d^3x\, \boldsymbol{j} \cdot \nabla\theta. \tag{3.8}$$

Their interpretation is easily obtained by an argument similar to the previous one. They represent the total electric current (i.e. the amount of charge per unit time) flowing through the surfaces Σ_T and Σ_P, respectively. They are clearly surface quantities. [The proof of this property requires the property $\nabla \cdot \boldsymbol{j} = 0$, which is a consequence of eq. (2.2)].

The various surface quantities introduced here (and many others) may each be used for labeling the magnetic surfaces. Most of these surface quantities are not independent of each other: a rather extensive list of their mutual relationships is found in the paper by Kruskal and Kulsrud (1958). A rather obvious conclusion follows from this discussion. Suppose we choose a surface quantity ψ as the radial coordinate for labeling the magnetic surfaces. (For instance, in section 8.2 we had made the choice: $\psi = -P$). Then *any surface quantity is necessarily a function of* ψ: $K(\psi)$, *independent of* θ *and* ζ.

8.4. *Natural coordinate systems*

We conclude the previous section with the remark that any surface quantity can be used as a radial coordinate in a coordinate system adapted to the toroidal symmetry of the confining field. We now concentrate on the angular coordinates. Till now, these were rather loosely defined in section 8.2. We now show that it is possible to restrict their definition in such a way as to obtain a coordinate system in which many properties are expressed in a particularly simple form.

Let ψ be a radial coordinate, θ a poloidal coordinate and ζ a toroidal coordinate. We assume, for simplicity, that (ψ, θ, ζ) form an *orthogonal coordinate system*. This is not an indispensable condition. If a non-orthogonal system is preferred for other reasons, the necessary changes in writing the formulae are easily made by using the results of Appendix G2. (The main difference consists of using covariant and contravariant components of vec-

tors, instead of the physical components used here). For a treatment of these matters in general coordinate systems, see the work by Soloviev and Shafranov (1970) and Hazeltine and Meiss (1985).

We first study the form of the magnetic field. We note that (1.2) reduces, in this coordinate system, to

$$\boldsymbol{B} \cdot \nabla \psi = \frac{B_\psi}{l_\psi} \frac{\partial}{\partial \psi} \psi = \frac{B_\psi}{l_\psi} = 0, \tag{4.1}$$

where $l_\psi = l_\psi(\psi, \theta, \zeta)$ is the scale factor for the coordinate ψ, defined in (**G2**.1.36). This equation clearly requires $B_\psi = 0$, hence the magnetic field has the form

$$\boldsymbol{B} = B_\theta \, \boldsymbol{e}_\theta + B_\zeta \, \boldsymbol{e}_\zeta. \tag{4.2}$$

Physically, it is clear that $B_\theta(\psi, \theta, \zeta)$ and $B_\zeta(\psi, \theta, \zeta)$ must be single-valued functions. As the angles θ and ζ are defined *modulo* 2π, the components of $\boldsymbol{B}$ must be periodic functions of these angles.

On the other hand, the magnetic field must be divergence-free. This introduces an additional condition (see **G2**.2.16):

$$\nabla \cdot \boldsymbol{B} = \frac{1}{l_\psi l_\theta l_\zeta} \left(\frac{\partial}{\partial \theta} \left(B_\theta l_\zeta l_\psi \right) + \frac{\partial}{\partial \zeta} \left(B_\zeta l_\psi l_\theta \right) \right) = 0. \tag{4.3}$$

As a result, it is clear that the two components of the magnetic field can be expressed in terms of a single function $\vartheta(\psi, \theta, \zeta)$,

$$B_\theta = -\frac{1}{2\pi l_\zeta l_\psi} \frac{\partial \vartheta}{\partial \zeta}, \qquad B_\zeta = \frac{1}{2\pi l_\psi l_\theta} \frac{\partial \vartheta}{\partial \theta}, \tag{4.4}$$

where the factor 2π has been written for further convenience.

The function $\vartheta(\psi, \theta, \zeta)$ is directly connected to the vector potential. Indeed, a magnetic field of the form (4.2) derives from a vector potential of the form

$$\boldsymbol{A} = A_\psi(\psi, \theta, \zeta) \, \boldsymbol{e}_\psi. \tag{4.5}$$

The expression of the magnetic field in the form

$$\boldsymbol{B} = \nabla \wedge \boldsymbol{A}, \tag{4.6}$$

written explicitly by using (**G2**.2.17) yields two relations of the same form as (4.4), with the identification

$$\vartheta = -2\pi A_\psi. \tag{4.7}$$

This is a good place for introducing a word of caution. It should be clear that (4.5) is not the only vector potential from which the magnetic field derives through (4.6). There are infinitely many vector potentials compatible with (4.2): they are mutually related by gauge transformations. We shall meet later with other convenient realizations. Let us just mention that the choice (4.5) is particularly simple, because this vector potential has only one radial component. On the other hand, this radial component must explicitly depend on all three variables in order to reproduce (4.2).

Consider now the equation of a field line (lying on the magnetic surface ψ):

$$\frac{l_\theta \, \mathrm{d}\theta}{B_\theta} = \frac{l_\zeta \, \mathrm{d}\zeta}{B_\zeta}. \tag{4.8}$$

Combining this with (4.4) we find

$$B_\zeta l_\theta \, \mathrm{d}\theta - B_\theta l_\zeta \, \mathrm{d}\zeta = \frac{1}{2\pi l_\psi}\left(\frac{\partial \vartheta}{\partial \theta}\mathrm{d}\theta + \frac{\partial \vartheta}{\partial \zeta}\mathrm{d}\zeta\right) = \frac{1}{2\pi l_\psi}\mathrm{d}\vartheta = 0,$$

where $\mathrm{d}\vartheta$ is the total differential of ϑ, with ψ held fixed. Thus, the equation of a field line is expressed by equating $\vartheta(\psi, \theta, \zeta)$ to a constant in θ and ζ:

$$\vartheta(\psi, \theta, \zeta) = c(\psi). \tag{4.9}$$

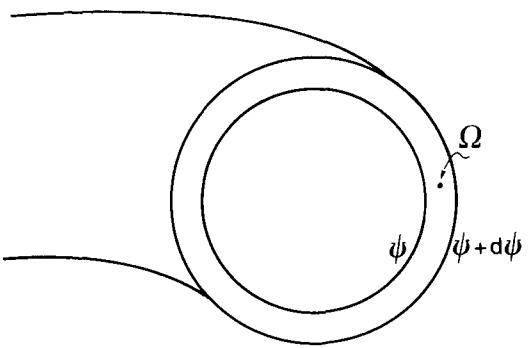

Fig 4 1. Shell Ω contained between two magnetic surfaces ψ and $\psi + \mathrm{d}\psi$

But, in order to ensure the single-valuedness of B_θ, B_ζ defined by (4.4), function ϑ can only depend on the angles as

$$\vartheta(\psi, \theta, \zeta) = a(\psi)\theta + b(\psi)\zeta + \tilde{\vartheta}(\psi, \theta, \zeta), \tag{4.10}$$

where $\tilde{\vartheta}(\psi, \theta, \zeta)$ is an arbitrary function, periodic in θ and ζ.

We now show that the coefficients $a(\psi)$, $b(\psi)$ have a simple physical meaning. To do so, we calculate the toroidal flux through the surface of an annular shell Ω contained between two adjacent magnetic surfaces ψ, $\psi + \mathrm{d}\psi$ (see fig. 4.1). Using (3.2), (4.4), (4.10) and the results of the General Appendix **G2**.2 (namely, **G2**.2.14, **G2**.2.15), we find

$$\begin{aligned}
\mathrm{d}\Phi_T &= \frac{1}{2\pi}\int_\Omega \mathrm{d}^3\boldsymbol{x}\, \boldsymbol{B}\cdot\nabla\zeta = \frac{1}{2\pi}\int_\psi^{\psi+\mathrm{d}\psi}\mathrm{d}\psi'\int_0^{2\pi}\mathrm{d}\theta\,\mathrm{d}\zeta\, l_\psi l_\theta l_\zeta \frac{B_\zeta}{l_\zeta} \\
&= \frac{1}{2\pi}\int_\psi^{\psi+\mathrm{d}\psi}\mathrm{d}\psi'\int_0^{2\pi}\mathrm{d}\theta\,\mathrm{d}\zeta \frac{1}{2\pi}\frac{\partial\vartheta(\psi', \theta, \zeta)}{\partial\theta} \\
&= \frac{1}{(2\pi)^2}\int_\psi^{\psi+\mathrm{d}\psi}\mathrm{d}\psi'\int_0^{2\pi}\mathrm{d}\theta\,\mathrm{d}\zeta \\
&\qquad \times\left(\frac{\partial\vartheta(\psi, \theta, \zeta)}{\partial\theta} + \mathrm{d}\psi\frac{\partial^2\vartheta(\psi, \theta, \zeta)}{\partial\psi\,\partial\theta} + \cdots\right) \\
&= \frac{\mathrm{d}\psi}{(2\pi)^2}\int_0^{2\pi}\mathrm{d}\theta\,\mathrm{d}\zeta\frac{\partial\vartheta}{\partial\theta} + \mathrm{O}(\mathrm{d}\psi^2) \\
&= \frac{\mathrm{d}\psi}{(2\pi)^2}\int_0^{2\pi}\mathrm{d}\theta\,\mathrm{d}\zeta\left(a(\psi) + \frac{\partial\tilde{\vartheta}(\psi, \theta, \zeta)}{\partial\theta}\right) \\
&= \frac{\mathrm{d}\psi}{(2\pi)^2}\left\{(2\pi)^2 a(\psi) + \int_0^{2\pi}\mathrm{d}\zeta\left[\tilde{\vartheta}(\psi, 2\pi, \zeta) - \tilde{\vartheta}(\psi, 0, \zeta)\right]\right\}.
\end{aligned} \tag{4.11}$$

As $\tilde{\vartheta}$ is periodic in θ, we conclude that

$$a(\psi) = \frac{\mathrm{d}\Phi_T(\psi)}{\mathrm{d}\psi}. \tag{4.12}$$

Calculating in the same way the poloidal flux through an annular ribbon, we find

$$b(\psi) = -\frac{\mathrm{d}\Phi_{\mathrm{P}}(\psi)}{\mathrm{d}\psi}. \tag{4.13}$$

Thus, the generating function ϑ is written explicitly as

$$\vartheta(\psi, \theta, \zeta) = \frac{\mathrm{d}\Phi_{\mathrm{T}}}{\mathrm{d}\psi}\theta - \frac{\mathrm{d}\Phi_{\mathrm{P}}}{\mathrm{d}\psi}\zeta + \tilde{\vartheta}(\psi, \theta, \zeta). \tag{4.14}$$

At this point, we make either one of the following changes of coordinates, which enables us to "get rid" of the periodic function $\tilde{\vartheta}$,

$$\theta = \bar{\theta} - \frac{\tilde{\vartheta}}{a(\psi)}, \qquad \zeta = \bar{\zeta}, \tag{4.15a}$$

or

$$\theta = \bar{\theta}, \qquad \zeta = \bar{\zeta} - \frac{\tilde{\vartheta}}{b(\psi)}. \tag{4.15b}$$

In terms of any of these variables, the generating function takes the simple form

$$\vartheta(\psi, \bar{\theta}, \bar{\zeta}) = \frac{\mathrm{d}\Phi_{\mathrm{T}}}{\mathrm{d}\psi}\bar{\theta} - \frac{\mathrm{d}\Phi_{\mathrm{P}}}{\mathrm{d}\psi}\bar{\zeta}. \tag{4.16}$$

The variables $(\psi, \bar{\theta}, \bar{\zeta})$ will be called *natural coordinates*. They have a number of remarkable properties. The most outstanding is obtained from (4.9), which yields the equation of a magnetic field line in the form

$$\frac{\mathrm{d}\Phi_{\mathrm{T}}}{\mathrm{d}\psi}\bar{\theta} - \frac{\mathrm{d}\Phi_{\mathrm{P}}}{\mathrm{d}\psi}\bar{\zeta} = c(\psi) \tag{4.17}$$

Thus expressed in terms of natural coordinates, *the field lines appear as straight lines*. To make this picture more precise, imagine that a magnetic surface is cut along a poloidal contour, then along a toroidal one, and the resulting surface is laid out on a plane.

If the angles $\bar{\theta}$ and $\bar{\zeta}$ are used as coordinates, the image of the surface is a square of side equal to 2π. A magnetic field line is represented by a set of *straight segments* constructed by the rule of periodic boundary conditions.

Whenever a segment hits the boundary, it re-enters the square from the opposite side (because $\bar{\theta}$ and $\bar{\theta} + 2\pi$ are equivalent, and similarly for $\bar{\zeta}$).

Another important result comes as a bonus. Consider definition (1.3) of the *rotational transform* ι. When expressed in terms of natural coordinates, $\Delta\theta_j$ is, of course, a constant (independent of j). Moreover, it is clear that the ratio $\iota/2\pi$ is none other than the slope of the straight line in the natural coordinates:

$$\frac{1}{q} = \frac{\iota}{2\pi} = \frac{\mathrm{d}\bar{\theta}}{\mathrm{d}\bar{\zeta}} = \frac{\mathrm{d}\Phi_P/\mathrm{d}\psi}{\mathrm{d}\Phi_T/\mathrm{d}\psi}. \tag{4.18}$$

This result provides the proof that the *rotational transform is a surface function.* If we choose, in particular, the toroidal flux as a radial coordinate, $\psi = \Phi_T$, we find the simple form

$$\frac{1}{q} = \frac{\iota}{2\pi} = \frac{\mathrm{d}\Phi_P}{\mathrm{d}\Phi_T}. \tag{4.19}$$

The difference between a rational and an ergodic surface is now easily visualized. If the slope of the line is a rational number, the graph of the field line consists of a finite number of segments (see fig. 4.2a); in the case of an irrational value of q, the graph consists of an infinite number of segments, densely covering the square (fig. 2.4b).

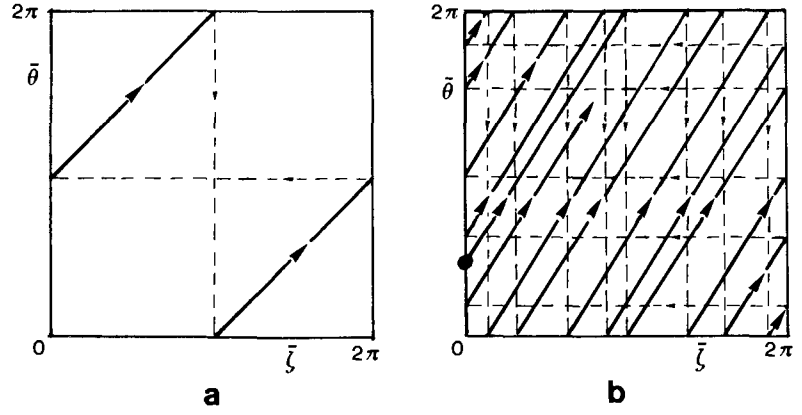

Fig. 4.2. Magnetic surface cut open and laid out flat. A magnetic field line, expressed in terms of $\bar{\theta}$, $\bar{\zeta}$ appears as a straight line. (a) Rational surface; (b) ergodic surface

We can now rewrite the equation of a field line as

$$\bar{\theta} = \bar{\theta}_0 + \frac{\iota}{2\pi}\bar{\zeta}. \tag{4.20}$$

Returning to eqs. (4.2), (4.4), (4.16), we find the following expression for the magnetic field vector:

$$\boldsymbol{B} = \frac{1}{2\pi l_\psi l_\theta l_\zeta}\left(\frac{\mathrm{d}\Phi_\mathrm{P}}{\mathrm{d}\psi} l_\theta \boldsymbol{e}_\theta + \frac{\mathrm{d}\Phi_\mathrm{T}}{\mathrm{d}\psi} l_\zeta \boldsymbol{e}_\zeta\right). \tag{4.21}$$

Let us finally mention the following point. It can be shown that the natural coordinates can be further restricted by requiring that the characteristic Jacobian defining the volume element be an absolute constant, e.g.

$$l_\psi l_\theta l_\zeta = 1. \tag{4.22}$$

Natural coordinates having the additional property (4.22) are called *Hamada coordinates* (Hamada 1962). The existence of such coordinates is a non-trivial property. We shall, however, not discuss this point, as we shall make no use of Hamada coordinates in this book.

8.5. Magnetic differential equations

We met in section 8.1 with the differential equation defining the magnetic surfaces $\psi(\boldsymbol{x}) = \psi$,

$$\boldsymbol{B} \cdot \nabla\psi = 0. \tag{5.1}$$

Clearly, any surface function $K(\psi)$ must satisfy this equation. In the forthcoming treatment we shall very often be faced with differential equations which are a direct generalization of (5.1),

$$\boldsymbol{B} \cdot \nabla f = \sigma, \tag{5.2}$$

where σ is a given scalar function of $\boldsymbol{x}$, or equivalently, of ψ, θ, ζ, and we want (5.2) to determine the unknown function f as a scalar, single-valued function of its arguments. A partial differential equation of the form (5.2) will be called a *magnetic differential equation*. This class of equations was first introduced by Kruskal and Kulsrud (1958).

A first, obvious statement is the following. If we know a particular solution $f_1(\psi, \theta, \zeta)$, then the general solution of (5.2) is of the form

$$f(\psi, \theta, \zeta) = f_1(\psi, \theta, \zeta) + K(\psi), \tag{5.3}$$

where $K(\psi)$ is an *arbitrary surface function*, obeying the homogeneous equation (5.1).

Next, we note that the partial differential equation (5.2) actually reduces to an ordinary differential equation. Let us recall, indeed, the results of section **1**.3. We introduce the *natural parameter* s (the proper length), in terms of which a magnetic field line is expressed as $\boldsymbol{x} = \boldsymbol{x}(s)$. We adopt again the notation of section **1**.3, putting a hat on a function $F(\boldsymbol{x})$ evaluated along the field line: $\hat{F}(s) \equiv F[\boldsymbol{x}(s)]$. Then (5.2) reduces to (see **1**.3.13)

$$\hat{B}(s)\frac{\mathrm{d}}{\mathrm{d}s}\hat{f}(s) = \hat{\sigma}(s). \tag{5.4}$$

If we prescribe a value $\hat{f}_0$ at a point s_0, the solution of this equation can be immediately written by integrating along the field line passing through s_0,

$$\hat{f}(s) = \hat{f}_0 + \int_{s_0}^{s} \mathrm{d}s' \frac{\hat{\sigma}(s')}{\hat{B}(s')}, \tag{5.5}$$

But this formal solution is not necessarily a satisfactory one. We now show that eq. (5.2) does not actually possess a solution meeting our requirements, unless the source term satisfies certain *integrability conditions* (Kruskal and Kulsrud 1958, Newcomb 1959, Soloviev and Shafranov 1970). In order to see this, we consider again the annular shell Ω defined in fig. 4.1. We integrate both sides of (5.2) over the volume of Ω, and apply the Gauss theorem. The left-hand side yields

$$\int_{\Omega} \mathrm{d}^3\boldsymbol{x}\, \boldsymbol{B}\cdot\nabla f = \int_{\Omega} \mathrm{d}^3\boldsymbol{x}\, \nabla\cdot(\boldsymbol{B}f) = \int_{\Sigma} \mathrm{d}\boldsymbol{S}\cdot(\boldsymbol{B}f) = 0, \tag{5.6}$$

where Σ is the surface bounding Ω, i.e. the set of two surfaces ψ, $\psi + \mathrm{d}\psi$. But clearly, by definition of a magnetic surface, we have $\mathrm{d}\boldsymbol{S}\cdot\boldsymbol{B} = 0$ everywhere, and the integral vanishes. * Therefore, for consistency, we must have

$$\int_{\Omega} \mathrm{d}^3\boldsymbol{x}\, \sigma(\boldsymbol{x}) = 0. \tag{5.7}$$

* Note that in (4.11) the calculation of a similar integral gave a non-vanishing result! The explanation is simple in that case we considered the function $f(\psi, \theta, \zeta) = \zeta$, i.e a *multivalued* function Any single-valued function f in (5.6) gives a zero flux.

We can transform this condition into a form related to a single magnetic surface. Using the result (**G2**.2.14), we express the integral in terms of the natural coordinates ψ, θ, ζ (see also 4.11):

$$\int_\Omega \mathrm{d}^3\boldsymbol{x}\ \sigma(\boldsymbol{x}) = \int_\psi^{\psi+\mathrm{d}\psi} \mathrm{d}\psi' \int \mathrm{d}\theta\ \mathrm{d}\zeta\ l_\psi l_\theta l_\zeta\ \sigma(\psi', \theta, \zeta)$$

$$= \mathrm{d}\psi \int \mathrm{d}\theta\ \mathrm{d}\zeta\ l_\psi l_\theta l_\zeta\ \sigma(\psi, \theta, \zeta) + \mathrm{O}(\mathrm{d}\psi^2) = 0. \tag{5.8}$$

We now note that (see **G2**.2.13)

$$\mathrm{d}\theta\ \mathrm{d}\zeta\ l_\theta l_\zeta = \mathrm{d}S_\psi$$

is just the surface element appropriate to the magnetic surface ψ, and that (see **G2**.1.15, **G2**.2.4)

$$l_\psi = |\nabla\psi|^{-1}.$$

Hence, dividing the integral in (5.8) by $\mathrm{d}\psi$ and letting $\mathrm{d}\psi \to 0$, we find

$$\int_\psi \mathrm{d}S_\psi \frac{\sigma}{|\nabla\psi|} = 0, \tag{5.9}$$

the integral being extended over the magnetic surface ψ. This *solubility condition* (Kruskal and Kulsrud 1958) must be satisfied on every magnetic surface, whether rational or ergodic.

It was pointed out later (Newcomb 1959) that (5.9) is a necessary solubility condition, but not a sufficient one. In order to understand this, we now fix our attention on a *rational* magnetic surface, on which all field lines are closed. We consider the formal solution (5.5), in which we integrate from s_0 up to the value $s = \bar{s}$ corresponding to a complete circuit around the field line passing through s_0. Thus, $\boldsymbol{x}(\bar{s}) = \boldsymbol{x}(s_0)$. It is clear that if the solution $f(\boldsymbol{x})$ of (5.2) is to be single-valued, we must have $\hat{f}(\bar{s}) = \hat{f}(s_0) \equiv \hat{f}_0$ and this is only possible if the source term σ satisfies the integrability condition

$$\oint \mathrm{d}s\ \frac{\hat{\sigma}(s)}{\hat{B}(s)} = 0. \tag{5.10}$$

This condition must be satisfied on all *rational* magnetic surfaces, i.e. on a dense set.

In order to clearly realize the relevance of these conditions, we write the solution of (5.2) in a more convenient form. Using the natural coordinates of section 8.4 (and omitting the overbars on θ and ζ), the left-hand side of the equation becomes

$$\boldsymbol{B}\cdot\nabla f=\left(\frac{B_\theta}{l_\theta}\frac{\partial}{\partial\theta}+\frac{B_\zeta}{l_\zeta}\frac{\partial}{\partial\zeta}\right)f=\frac{\Phi_T'}{2\pi l_\psi l_\theta l_\zeta}\left(\frac{\iota}{2\pi}\frac{\partial}{\partial\theta}+\frac{\partial}{\partial\zeta}\right)f, \tag{5.11}$$

where we used (4.21) and (4.18), and where $\Phi_T'\equiv \mathrm{d}\Phi_T/\mathrm{d}\psi$. Hence, (5.2) becomes

$$\left(\frac{\iota}{2\pi}\frac{\partial}{\partial\theta}+\frac{\partial}{\partial\zeta}\right)f=\frac{2\pi}{\Phi_T'}\,l_\psi l_\theta l_\zeta\,\sigma. \tag{5.12}$$

This equation is easily solved by Fourier transformation. We define

$$f(\psi,\theta,\zeta)=\sum_m\sum_n f_{mn}(\psi)\,\mathrm{e}^{\mathrm{i}(m\theta-n\zeta)},$$

$$l_\psi l_\theta l_\zeta\,\sigma=\sum_m\sum_n \rho_{mn}(\psi)\,\mathrm{e}^{\mathrm{i}(m\theta-n\zeta)}. \tag{5.13}$$

Then (5.12) reduces to

$$\mathrm{i}\left(\frac{\iota}{2\pi}m-n\right)f_{mn}(\psi)=\frac{2\pi}{\Phi_T'(\psi)}\rho_{mn}(\psi). \tag{5.14}$$

This equation makes sense only if, for $m=n=0$, the right-hand side vanishes:

$$\rho_{00}(\psi)=0. \tag{5.15}$$

It is immediately seen that this equation is equivalent to (5.8) and (5.9), i.e. the Kruskal–Kulsrud condition. If this condition is satisfied, the formal solution of (5.14) is

$$f_{mn}(\psi)=-\frac{2\pi\mathrm{i}}{\Phi_T'(\psi)}\frac{\rho_{mn}(\psi)}{(2\pi)^{-1}\iota(\psi)\;m-n}+\delta_{m,0}\delta_{n,0}K(\psi), \tag{5.16}$$

where $K(\psi)$ is the arbitrary surface function of (5.3).

The characteristic feature of this solution is the presence of *resonant denominators*, a very common feature in many problems of mechanics and statistical mechanics. Consider, indeed, what happens when ψ takes a value ψ_{pq} corresponding to a rational magnetic surface, for which the rotational transform has the value

$$\frac{\iota(\psi_{pq})}{2\pi} = \frac{q}{p}, \tag{5.17}$$

where q and p are integers. For that value of ψ, the denominator in (5.16) vanishes for $m = p$, $n = q$. Hence, a finite solution of (5.2) exists only if

$$\rho_{pq}(\psi_{pq}) = 0 \tag{5.18}$$

for all rational surfaces $\psi = \psi_{pq}$. It is an easy exercise, left to the reader, to show that (5.18) is equivalent to the Newcomb solubility condition (5.10).

This condition is very stringent, as it imposes conditions on the Fourier components on a dense set of values of ψ. Let us note that there is one case, of great physical importance, for which the Newcomb condition (5.18) is irrelevant and the Kruskal–Kulsrud criterion is sufficient for solubility.

We might be interested in *solutions of* (5.2) *which are independent of the toroidal angle* ζ. Such solutions are quite natural in a study of axisymmetric systems, such as the tokamak. In this case, (5.2) or, equivalently, (5.12) reduces to

$$\frac{\iota}{2\pi}\frac{\partial}{\partial\theta} f(\psi, \theta) = \frac{2\pi}{\Phi_T'} l_\psi l_\theta l_\zeta \sigma(\psi, \theta), \tag{5.19}$$

where, of course, the source term σ as well as the scale factors are assumed to be independent of ζ. In the Fourier representation (5.13), this means that only the coefficients f_{m0}, ρ_{m0} appear in the Fourier expansion. Equation (5.16) then reduces to

$$f_{m0}(\psi) = -\frac{2\pi_{\mathrm{i}}}{\Phi_T'(\psi)}\frac{2\pi}{\iota(\psi)}\frac{\rho_{m0}(\psi)}{m} + \delta_{m,0}K(\psi). \tag{5.20}$$

We see that there are no resonant denominators anymore, and solution (5.15) is well-behaved, provided (5.15) is met. Thus, for this class of solutions, the Kruskal–Kulsrud criterion is necessary and sufficient for the solubility.

8.6. Surface averages

We introduce here a concept which will prove very useful, especially in transport theory. Consider, for instance, the flux of particles across the magnetic surfaces. This quantity is of primordial importance for evaluating the quality of a confinement device, as it measures the leakage of matter through the magnetic "walls". Such a quantity depends, in general, on the position of the toroidal surface, i.e. on θ and ζ. However, the experimentalist is usually interested in a more global quantity which can be directly measured, i.e. an average over the surface. We now construct a theoretical tool which produces this effect, and has some additional desirable properties.

Let $A = A(\psi, \theta, \zeta)$ be an arbitrary, single-valued function of ψ, θ, ζ. We define an averaging (or *surface averaging*) operation $\langle \cdots \rangle$ which has the following properties (Hinton and Hazeltine 1976):

(a) *The average of any function is a surface quantity*:

$$\langle A(\psi, \theta, \zeta) \rangle = a(\psi). \tag{6.1}$$

(b) *The average of any surface quantity* $b(\psi)$ *reproduces the quantity itself*:

$$\langle b(\psi) \rangle = b(\psi). \tag{6.2}$$

(c) *The averaging operation annihilates the effect of the operator* $\boldsymbol{B} \cdot \nabla$ *applied to an arbitrary function*

$$\langle \boldsymbol{B} \cdot \nabla A(\psi, \theta, \zeta) \rangle = 0. \tag{6.3}$$

We now look for an explicit realization of this operator. Condition (c) gives us a clue, if we remember (5.6). Let us define

$$\langle A(\psi, \theta, \zeta) \rangle = \frac{\int_\Omega \mathrm{d}^3 x \, A}{\int_\Omega \mathrm{d}^3 x}, \tag{6.4}$$

where Ω is an annular shell contained between two adjacent magnetic surfaces ψ, $\psi + \mathrm{d}\psi$ (see fig. 4.1); the limit $\mathrm{d}\psi \to 0$ is understood in (6.4).

This definition produces an averaged quantity which has the same dimensions as A, a desirable feature. Let us verify the fulfillment of our three requirements.

The numerator is easily evaluated [see eq. (4.11) for a similar calculation]:

$$\int_\Omega \mathrm{d}^3\boldsymbol{x}\, A = \int_\psi^{\psi+\mathrm{d}\psi} \mathrm{d}\psi' \int_0^{2\pi} \mathrm{d}\theta\, \mathrm{d}\zeta\, l_{\psi'} l_\theta l_\zeta\, A(\psi', \theta, \zeta)$$

$$= \int_\psi^{\psi+\mathrm{d}\psi} \mathrm{d}\psi' \int_0^{2\pi} \mathrm{d}\theta\, \mathrm{d}\zeta \Big\{ l_\psi l_\theta l_\zeta\, A(\psi, \theta, \zeta)$$

$$+ (\mathrm{d}\psi) \frac{\partial}{\partial \psi'} \left[l_{\psi'} l_\theta l_\zeta\, A(\psi', \theta, \zeta) \right] + \cdots \Big\}$$

$$= \mathrm{d}\psi \int_0^{2\pi} \mathrm{d}\theta\, \mathrm{d}\zeta\, l_\psi l_\theta l_\zeta\, A(\psi, \theta, \zeta) + \mathrm{O}(\mathrm{d}\psi^2). \tag{6.5}$$

We now note that (see eq. **G2**.2.13)

$$\mathrm{d}\theta\, \mathrm{d}\zeta\, l_\theta l_\zeta = \mathrm{d}S_\psi,$$

where $\mathrm{d}S_\psi$ is the absolute value of the surface element perpendicular to the magnetic surface. Further, we note from (**G2**.2.15) that $l_\psi = |\nabla\psi|^{-1}$; thus

$$\int_\omega \mathrm{d}^3\boldsymbol{x}\, A = \mathrm{d}\psi \int_\psi \mathrm{d}S_\psi \frac{1}{|\nabla\psi|} A(\psi, \theta, \zeta). \tag{6.6}$$

The denominator of (6.4) is also very simple. Recalling the definition of the volume V enclosed by a magnetic surface, (3.1), we have

$$\int_\Omega \mathrm{d}^3\boldsymbol{x} = V(\psi + \mathrm{d}\psi) - V(\psi) = \mathrm{d}\psi \frac{\mathrm{d}V}{\mathrm{d}\psi} + \mathrm{O}(\mathrm{d}\psi^2). \tag{6.7}$$

We thus obtain an alternative expression for the surface average:

$$\langle A(\psi, \theta, \zeta) \rangle = \left(\frac{\mathrm{d}V}{\mathrm{d}\psi} \right)^{-1} \int_\psi \mathrm{d}S_\psi \frac{1}{|\nabla\psi|} A(\psi, \theta, \zeta), \tag{6.8}$$

a form which is often met in the literature. *

In the forthcoming sections we shall transform (6.8) into other, equivalent forms, valid in specific cases. Before concluding this section, we prove two

* We may note that Bernstein (1974) adopts this definition *without* the factor $(\mathrm{d}V/\mathrm{d}\psi)^{-1}$. This has the disadvantage that the dimensions of A and of $\langle A \rangle$ are different

other useful properties of the averaging operation (Hinton and Hazeltine 1976). For an arbitrary vector function $\boldsymbol{F}(\psi, \theta, \zeta)$ we have

$$\langle \nabla \cdot \boldsymbol{F} \rangle = \frac{1}{V'} \frac{\mathrm{d}}{\mathrm{d}\psi} V' \langle \boldsymbol{F} \cdot \nabla \psi \rangle, \tag{6.9}$$

$$\langle (\nabla \wedge \boldsymbol{F}) \cdot \nabla \psi \rangle = 0, \tag{6.10}$$

where $V' \equiv \mathrm{d}V/\mathrm{d}\psi$.

The proof of (6.9) uses eq. (6.8) and the expression of the divergence in curvilinear coordinates (**G**2.2.16):

$$\langle \nabla \cdot \boldsymbol{F} \rangle = \frac{1}{V'} \int \mathrm{d}\theta \, \mathrm{d}\zeta \, l_\psi l_\theta l_\zeta \frac{1}{l_\psi l_\theta l_\zeta} \left(\frac{\partial}{\partial \psi} l_\theta l_\zeta F_\psi + \frac{\partial}{\partial \theta} \cdots + \frac{\partial}{\partial \zeta} \cdots \right).$$

The terms in $\partial/\partial\theta$ and $\partial/\partial\zeta$ vanish by integration over the angles, and we are left with

$$\begin{aligned} \langle \nabla \cdot \boldsymbol{F} \rangle &= \frac{1}{V'} \frac{\mathrm{d}}{\mathrm{d}\psi} \int \mathrm{d}\theta \, \mathrm{d}\zeta \, l_\theta l_\zeta F_\psi \\ &= \frac{1}{V'} \frac{\mathrm{d}}{\mathrm{d}\psi} V' \left\langle l_\psi^{-1} F_\psi \right\rangle = \frac{1}{V'} \frac{\mathrm{d}}{\mathrm{d}\psi} V' \left\langle l_\psi^{-1} \boldsymbol{e}_\psi \cdot \boldsymbol{F} \right\rangle \\ &= \frac{1}{V'} \frac{\mathrm{d}}{\mathrm{d}\psi} V' \langle \boldsymbol{F} \cdot \nabla \psi \rangle. \end{aligned}$$

Equation (6.10) is a consequence of (6.9),

$$\begin{aligned} \langle (\nabla \wedge \boldsymbol{F}) \cdot \nabla \psi \rangle &= \langle \nabla \cdot (\boldsymbol{F} \wedge \nabla \psi) \rangle + \langle \boldsymbol{F} \cdot \nabla \wedge (\nabla \psi) \rangle \\ &= \frac{1}{V'} \frac{\mathrm{d}}{\mathrm{d}\psi} V' \langle \nabla \psi \cdot (\boldsymbol{F} \wedge \nabla \psi) \rangle = 0. \end{aligned}$$

8.7. *The Clebsch representation*

In this section we discuss several useful (and very widely used) representations of a toroidal confining magnetic field. Such a field must satisfy the following equations (see eqs. 1.2, 2.3):

$$\nabla \cdot \boldsymbol{B} = 0, \tag{7.1}$$

$$\boldsymbol{B} \cdot \nabla \psi = 0, \tag{7.2}$$

where ψ is any surface function. We parametrize the space by a set of toroidal coordinates in which ψ is the radial coordinate, θ the poloidal angle and ζ the toroidal angle. (ψ, θ, ζ) is not necessarily a natural coordinate system in the sense of section 8.4. * It is characterized by a set of scale factors l_ψ, l_θ, l_ζ (**G2**.1.36), functions of ψ, θ, ζ, and such that

$$\nabla\psi = l_\psi^{-1}\, \boldsymbol{e}_\psi = \hat{\boldsymbol{e}}^\psi, \quad \nabla\theta = l_\theta^{-1}\, \boldsymbol{e}_\theta = \hat{\boldsymbol{e}}^\theta, \quad \nabla\zeta = l_\zeta^{-1}\, \boldsymbol{e}_\zeta = \hat{\boldsymbol{e}}^\zeta. \tag{7.3}$$

Note that these are simply the three contravariant basis vectors (**G2**.1.15).

From (7.2) follows the condition (see eq. 4.2)

$$\boldsymbol{B} = B_\theta\, \boldsymbol{e}_\theta + B_\zeta\, \boldsymbol{e}_\zeta. \tag{7.4}$$

We now note that (7.1) is automatically satisfied if we take $\boldsymbol{B}$ of the form

$$\boldsymbol{B} = (\nabla\alpha) \wedge (\nabla\beta), \tag{7.5}$$

where α, β are two arbitrary scalar fields. This form appears very frequently in the literature: it is sometimes called the *Clebsch representation*. One can also use a more general form

$$\boldsymbol{B} = \sum_i \left[(\nabla\alpha_i) \wedge (\nabla\beta_i)\right], \tag{7.6}$$

where α_i, β_i are arbitrary scalar fields, and the sum contains any number of terms. The proof is elementary: it suffices to note that (7.6) is equivalent to

$$\boldsymbol{B} = \nabla \wedge \boldsymbol{A}, \tag{7.7}$$

together with

$$\boldsymbol{A} = \sum_i \alpha_i \nabla\beta_i. \tag{7.8}$$

The large degree of arbitrariness in (7.6) allows us to use a wide variety of representations. A particularly useful one consists of taking two terms in (7.6) and choosing $\beta_2 = -\zeta$ and $\beta_3 = \theta$:

$$\boldsymbol{B} = -(\nabla\alpha_2) \wedge (\nabla\zeta) + (\nabla\alpha_3) \wedge (\nabla\theta). \tag{7.9}$$

The minus sign is introduced here for later convenience of the physical interpretation.

* We assume, for simplicity, that it is an *orthogonal coordinate system*

Expressing the fact that the field $\boldsymbol{B}$ must satisfy (7.2) puts a restriction on the functions α_2, α_3. Indeed,

$$(\nabla\alpha_3)\wedge\nabla\theta=\frac{\partial\alpha_3}{\partial\zeta}\nabla\zeta\wedge\nabla\theta+\frac{\partial\alpha_3}{\partial\psi}\nabla\psi\wedge\nabla\theta$$

$$=-\frac{\partial\alpha_3}{\partial\zeta}\frac{1}{l_\zeta l_\theta}\boldsymbol{e}_\psi+\frac{\partial\alpha_3}{\partial\psi}\frac{1}{l_\psi l_\theta}\boldsymbol{e}_\zeta,$$

where we used (7.3) and (2.7). Comparing with (7.4), we see that α_3 must be independent of ζ. It is similarly shown that α_2 must be independent of θ. As l_ψ, l_θ, l_ζ may still depend on the angles, we do not loose any generality by choosing the functions α_2 and α_3 to depend only on ψ, i.e. to be surface functions; thus

$$\boldsymbol{B}=-\nabla\alpha_2(\psi)\wedge\nabla\zeta+\nabla\alpha_3(\psi)\wedge\nabla\theta. \tag{7.10}$$

This is the form chosen, among others, by Hinton and Hazeltine (1976). This representation can be written more explicitly as

$$\boldsymbol{B}=\frac{1}{l_\psi l_\zeta}\frac{\mathrm{d}\alpha_2(\psi)}{\mathrm{d}\psi}\boldsymbol{e}_\theta+\frac{1}{l_\psi l_\theta}\frac{\mathrm{d}\alpha_3(\psi)}{\mathrm{d}\psi}\boldsymbol{e}_\zeta. \tag{7.11}$$

This formula shows that, in this representation, the dependence of $\boldsymbol{B}$ on θ and ζ is entirely contained in the scale factors l_i and in the local unit vectors $\boldsymbol{e}_\theta$, $\boldsymbol{e}_\zeta$. *

Conversely, it may be said that representation (7.10) *amounts to constructing a coordinate system intrinsically bound to the given magnetic field* $\boldsymbol{B}$. Indeed, we see here a simple relation between the geometrical scale factors and the components of the field,

$$\begin{aligned}\frac{1}{l_\psi(\psi,\theta,\zeta)\,l_\zeta(\psi,\theta,\zeta)}&=\frac{1}{\mathrm{d}\alpha_2/\mathrm{d}\psi}B_\theta(\psi,\theta,\zeta),\\ \frac{1}{l_\psi(\psi,\theta,\zeta)\,l_\theta(\psi,\theta,\zeta)}&=\frac{1}{\mathrm{d}\alpha_3/\mathrm{d}\psi}B_\zeta(\psi,\theta,\zeta).\end{aligned} \tag{7.12}$$

* This would not be the case, had we chosen to use the complete freedom available in (7.9), i.e. had we chosen $\alpha_2=\alpha_2(\psi,\theta)$, $\alpha_3=\alpha_3(\psi,\zeta)$ In this case we might, in principle, choose an arbitrary coordinate system and introduce the correct angle dependence of $\boldsymbol{B}$ through functions α_2, α_3.

There is an analogy between the present toroidal coordinate system and the coordinate system used in section 1.3, in the sense that both are intrinsically determined by the magnetic field. But in the latter case, the local triad was fixed by the geometrical properties of a *curve*, the magnetic field line, whereas here the local triad is determined by the properties of a *surface*. This is natural in the confinement situation, where on almost every surface, each magnetic field line covers densely the entire surface. In subsequent chapters, we shall have several opportunities of discussing further relations between the various intrinsic coordinate systems.

A second important remark concerns the physical meaning of functions α_2, α_3 in representation (7.10). Let us calculate the toroidal flux, defined by (3.2),

$$\begin{aligned}\Phi_T(\psi) &= \frac{1}{2\pi}\int_\psi d^3x\, \boldsymbol{B}\cdot\nabla\zeta \\ &= \frac{1}{2\pi}\int_\psi d\psi'\, d\theta\, d\zeta\, l_{\psi'} l_\theta l_\zeta \frac{1}{l_\zeta} B_\zeta \\ &= \frac{1}{2\pi}\int_\psi d\psi'\, d\theta\, d\zeta\, l_{\psi'} l_\theta \frac{1}{l_\psi l_\theta}\frac{d\alpha_3(\psi')}{d\psi'} \\ &= \frac{4\pi^2}{2\pi}\int_\psi d\psi' \frac{d\alpha_3}{d\psi'} = 2\pi\alpha_3(\psi). \end{aligned} \tag{7.13}$$

A similar calculation shows that

$$\Phi_P(\psi) = 2\pi\alpha_2(\psi). \tag{7.14}$$

In these calculations it is assumed that both α_2 and α_3 vanish at the magnetic axis ($\psi = 0$). It is always possible to meet this condition, because these functions enter (7.10) only through their gradients.

Thus, *the surface functions* $\alpha_2(\psi)$ *and* $\alpha_3(\psi)$ *in* (7.10) *and* (7.11) *are, respectively, the poloidal flux and the toroidal flux* (up to a factor 2π).

We now demonstrate another property, by looking at the field lines defined by (7.11). From (4.8) we write the equation of a field line as

$$\frac{d\theta}{d\zeta} = \frac{l_\zeta}{l_\theta}\frac{B_\theta}{B_\zeta} = \frac{d\alpha_2(\psi)/d\psi}{d\alpha_3(\psi)/d\psi}. \tag{7.15}$$

This result shows that *the slope of any field line is constant on each magnetic surface*. Looking at this result from the point of view of the discussion in section 8.4, we arrive at the following, very important conclusion: *The intrinsic*

coordinate system defined by (7.10) *is automatically a natural coordinate system* (in the sense defined in section 8.4). The value (7.15) for the slope of a field line, which is simply the rotational transform, coincides with (4.18): this property follows from (7.13), (7.14).

We now show that definition (6.8) of the averaging operation can be written in two interesting forms when representation (7.10) is used for the magnetic field. From (7.12) we find the following expression for the Jacobian of our coordinate system [using also eq. (7.3)]:

$$l_\psi l_\theta l_\zeta = \frac{\mathrm{d}\alpha_2}{\mathrm{d}\psi}\frac{l_\theta}{B_\theta} = \frac{\mathrm{d}\alpha_2}{\mathrm{d}\psi}\frac{1}{\boldsymbol{B}\cdot\nabla\theta}.$$

Hence, (6.8) can be written in the form

$$\langle A\rangle = \frac{1}{2\pi}\frac{\mathrm{d}\Phi_P/\mathrm{d}\psi}{\mathrm{d}V/\mathrm{d}\psi}\int \mathrm{d}\theta\,\mathrm{d}\zeta\,\frac{A}{\boldsymbol{B}\cdot\nabla\theta}. \tag{7.16}$$

Alternatively, we may write

$$\langle A\rangle = \frac{1}{2\pi}\frac{\mathrm{d}\Phi_T/\mathrm{d}\psi}{\mathrm{d}V/\mathrm{d}\psi}\int \mathrm{d}\theta\,\mathrm{d}\zeta\,\frac{A}{\boldsymbol{B}\cdot\nabla\zeta}. \tag{7.17}$$

8.8. Axisymmetric systems

We now discuss a slightly different representation of the magnetic field, which is widely used in the literature in the case of *axisymmetric systems* (fig. 8.1). The idea of this representation will be again to construct a coordinate system ψ, θ, ζ adapted to the symmetry of the configuration. This requires the axisymmetry of the coordinate system itself: the scale factors l_ψ, l_θ, l_ζ must be functions of ψ and θ, but not of ζ. We describe in the present section two particular choices of coordinate systems: the first one is very widely used in the literature; the second one is of special interest in transport theory and will be used in most of the forthcoming chapters of this book.

(A) The poloidal flux as a magnetic surface label

In the first alternative we start from the general representation (7.10). We specifically choose as the coordinate ψ labelling the magnetic surfaces, the *poloidal flux divided by* 2π, i.e. we take $\alpha_2(\psi) = (2\pi)^{-1}\Phi_P \equiv \psi$. Moreover, we write the second term, i.e. the toroidal field, in a different form,

$$\boldsymbol{B} = -\nabla\psi\wedge\nabla\zeta + \mathscr{I}\nabla\zeta. \tag{8.1}$$

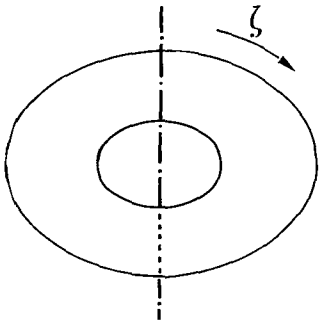

Fig. 8 1 In an axisymmetric system, the central axis through the hole of the toroid is a symmetry axis for all physical quantities

This widely used representation will now be further analyzed. Writing (8.1) more explicitly, we find

$$\boldsymbol{B} = \frac{1}{l_\psi l_\zeta}\boldsymbol{e}_\theta + \frac{\mathscr{I}}{l_\zeta}\boldsymbol{e}_\zeta, \tag{8.2}$$

where the scale factors, as well as the function $\mathscr{I}$ must be independent of ζ in order to describe an axisymmetric field. It is easily checked that these conditions are sufficient in order to ensure automatically that the field $\boldsymbol{B}$ is divergence-free.

We find some further interesting relations by studying the electric current density, which is related to $\boldsymbol{B}$ through the Maxwell equation (2.2),

$$\boldsymbol{j} = \frac{c}{4\pi}(\nabla \wedge \boldsymbol{B}). \tag{8.3}$$

The components of the curl of $\boldsymbol{B}$ are easily calculated by using (**G2**.2.17)

$$j_\psi = \frac{c}{4\pi}\frac{1}{l_\theta l_\zeta}\frac{\partial \mathscr{I}}{\partial \theta}, \tag{8.4}$$

$$j_\theta = -\frac{c}{4\pi}\frac{1}{l_\zeta l_\psi}\frac{\partial \mathscr{I}}{\partial \psi}, \tag{8.5}$$

$$j_\zeta = \frac{c}{4\pi}\frac{1}{l_\psi l_\theta}\frac{\partial}{\partial \psi}\frac{l_\theta}{l_\psi l_\zeta}. \tag{8.6}$$

We now note that, *in a situation of MHD-equilibrium*, the current and the magnetic field are related to the pressure which is a surface function (see eq. 2.1),

$$\boldsymbol{j} \wedge \boldsymbol{B} = c\, \nabla P(\psi). \tag{8.7}$$

Consider the θ-component of this equation: the right-hand side is zero, and we obtain

$$(\boldsymbol{j} \wedge \boldsymbol{B})_\theta = j_\zeta B_\psi - j_\psi B_\zeta = 0.$$

But B_ψ is necessarily zero, whereas B_ζ is different from zero in all physically interesting cases; we thus conclude that

$$j_\psi = 0. \tag{8.8}$$

Thus, *the current, in an equilibrium configuration, flows only along the magnetic surfaces, not across them*. This condition can also be written as

$$\boldsymbol{j} \cdot \nabla\psi = 0. \tag{8.9}$$

Thus, in equilibrium, *the magnetic surfaces are also current surfaces.*

Considering now (8.4), we see that this condition puts an additional constraint on the function $\mathscr{I}$, which must be independent of θ. Thus $\mathscr{I}$ *must be a surface function*:

$$\mathscr{I} = \mathscr{I}(\psi). \tag{8.10}$$

The ζ-component of (8.7) tells us nothing new, but the ψ-component provides us with the relation

$$j_\theta B_\zeta - j_\zeta B_\theta = \frac{c}{l_\psi} \frac{\mathrm{d}P}{\mathrm{d}\psi}, \tag{8.11}$$

which is transformed by using (8.2), (8.5), (8.6) and (8.10):

$$\frac{l_\zeta}{l_\psi l_\theta} \frac{\partial}{\partial\psi} \frac{l_\theta}{l_\psi l_\zeta} = -\mathscr{I} \frac{\mathrm{d}\mathscr{I}}{\mathrm{d}\psi} - 4\pi l_\zeta^2 \frac{\mathrm{d}P}{\mathrm{d}\psi}. \tag{8.12}$$

We note that the expression on the left-hand side can be understood by calculating the quantity (using the formulae of the General Appendix G2)

$$l_\zeta^2\,\nabla\cdot\left(l_\zeta^{-2}\,\nabla\psi\right)=l_\zeta^2\,\nabla\cdot\left[l_\zeta^{-2}\left(l_\psi^{-1}\boldsymbol{e}_\psi\right)\right]$$

$$=l_\zeta^2\frac{1}{l_\psi l_\theta l_\zeta}\frac{\partial}{\partial\psi}l_\theta l_\zeta\left(l_\zeta^{-2}l_\psi^{-1}\right)=\frac{l_\zeta}{l_\psi l_\theta}\frac{\partial}{\partial\psi}\frac{l_\theta}{l_\zeta l_\psi}. \tag{8.13}$$

Thus, introducing the second-order differential operator

$$\Delta^* A\equiv l_\zeta^2\,\nabla\cdot l_\zeta^{-2}\,\nabla A, \tag{8.14}$$

eq. (8.12) is rewritten in the form

$$\Delta^*\psi=-\mathscr{I}\frac{\mathrm{d}\mathscr{I}}{\mathrm{d}\psi}-4\pi l_\zeta^2\frac{\mathrm{d}P}{\mathrm{d}\psi}. \tag{8.15}$$

This non-linear equation relating the characteristics of the magnetic field to the pressure and current distributions is the celebrated *Grad–Shafranov equation* (Lüst and Schlüter 1957, Grad and Rubin 1958, Shafranov 1960). It provides the basis of the self-consistent determination of axisymmetric equilibria, and is widely used for the design of confinement devices. More precisely, the solution of (8.15) yields the shape of the (axisymmetric) magnetic surfaces, $\psi(\boldsymbol{x})$, whenever the surface functions $\mathscr{I}(\psi)$ and $P(\psi)$ are given. It is not our purpose in this book to discuss the many existing methods of solution of this equation; the reader will find an excellent recent review in the paper by Freidberg (1982).

We now calculate other interesting quantities related to the magnetic field. We already know that

$$\Phi_{\mathrm{P}}(\psi)=2\pi\psi, \tag{8.16}$$

The toroidal flux is given by

$$\Phi_{\mathrm{T}}(\psi)=\int_0^\psi\mathrm{d}\psi'\int_0^{2\pi}\mathrm{d}\theta\; l_{\psi'}l_\theta\frac{\mathscr{I}(\psi')}{l_\zeta}. \tag{8.17}$$

Calculating the total poloidal current I_{P} by means of (3.8) and (8.5), we find

$$I_{\mathrm{P}}(\psi)=\int_0^\psi\mathrm{d}\psi'\int_0^{2\pi}\mathrm{d}\theta\; l_{\psi'}l_\theta l_\zeta\frac{1}{l_\theta}j_\theta$$

$$=-\frac{c}{4\pi}\int_0^\psi\mathrm{d}\psi'\int_0^{2\pi}\mathrm{d}\theta\frac{\mathrm{d}\mathscr{I}(\psi')}{\mathrm{d}\psi'}=-\frac{c}{2}\mathscr{I}(\psi). \tag{8.18}$$

Thus, the coefficient $\mathscr{I}$ entering eq. (8.1) is nothing other (up to a constant factor) than the *poloidal current*, a surface quantity.

Another interesting quantity is the slope of the magnetic field lines,

$$\frac{\mathrm{d}\zeta}{\mathrm{d}\theta} = \frac{l_\theta}{l_\zeta}\frac{B_\zeta}{B_\theta} = \mathscr{I}\frac{l_\theta l_\psi}{l_\zeta}. \tag{8.19}$$

Here we see an interesting difference between representations (8.2) and (7.10). In the latter case, this slope was automatically independent of the angular coordinates, and could therefore be identified with the rotational transform. The coordinate system defined by (7.10) is therefore automatically a natural one. This is no longer true for representation (8.2): the right-hand side of (8.19) depends, in general, on the poloidal angle θ through the scale factors. Thus, in general, the coordinate system attached to the field configuration (8.1) is *not* a natural coordinate system. As a result, the slope of the magnètic field lines is not constant (in θ) and cannot be identified with the rotational transform or the safety factor (which are surface quantities). The latter is defined by the more general equation (4.19), using (8.17),

$$q(\psi) \equiv \frac{2\pi}{\iota(\psi)} = \frac{1}{2\pi}\frac{\mathrm{d}\Phi_\mathrm{T}}{\mathrm{d}\psi} = \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\theta\, l_\psi l_\theta \frac{\mathscr{I}(\psi)}{l_\zeta}. \tag{8.20}$$

Finally, we derive a particular form for the averaging operation in the case of axisymmetric systems represented by (8.2). From (7.16) and (8.16) we obtain

$$\langle A\rangle = \left(\frac{\mathrm{d}V}{\mathrm{d}\psi}\right)^{-1}\int_0^{2\pi}\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\zeta\,\frac{l_\theta}{B_\theta}A(\psi,\theta,\zeta).$$

The volume enclosed by the magnetic surface is easily evaluated and this expression is rewritten explicitly as

$$\langle A\rangle = \frac{\displaystyle\int_0^{2\pi}\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\zeta\,\frac{l_\theta}{B_\theta}A(\psi,\theta,\zeta)}{\displaystyle 2\pi\int_0^{2\pi}\mathrm{d}\theta\,\frac{l_\theta}{B_\theta}} \tag{8.21}$$

Using (8.2), this expression can also be written in the form

$$\langle A\rangle = \frac{\displaystyle\int_0^{2\pi}\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\zeta\, l_\psi l_\theta l_\zeta\, A(\psi,\theta,\zeta)}{\displaystyle 2\pi\int_0^{2\pi}\mathrm{d}\theta\, l_\psi l_\theta l_\zeta}. \tag{8.22}$$

In particular, we note the simple form

$$\frac{dV}{d\psi} = 2\pi \int_0^{2\pi} d\theta \; l_\psi l_\theta l_\zeta . \tag{8.23}$$

Note that eq. (8.20), combined with (8.22) yields a very general and elegant expression of the safety factor in any axisymmetric configuration:

$$q(\psi) = \frac{\mathcal{I}(\psi)}{(2\pi)^2} \left(\frac{dV}{d\psi} \right) \left\langle l_\zeta^{-2} \right\rangle . \tag{8.24}$$

(B) The topological radius as a magnetic surface label

The coordinate system described above is perfectly coherent and is very widely used in the literature on fusion theory. It has, however, the disadvantage of being very "abstract". We would rather like to characterize the magnetic surfaces by a label describing their "distance" from the magnetic axis, rather than the value of the poloidal flux, which is not directly related to a geometrical concept.

Such a "geometrical" description is strongly motivated by the fact that the experimental data from tokamaks consist of density, pressure or temperature *profiles*, i.e. functions of the type $n_\alpha(r)$, $P(r)$, $T_\alpha(r)$, describing the variation of these quantities along a chord traversing the plasma cross-section explored by the diagnostic device. Note also that, as the plasma in the tokamak has a rotating motion, the diagnostics actually measure averages over the poloidal angle, i.e. *surface averages* of the densities, pressure or temperatures.

Of course, it is always possible to translate any quantities expressed as functions of ψ into quantities defined as functions of r [by solving the Grad–Shafranov equation (8.15)]. Nevertheless, it is desirable to construct a description giving a more immediate "feeling" of the spatial localization of the phenomena.

The difficulty lies in the following point. The ordinary radial coordinate is based on the idea of the (Euclidean) distance from a fixed point. Unfortunately, the cross-sections of the magnetic surfaces are *not*, in general, concentric circles around the magnetic axis. They can only be characterized by the value of some (arbitrary) surface quantity, i.e. a *physical*, rather than a *geometrical* quantity. It is possible, however, to reach a compromise between physics and geometry, as was first shown by Hinton and Hazeltine (1976).

Consider first a purely *cylindrical geometry*, with a constant magnetic field B_T (fig. 8.2a). The magnetic surfaces are straight coaxial cylinders, char-

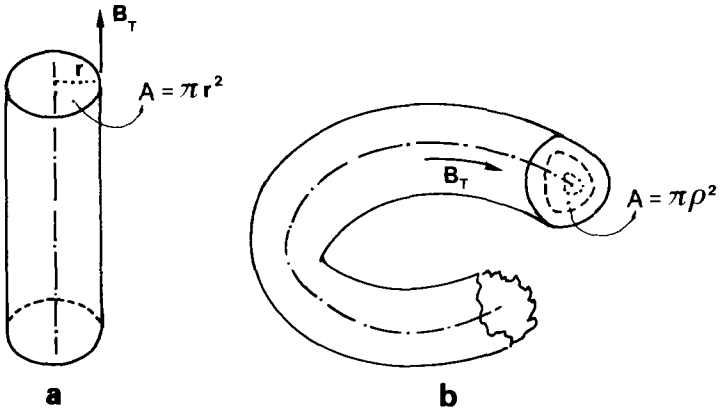

Fig. 8 2 Definition of the toroidal flux. (a) Cylindrical geometry, (b) Toroidal geometry.

acterized by the radial distance r from the (arbitrarily located) magnetic axis. In such a simple configuration, the magnetic flux through the orthogonal cross-section of a given magnetic surface of radius r is simply

$$\Phi_T^{cyl} = B_T \pi r^2. \tag{8.25}$$

We now compare this situation with a tokamak geometry, in which the cylinder is bent into a torus and the magnetic surfaces are neither concentric nor circular (fig. 8.2b). These surfaces can be characterized by value (3.2) of their *toroidal flux*, which is the analog of (8.25). This analogy is exploited by defining a quantity ρ, having the dimension of a length, and a quantity $\mathscr{B}_T$, having the dimension of a magnetic field, in such a way that the toroidal flux is represented as

$$\Phi_T = \mathscr{B}_T \pi \rho^2. \tag{8.26}$$

For consistency, $\mathscr{B}_T$ must, in general, be a function of ρ. It may be said that $\mathscr{B}_T$ is "representative" of the toroidal magnetic field and $\pi\rho^2$ measures the area of the cross-section of the magnetic surface. Thus, ρ is "representative" of the distance from the magnetic axis in the real field configuration. This appears to be the most effective way of introducing the concept of a "radius", while satisfying all the constraints of the problem. Note that it is important to use the *toroidal* flux as a starting point: the poloidal flux could not do the job!

We now proceed to the construction of a coordinate system. Nothing prevents us from choosing the toroidal flux, a surface quantity, for labelling the magnetic surfaces (rather than the poloidal flux used in subsection A).

Next, by using (8.26), we switch over from Φ_T to ρ as a surface label. We thus decide to use (ρ, θ, ζ) as the coordinate system for the description of an axisymmetric configuration. (ρ, θ, ζ) will be called the *Hinton–Hazeltine (or, simply, HH) coordinates*. Their definition is completed by assigning in each point a right-handed triad of unit vectors, $\boldsymbol{e}_\rho$, $\boldsymbol{e}_\theta$, $\boldsymbol{e}_\zeta$,

$$\boldsymbol{e}_\rho \wedge \boldsymbol{e}_\theta = \boldsymbol{e}_\zeta. \tag{8.27}$$

Clearly, $\boldsymbol{e}_\rho = \boldsymbol{e}_\psi$. Moreover, a scale factor $l_\rho(\rho, \theta)$ is defined in such a way that, for any scalar function $A(\rho, \theta, \zeta)$,

$$\nabla_\rho A(\rho, \theta, \zeta) = \frac{1}{l_\rho} \frac{\partial A(\rho, \theta, \zeta)}{\partial \rho}. \tag{8.28}$$

The toroidal flux is (in any non-pathological situation) a monotonously increasing quantity for successive magnetic surfaces starting from the magnetic axis outward. Therefore the coordinate ρ can be thought of as a (non-Euclidean) measure of the distance from the magnetic axis. We name it the *topological radius* (In the forthcoming text, whenever no confusion is possible, ρ will also briefly be called the "radial coordinate" or, simply, the "radius".)

Our next goal is the transformation of the expressions obtained in subsection A to the new coordinates. In particular, considering expression (8.1), we must transform the gradient $\nabla\psi$, where ψ is now considered as a function of ρ (alone!),

$$\nabla\psi \equiv \frac{1}{l_\psi} \boldsymbol{e}_\psi = \frac{\mathrm{d}\psi}{\mathrm{d}\rho} \nabla\rho \equiv \frac{1}{l_\rho} \frac{\mathrm{d}\psi}{\mathrm{d}\rho} \boldsymbol{e}_\rho. \tag{8.29}$$

It is very convenient, following Hinton and Hazeltine (1976), to introduce a surface quantity $\mathscr{B}_P(\rho)$, called the *effective poloidal field*, through the definition

$$\mathscr{B}_P(\rho) = \frac{1}{R_0} \frac{\mathrm{d}\psi(\rho)}{\mathrm{d}\rho}, \tag{8.30}$$

where R_0 is the major radius of the torus (i.e. the radius of the magnetic axis). Note that this is a true "geometrical radius", because in an axisymmetric configuration the magnetic axis is a circle. Clearly, $\mathscr{B}_P$ has the dimension of a magnetic field. Combining now (8.1) with these results, we obtain the following representation of the *magnetic field in HH coordinates*:

$$\boldsymbol{B}(\rho, \theta) = -R_0 \mathscr{B}_P(\rho) \nabla\rho \wedge \nabla\zeta + \mathscr{I}(\rho) \nabla\zeta. \tag{8.31}$$

Alternatively, we exhibit the poloidal and toroidal (physical) components in the following form, analogous to (8.2),

$$\boldsymbol{B} = \frac{R_0 \mathscr{B}_{\mathrm{P}}}{l_\rho l_\zeta}\, \boldsymbol{e}_\theta + \frac{\mathscr{I}}{l_\zeta}\, \boldsymbol{e}_\zeta. \tag{8.32}$$

A magnetic field configuration is thus specified by the two surface functions $\mathscr{B}_{\mathrm{P}}(\rho)$, $\mathscr{I}(\rho)$, as well as the geometrical scale factors l_ρ, l_ζ.

We also note the following relation between the scale factors, obtained from (8.29), (8.30):

$$l_\rho = R_0 \mathscr{B}_{\mathrm{P}}(\rho)\, l_\psi. \tag{8.33}$$

The components of the electric current are now expressed in the HH coordinates (see eqs. 8.4–8.6) as

$$j_\rho = 0, \qquad j_\theta = -\frac{c}{4\pi}\frac{1}{l_\zeta l_\rho}\frac{\mathrm{d}\mathscr{I}(\rho)}{\mathrm{d}\rho}, \qquad j_\zeta = \frac{cR_0}{4\pi}\frac{1}{l_\rho l_\theta}\frac{\partial}{\partial\rho}\mathscr{B}_{\mathrm{P}}(\rho)\frac{l_\theta}{l_\rho l_\zeta}. \tag{8.34}$$

The *surface-average* operation is defined by analogy with (8.22) as

$$\langle A\rangle = [V'(\rho)]^{-1}\int_0^{2\pi}\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\zeta\; l_\rho l_\theta l_\zeta\, A(\rho, \theta, \zeta). \tag{8.35}$$

Here $V(\rho)$ is the volume enclosed by a magnetic surface, and

$$V'(\rho) \equiv \frac{\mathrm{d}V(\rho)}{\mathrm{d}\rho} = 2\pi\int_0^{2\pi}\mathrm{d}\theta\; l_\rho l_\theta l_\zeta. \tag{8.36}$$

As a quick exercise, we may check explicitly that the fundamental property (6.3) is indeed satisfied in the present (axisymmetric) case. For an arbitrary function $A(\rho, \theta)$, we have

$$\langle \boldsymbol{B}\cdot\nabla A(\rho, \theta)\rangle$$

$$= 2\pi[V'(\rho)]^{-1}\int_0^{2\pi}\mathrm{d}\theta\; l_\rho l_\theta l_\zeta\left(\frac{B_\rho}{l_\rho}\frac{\partial}{\partial\rho} + \frac{B_\theta}{l_\theta}\frac{\partial}{\partial\theta} + \frac{B_\zeta}{l_\zeta}\frac{\partial}{\partial\zeta}\right)A(\rho, \theta).$$

We now use the following facts:

(1) $B_\rho = 0$.

(2) $\partial A/\partial\zeta = 0$.

(3) $A(\rho, \theta)$ is a single-valued function, i.e. a 2π-periodic function of θ. Thus

$$\langle \boldsymbol{B}\cdot\nabla A(\rho,\theta)\rangle = 2\pi[V'(\rho)]^{-1}\int_0^{2\pi} d\theta\, l_\rho l_\theta l_\zeta \frac{B_\theta}{l_\theta}\frac{\partial A}{\partial\theta}$$

$$= 2\pi[V'(\rho)]^{-1} R_0 \mathscr{B}_P(\rho)\int_0^{2\pi} d\theta\, l_\rho l_\theta l_\zeta \frac{1}{l_\theta l_\rho l_\zeta}\frac{\partial A}{\partial\theta}$$

$$= 2\pi[V'(\rho)]^{-1} R_0 \mathscr{B}_P(\rho)[A(\rho, 2\pi) - A(\rho, 0)] = 0.$$

The remaining important surface quantities are easily obtained from the formulae of subsection A. The *safety factor* $q(\rho)$ is found from (8.24):

$$q(\rho) = \frac{\mathscr{I}(\rho)}{(2\pi)^2 R_0 \mathscr{B}_P(\rho)} V'(\rho)\langle l_\zeta^{-2}\rangle. \tag{8.37}$$

The *toroidal flux* $\Phi_T(\rho)$ is given by eq. (8.26); but this definition involves the additional surface function $\mathscr{B}_T(\rho)$. The latter is, however, easily related to the quantities entering the magnetic field (8.32). Indeed, the general definition (3.2) yields

$$\Phi_T(\rho) = \frac{1}{2\pi}\int_0^\rho d\rho'\, d\theta\, d\zeta\, l_{\rho'} l_\theta l_\zeta \frac{B_\zeta}{l_\zeta} = \int_0^\rho d\rho'\, d\theta\, l_{\rho'} l_\theta l_\zeta \frac{\mathscr{I}(\rho')}{l_\zeta^2}$$

$$= \frac{1}{2\pi}\int_0^\rho d\rho'\, \mathscr{I}(\rho')\, V'(\rho')\langle l_\zeta^{-2}\rangle.$$

Combing this result with (8.37) and (8.26), we find the following expression for the *effective toroidal field*:

$$\mathscr{B}_T(\rho) = \frac{2R_0}{\rho^2}\int_0^\rho d\rho'\, \mathscr{B}_P(\rho')\, q(\rho'). \tag{8.38}$$

Finally, the *poloidal flux* is obtained from (3.5) and (8.32) as

$$\Phi_P(\rho) = \frac{1}{2\pi}\int_0^\rho d\rho' \int d\theta\, d\zeta\, l_{\rho'} l_\theta l_\zeta \frac{B_\theta}{l_\theta} = 2\pi R_0 \int_0^\rho d\rho'\, \mathscr{B}_P(\rho'). \tag{8.39}$$

8.9. The standard model

From the discussions of the previous sections follows that the determination of a magnetic field appropriate for a confinement system and the construction of an intrinsic coordinate system are related problems. The scale factors which characterize the latter are intimately associated with the components of the magnetic field, as can be seen from (7.11) or (8.2). The determination of these quantities in a realistic system requires rather involved calculations, which must be done self-consistently (for instance, the solution of the Grad–Shafranov equation with appropriate source terms and boundary conditions).

Many properties of a confining toroidal system can be studied by using a *model magnetic field* associated with a particularly simple geometry, which retains the essential features of a real system. This *standard model* has the main advantage of being amenable to analytic treatment. It has been widely used in the literature, particularly in connection with transport theory.

The logical argument in introducing the standard model is, in a sense, opposite to the one used in previous sections. There, the magnetic field was considered as a given datum, and the coordinate system was adapted to it. Here, we choose to use a particular coordinate system, and to define a magnetic field which has a simple expression in these coordinates. This model field is not a self-consistent solution of the equations of MHD equilibrium; but in a certain limit, to be specified below, it is a good approximation to a realistic system.

Specifically, we want to construct a set of magnetic surfaces which are *nested, concentric circular tori* (fig. 9.1). For convenience, the symmetry axis

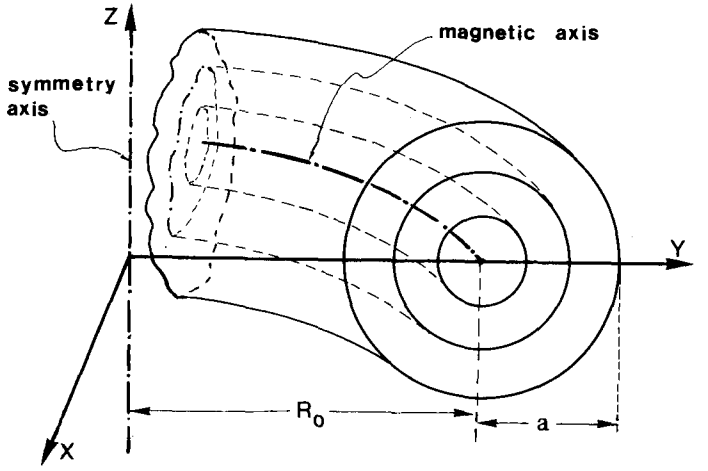

Fig. 9 1. The standard model. Its magnetic surfaces are concentric circular tori.

will be taken as the z-axis of a Cartesian reference frame. The major radius of the torus is denoted by R_0: the magnetic axis is therefore a circle of radius R_0. The plane of this circle is taken as the x–y plane of the reference frame.

It is assumed that the region of interest is bounded by a torus of inner radius a. The quantities R_0 and a are called, respectively, the major and minor radii of the toroidal configuration. It is customary to call their ratio, A, the *aspect ratio* of the configuration,

$$A = \frac{R_0}{a} > 1. \tag{9.1}$$

It is a number which, for obvious reasons, must exceed one. More useful for the theory is the *inverse aspect ratio*, $\bar{\eta}$. If the configuration is such that

$$\bar{\eta} \equiv \frac{a}{R_0} \ll 1, \tag{9.2}$$

which is realized for a very thin torus, we have a natural small parameter in the problem. It is precisely this feature which enables us to perform explicit analytical calculations.

The best adapted coordinate system in this geometry is a *toroidal coordinate system*, described in detail in Appendix G2.3 (fig. 9.2). In order to define the coordinates of a point P in this system, we imagine a section of the torus by a plane passing through P and through the symmetry axis. The *radial coordinate* r is defined as the distance from the intercept O of the magnetic axis to the point P. The *poloidal angle* θ is defined counter-clockwise in the plane of section. By convention, a point lying in the x–y plane on the exterior side with respect to O, is given the value $\theta = 0$. Finally, the *toroidal angle* ζ runs around

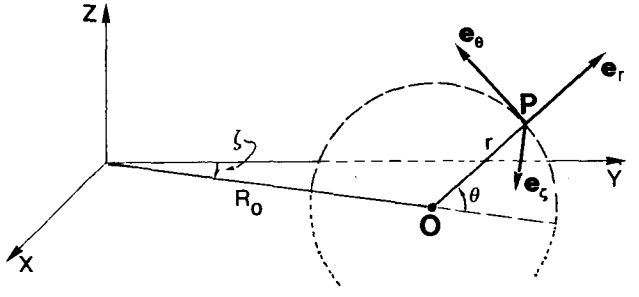

Fig. 9.2 Toroidal coordinate system (r, θ, ζ) and local triad (e_r, e_θ, e_ζ).

the torus, starting from the (arbitrary) y-axis. With these definitions, we can construct at each point a *local orthogonal right-handed triad* $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$, $\boldsymbol{e}_\zeta$ such that

$$\boldsymbol{e}_r \wedge \boldsymbol{e}_\theta = \boldsymbol{e}_\zeta. \tag{9.3}$$

The main advantage of this coordinate system is the trivial simplicity of the equation of the magnetic surfaces,

$$r = r_0. \tag{9.4}$$

where r_0 is a constant. The normal sections of these surfaces are, of course, concentric circles. The unit vector $\boldsymbol{e}_r$ points radially outward from the center. The vector $\boldsymbol{e}_\theta$ is tangent to the circle, and the vector $\boldsymbol{e}_\zeta$ is perpendicular to the plane of the section. The coordinate system (r, θ, ζ) is obviously orthogonal, a property formally proven in Appendix **G2**.3.

We now define the magnetic field in the standard model as

$$\boldsymbol{B}(r, \theta) = \frac{B_0}{q(r)} \frac{r}{R_0} \boldsymbol{e}_\theta + \frac{B_0}{1 + (r/R_0) \cos \theta} \boldsymbol{e}_\zeta, \tag{9.5}$$

where B_0 is a constant (having the dimension of a magnetic field intensity) and $q(r)$ is a dimensionless function of r alone. Equation (9.5) defines an axisymmetric magnetic field, compatible with the magnetic surfaces (9.4) (because $B_r = 0$). It is important to note that, if condition (9.2) is satisfied, then necessarily

$$\eta \equiv \frac{r}{R_0} \leqslant \bar{\eta} \ll 1. \tag{9.6}$$

Hence, the ratio $\eta \equiv r/R_0$ will be treated as a uniformly small parameter.

Let us now calculate some of the characteristic quantities of this configuration. We first note that all surface quantities are functions of r alone. We may therefore compare (9.5) with the general form (8.31) of an axisymmetric field in HH coordinates, by expressing the latter in toroidal coordinates, and using the results of Appendices **G2**.2 and **G2**.3,

$$\begin{aligned} \boldsymbol{B} &= -R_0 \mathscr{B}_P(r) \frac{\mathrm{d}\rho}{\mathrm{d}r} \nabla r \wedge \nabla \zeta + \mathscr{I}(r) \nabla \zeta \\ &= \frac{R_0 \mathscr{B}_P(r)}{R_0 + r \cos \theta} \frac{\mathrm{d}\rho}{\mathrm{d}r} \boldsymbol{e}_\theta + \mathscr{I}(r) \frac{1}{R_0 + r \cos \theta} \boldsymbol{e}_\zeta. \end{aligned} \tag{9.7}$$

Comparing now the separate components of (9.5) and (9.7) we find

$$\frac{\mathrm{d}\rho}{\mathrm{d}r} = \frac{B_0}{\mathscr{B}_\mathrm{P}(r)\, q(r)} \frac{r}{R_0} + \mathrm{O}(\eta^2) \tag{9.8}$$

$$\mathscr{I}(r) = B_0 R_0. \tag{9.9}$$

The latter expression is (up to the factor $-c/2$) the total poloidal current [as we know from eq. (8.18)]; in the standard model, this quantity is independent of r.

Another quantity which is required in translating general expressions into the standard model is the scale factor l_ρ, obtained from the identities

$$\nabla\rho = \frac{\mathrm{d}\rho}{\mathrm{d}r}\nabla r = \frac{\mathrm{d}\rho}{\mathrm{d}r}\boldsymbol{e}_r, \qquad \nabla\rho = \frac{1}{l_\rho}\boldsymbol{e}_\rho = \frac{1}{l_\rho}\boldsymbol{e}_r.$$

Hence,

$$l_\rho = \left(\frac{\mathrm{d}\rho}{\mathrm{d}r}\right)^{-1} = \frac{\mathscr{B}_\mathrm{P}(r)\, q(r)}{B_0} \frac{R_0}{r}. \tag{9.10}$$

The volume enclosed by a magnetic surface ($r = \text{const.}$) is (see **G2**.3.4)

$$V(r) = 2\pi \int_0^r \mathrm{d}r' \int_0^{2\pi} \mathrm{d}\theta\, l_r l_\theta l_\zeta$$

$$= 2\pi \int_0^r \mathrm{d}r' \int_0^{2\pi} \mathrm{d}\theta\, r'(R_0 + r'\cos\theta) = 2\pi^2 R_0 r^2. \tag{9.11}$$

The *averaging formula* for the standard model is obtained from (8.35), using (9.11) and (**G2**.3.4),

$$\langle A(r,\theta)\rangle = \left(\frac{\mathrm{d}V}{\mathrm{d}r}\right)^{-1} 2\pi \int_0^{2\pi} \mathrm{d}\theta\, l_r l_\theta l_\zeta A(r,\theta)$$

$$= \frac{1}{2\pi R_0 r} \int_0^{2\pi} \mathrm{d}\theta\, r(R_0 + r\cos\theta)\, A(r,\theta),$$

and, finally,

$$\langle A(r,\theta)\rangle = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\theta \left(1 + \frac{r}{R_0}\cos\theta\right) A(r,\theta). \tag{9.12}$$

Next, we calculate the slope of the magnetic field lines (to dominant order in η), using (9.5),

$$\frac{\mathrm{d}\zeta}{\mathrm{d}\theta} = \frac{l_\theta}{l_\zeta}\frac{B_\zeta}{B_\theta} = q(r). \tag{9.13}$$

As this quantity depends only on r, it can be identified with the *safety factor* (4.18): this justifies the choice of the symbol q for this coefficient in (9.5). This result also shows that the toroidal coordinates are *natural coordinates* for the standard model, in the sense of section 8.4.

We now identify the safety factor with its alternative expression, calculated by the general formula (8.37), in which $l_\zeta = R_0 + \mathrm{O}(\eta)$. We find, to leading order in η, using (9.9) and (9.11),

$$q(r) = \frac{1}{(2\pi)^2}\frac{\mathscr{I}(r)}{R_0\mathscr{B}_\mathrm{P}(r)}V'(r)\,R_0^{-2} = \frac{B_0}{\mathscr{B}_\mathrm{P}(r)}\frac{r}{R_0}.$$

This result actually determines the surface function $\mathscr{B}_\mathrm{P}(r)$,

$$\mathscr{B}_\mathrm{P}(r) = \frac{B_0}{q(r)}\frac{r}{R_0}. \tag{9.14}$$

We see that, in the standard model, $\mathscr{B}_\mathrm{P}(r)$ is precisely the poloidal component of the magnetic field, defined in (9.5),

$$\mathscr{B}_\mathrm{P}(r) = B_\theta. \tag{9.15}$$

Moreover, upon substitution of (9.14) into (9.8) and (9.10), we find

$$\frac{\mathrm{d}\rho}{\mathrm{d}r} = 1, \tag{9.16}$$

$$l_\rho = 1. \tag{9.17}$$

These equations are completed by the natural condition

$$\rho(r=0) = 0.$$

This shows that, in the standard model (i.e. in the limit of a very large aspect ratio), *the topological radius ρ reduces to the geometrical radius r*:

$$\rho = r. \tag{9.18}$$

Next, we calculate the *toroidal flux* Φ_T,

$$\Phi_T(r) = \frac{1}{2\pi}\int d^3x\, \boldsymbol{B}\cdot\nabla\zeta = \pi r^2 B_0.$$

Comparing this formula with (8.26) and (9.18) we find that the *effective toroidal field* $\mathscr{B}_T(r)$ reduces to the constant B_0 of the standard model,

$$\mathscr{B}_T(r) = B_0. \tag{9.19}$$

Finally, the *poloidal flux* is easily obtained from (8.39) and (9.14):

$$\Phi_P(r) = 2\pi\int_0^r dr' \frac{r'}{q(r')}. \tag{9.20}$$

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9

The motion of charged particles in toroidal magnetic fields

9.1. Introduction. Qualitative description of the motion

In chapter 8 we have "set the stage" for the theory of magnetically confined plasmas. The geometrical aspects of the toroidal field configuration was described in some detail.

Our next goal, to be realized in the forthcoming chapters, is the discussion of the transport theory within this framework. It will be shown how the geometrical constraints are able to strongly influence the transport behaviour of a confined plasma. It will be seen that the effects are *surprisingly large*, even in the limiting cases where the toroidal geometrical features are intrinsically weak.

The reader must be warned that our study in the forthcoming chapters by no means exhausts the problem. The characteristic feature of the theory presented here is the following. The behaviour of the plasma is studied in a *given* magnetic field geometry: this geometry produces a certain type of (strongly modified) transport behaviour, i.e. a particle flux, an energy flux and a momentum flux. The study of these fluxes in a *given* toroidal magnetic geometry is the object of the *neoclassical transport theory* developed in this book.

However, it is easily conceived that the motion of the plasma, and in particular these fluxes, produce a "backlash" on the magnetic field configuration. If, moreover, we remember how fragile are the toroidal confinement configurations, it is easily understood that such retroactions may possibly destabilize the initial magnetic geometry. The result may be a slight distortion of the magnetic surfaces, but in some cases it may also lead to a complete destruction of the latter. As a result, the confining effect in the radial direction is lost, and the radial transport is very strongly enhanced. This, in turn, produces a new retroaction on the magnetic field, and so on...

We see that a truly complete transport theory of plasmas must self-consistently incorporate the "equilibrium" theory, the stability (or, rather, the instability!) theory and the effects of the possible presence of external heating

beams. This global approach is the purpose of the so-called *anomalous transport theory*. Needless to say that we are still far from such a complete and comprehensive theory. This type of problems will however not be treated in this volume.

In order to present the more modest neoclassical theory, we follow the same general line as in the first part of the book. We first discuss (in the present chapter) the motion of a single charged particle in a *toroidal* magnetic field. Then we turn to the effect of the toroidal geometry on the microscopic, kinetic description of the plasma. Finally, we go over to the macroscopic level and construct the transport theory.

In the absence of external fields, the orbit of a particle between two collisions is simply a straight line. But we know from chapter 1 that in the presence of inhomogeneous and curved magnetic and electric fields, the trajectory of such a particle becomes completely different and is, in general, extremely complicated. The only case in which something general can be determined without going blindly to the computer, is the regime in which the *drift approximation* (**1**.5.26) is applicable. In this case we know that the motion consists of a *fast gyration* around a magnetic field line and of a *slow drift* of the guiding centre. An appropriate pseudo-canonical transformation allows us to eliminate the fast gyration, and we are left with much simpler (approximate) equations of motion, which do not contain any rapidly oscillating coefficients.

Nevertheless, when the confining magnetic field has such a complicated shape as in a tokamak, even these gyro-averaged equations of motion cannot be solved exactly. A good idea about the nature and the shape of the guiding centre trajectories can, however, be obtained by using the *standard model* defined and discussed in section **8**.9. In fact, the overwhelming majority of the analytical work published to-date on the particle motions is done within that framework.

In the forthcoming sections of this chapter, the equations of motion of a charged particle in the standard toroidal model will be derived. The Hamiltonian formalism and the method of pseudo-canonical transformation developed in chapter 1 will again be of great use. They allow us to obtain a clear and rigorous derivation and also point out some important facts which were not previously discussed in the literature. Thereafter, the equations of motion are solved and the nature of the trajectories is discussed in detail.

Before starting this mathematical programme, it is useful to get a qualitative picture of the problem by using some simple physical ideas. We follow here an argument developed by Golant et al. (1980) (see also Stacey 1981).

The magnetic field in a tokamak configuration is of the form

$$\boldsymbol{B} = B_\theta \boldsymbol{e}_\theta + B_\zeta \boldsymbol{e}_\zeta,$$

where $\boldsymbol{e}_\theta$, $\boldsymbol{e}_\zeta$ are the local basis vectors in toroidal geometry, as defined in eq. (**8**.9.3). It is also known that, for large aspect ratio, $|B_\zeta| \gg |B_\theta|$. In a toroidal configuration, the magnetic field intensity $B \cong |B_\zeta|$ is always larger on the side of the torus pointing toward the symmetry axis ($\theta = \pi$, $\cos\theta = -1$): this is clear from eq. (**8**.9.5). Hence, there is a gradient ∇B directed everywhere towards the symmetry axis.

Consider now the motion of particles in such a field. To zeroth order, they gyrate around the magnetic field lines. As the latter are wrapped around the torus, the particles travel in succession through regions of weak and strong fields. The situation is similar to the one prevailing in a *magnetic mirror*. As a result, according to the ratio of their parallel and perpendicular velocities, some particles can travel "freely", while others are reflected upon entering a strong field region.

A more detailed picture is obtained by considering the motion of the associated guiding centre (in the drift approximation). Because of the existence of the gradient ∇B, the guiding centre will have a grad-B drift velocity (**1**.5.24)*,

$$\boldsymbol{w}_D \sim \boldsymbol{B} \wedge \nabla B.$$

Consider now a guiding centre starting at a given point 1; we construct qualitatively the projection of its trajectory on the section of the torus corresponding to its initial position (see fig. 1.1). We first assume that the particle has a relatively large parallel velocity. If the magnetic field is entering the plane of the paper from above, the grad-B drift is pointing vertically upward throughout the section of the torus. As the guiding centre starts moving around the torus, it drifts upward: as a result, it leaves its original magnetic surface and moves towards more and more exterior surfaces ($1 \to 2 \to 3$) until it reaches $\theta = \pi$. As it goes on moving on the lower side of the torus, the drift, (which is still pointing upwards) brings the guiding centre on more and more interior magnetic surfaces ($3 \to 4 \to 5$), thus reversing the previous effect. Finally, the projection of the trajectory closes on itself. (The real three-dimensional trajectory is a helicoidal curve wrapped around the torus.) It is precisely because of this compensation of the drift velocity in the upper and the lower parts of the torus that the particles are *confined* in a tokamak configuration, by remaining close to their initial magnetic surface.

Consider now (fig. 1.2) a particle having a relatively small parallel velocity (this statement will be made precise in section 9.5). Its guiding centre moves in the same fashion under the action of the upward drift ($1 \to 2 \to 3$). When it

* It turns out that the curvature drift velocity points in the same direction as the grad-B drift (see section 9.3)· its effect reinforces the features described below

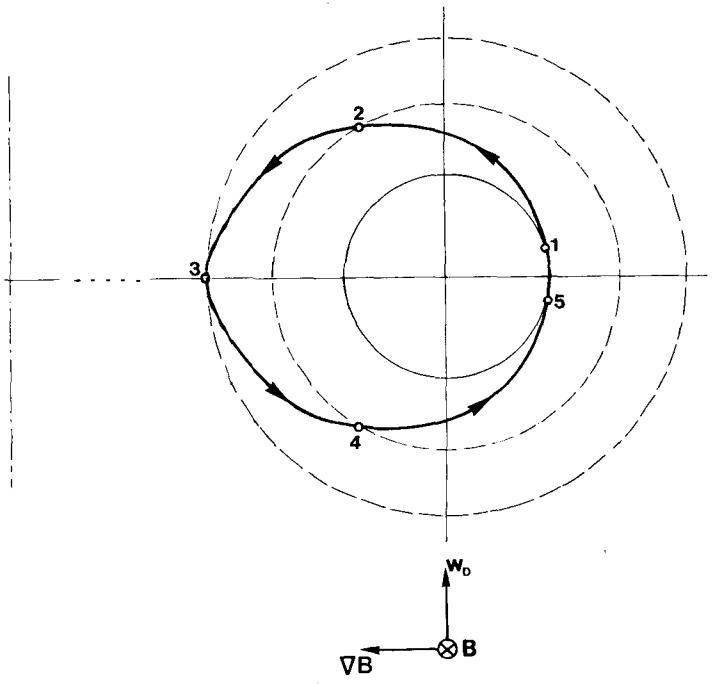

Fig. 1.1 Projected trajectory of a passing particle.

arrives at position 3, however, its whole parallel kinetic energy is converted into perpendicular energy, and the particle is reflected, as in a mirror. It starts moving back, but is still pushed towards exterior surfaces by the grad-B drift ($3 \rightarrow 4 \rightarrow 5$). As it arrives at $\theta = 0$, it is displaced outward by a certain amount, compared to its initial position. As it goes on moving in the lower section of the torus, the vertical drift brings it to more and more interior surfaces ($5 \rightarrow 6 \rightarrow 7$). At point 7 the particle suffers again a reflection and starts moving back, passing again through more and more interior surfaces ($7 \rightarrow 8 \rightarrow 9$) until the curve closes on itself. Note that in this case, not only the projection, but also the real three-dimensional trajectory is closed (in first approximation; see section 9.8). Such a particle is *trapped*: it never enters the central region of strong magnetic field. Because of its picturesque shape, the orbit of a trapped particle in a tokamak has come to be called a "*banana orbit*".

This qualitative picture of the motion of the particles and the important topological distinction between passing and trapped particle motion will be made precise in the forthcoming sections of this chapter.

The general nature of the motion of particles in confining systems (including the existence of reflecting barriers in toroidal configurations like the

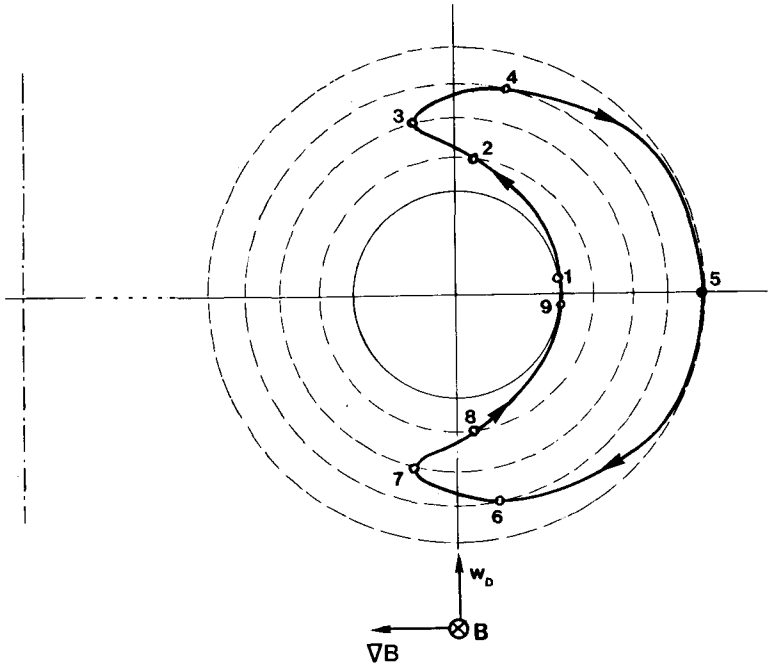

Fig. 1.2. Projected trajectory of a trapped particle.

stellarator) was known, of course, to the pioneers of the controlled thermonuclear fusion programme (see e.g. Spitzer 1958). However, it appears that the first (published) systematic study of the motion of particles in a toroidal magnetic field configuration was done by Morozov and Soloviev (1959, 1960a, b), who clearly stressed the topologically distinct classes of trajectories. They used specific models of magnetic fields relevant for fusion ("magnetic traps"): in their (1959) paper, an axisymmetric configuration was considered, whereas their (1960a) paper treats a bumpy torus and the (1960b) a stellarator-like geometry. This work is also discussed in the review paper by Morozov and Soloviev (1966).

9.2. *Exact equations of the charged particle motion in toroidal geometry*

We now derive the equations of motion of a charged particle, expressed in the toroidal coordinate system adapted to the standard model. This coordinate system is described in detail in the General Appendix **G2**.3.

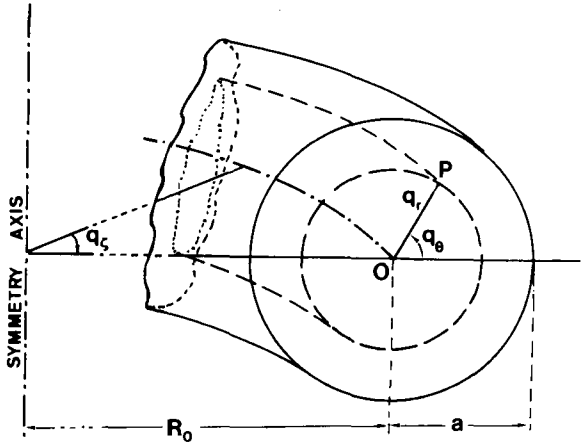

Fig. 2.1. Position variables q_r, q_θ, q_ζ of particle P in toroidal coordinates.

Following closely the developments of section **1**.3, we introduce coordinates (q_r, q_θ, q_ζ) to describe the position of a particle (see fig. 2.1). Here q_r is the radial distance, measured from the magnetic axis, q_θ is the poloidal angle and q_ζ the toroidal angle. These coordinates are related to the Cartesian coordinates q_x, q_y, q_z by eqs. (**G2**.3.1),

$$q_x = (R_0 + q_r \cos q_\theta) \sin q_\zeta,$$

$$q_y = (R_0 + q_r \cos q_\theta) \cos q_\zeta,$$

$$q_z = q_r \sin q_\theta. \tag{2.1}$$

In the first stage of the Hamiltonian formalism, we need to define a set of momentum variables which are canonically conjugate to q_r, q_θ, q_ζ. It turns out that the conjugate momenta, denoted by $\hat{p}_1$, $\hat{p}_2$, $\hat{p}_3$, are obtained by the transformation

$$p_x = \hat{p}_1 \cos q_\theta \sin q_\zeta - \hat{p}_2 \frac{1}{q_r} \sin q_\theta \sin q_\zeta + \hat{p}_3 \frac{1}{R_0 + q_r \cos q_\theta} \cos q_\zeta,$$

$$p_y = \hat{p}_1 \cos q_\theta \cos q_\zeta - \hat{p}_2 \frac{1}{q_r} \sin q_\theta \cos q_\zeta - \hat{p}_3 \frac{1}{R_0 + q_r \cos q_\theta} \sin q_\zeta$$

$$p_z = \hat{p}_1 \sin q_\theta + \hat{p}_2 \frac{1}{q_r} \cos q_\theta. \tag{2.2}$$

Before checking this result, we must clearly understand the nature of the variables $\hat{p}_i$. If we consider the vector $\boldsymbol{p}$ and expand it along the Cartesian basis $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$, and then use (2.2), we find

$$\begin{aligned}\boldsymbol{p} &= p_x \boldsymbol{i} + p_y \boldsymbol{j} + p_z \boldsymbol{k} \\ &= \hat{p}_1 \cos q_\theta \sin q_\zeta \, \boldsymbol{i} - \hat{p}_2 \frac{1}{q_r} \sin q_\theta \sin q_\zeta \, \boldsymbol{i} + \hat{p}_3 \frac{1}{R_0 + q_r \cos q_\theta} \cos q_\zeta \, \boldsymbol{i} \\ &\quad + \hat{p}_1 \cos q_\theta \cos q_\zeta \, \boldsymbol{j} - \hat{p}_2 \frac{1}{q_r} \sin q_\theta \cos q_\zeta \, \boldsymbol{j} - \hat{p}_3 \frac{1}{R_0 + q_r \cos q_\theta} \sin q_\zeta \, \boldsymbol{j} \\ &\quad + \hat{p}_1 \sin q_\theta \, \boldsymbol{k} + \hat{p}_2 \frac{1}{q_r} \cos q_\theta \, \boldsymbol{k}.\end{aligned}$$

Noting eq. (**G2**.3.6), we rewrite this as

$$\boldsymbol{p} = \hat{p}_1 \boldsymbol{e}_r + \hat{p}_2 \frac{1}{q_r} \boldsymbol{e}_\theta + \hat{p}_3 \frac{1}{R_0 + q_r \cos q_\theta} \boldsymbol{e}_\zeta. \tag{2.3}$$

Hence, the *physical components of the momentum vector* in the toroidal coordinate system are

$$p_r = \hat{p}_1, \qquad p_\theta = \frac{1}{q_r} \hat{p}_2, \qquad p_\zeta = \frac{1}{R_0 + q_r \cos q_\theta} \hat{p}_3. \tag{2.4}$$

On the other hand, noting definition (**G2**.2.4) of the contravariant basis vectors $\hat{\boldsymbol{e}}^i$, as well as eq. (**G2**.3.4), we may also write (2.3) in the form

$$\boldsymbol{p} = \hat{p}_1 \hat{\boldsymbol{e}}^1 + \hat{p}_2 \hat{\boldsymbol{e}}^2 + \hat{p}_3 \hat{\boldsymbol{e}}^3. \tag{2.5}$$

Thus, the quantities $\hat{p}_i$ are the *covariant components of the momentum vector* in the toroidal coordinate system. (It also follows that the "hat" in the symbol $\hat{p}_i$ is consistent with the conventions of notation adopted in the General Appendix G2.)

A simple argument allows us to understand why the momentum is "naturally" a covariant vector. Noting that $\hat{q}^i$ as well as $\dot{\hat{q}}^i$ are contravariant vectors, the momentum is obtained from the scalar Lagrangian $L(\hat{q}^i, \dot{\hat{q}}^i)$ by

$$\hat{p}_i = \frac{\partial L}{\partial \dot{\hat{q}}^i},$$

hence it transforms like a gradient, i.e. as a covariant vector.

This simple, but important point is not often discussed in the literature; see some indications in the book by Fock (1964). (Very unfortunately, in the latter book, the contravariant position coordinates are "exceptionally" denoted by subscripts instead of superscripts, see the remarks on pp. 60 and 66 of that reference).

In order to enhance the symmetry of the notations, we shall write the variables q_r, q_θ, q_ζ in the form of contravariant coordinates of the position vector, in accordance with General Appendix **G2**.2:

$$\hat{q}^1 = q_r, \qquad \hat{q}^2 = q_\theta, \qquad \hat{q}^3 = q_\zeta. \tag{2.6}$$

It is now easily checked that the fundamental Lie brackets of table **1**.4.1. imply

$$[\hat{q}^i, \hat{q}^j] = 0, \qquad [\hat{p}_i, \hat{p}_j] = 0. \qquad [\hat{q}^i, \hat{p}_j] = \delta^i_j. \tag{2.7}$$

We note that transformations (2.1), (2.2) are highly non-linear; thus relations (2.7) are by no means trivial. As an example, we check the following bracket (abbreviating $R_0 + q_r \cos q_\theta \equiv R$):

$$\begin{aligned} 1 &= [q_y, p_y] \\ &= \left[R \cos \hat{q}^3, \hat{p}_1 \cos \hat{q}^2 \cos \hat{q}^3 - \hat{p}_2 \frac{1}{\hat{q}^1} \sin \hat{q}^2 \cos \hat{q}^3 - \hat{p}_3 \frac{1}{R} \sin \hat{q}^3 \right] \\ &= R \sin \hat{q}^3 [\hat{q}^3, \hat{p}_3] \frac{1}{R} \sin \hat{q}^3 + \cos \hat{q}^3 \cos \hat{q}^2 [\hat{q}^1, \hat{p}_1] \cos \hat{q}^2 \cos \hat{q}^3 \\ &\quad + \hat{q}^1 \cos \hat{q}^3 \sin \hat{q}^2 [\hat{q}^2, \hat{p}_2] \frac{1}{\hat{q}^1} \sin \hat{q}^2 \cos \hat{q}^3. \end{aligned}$$

The right-hand side equals one, provided (2.7) holds true.

The conclusion of this discussion is the following. The fundamental canonically conjugate variables in toroidal coordinates are the *contravariant position components* $\hat{q}^i$ and the *covariant momentum components* $\hat{p}_i$. These must be carefully distinguished from the physical components (see eq. **G2**.2.7) in the present curvilinear coordinates.

Once this important point is established, the Hamiltonian (a scalar) is easily written as

$$H = \frac{1}{2m}\left(\hat{p}_i - \frac{e}{c}\hat{A}_i\right)\left(\hat{p}^i - \frac{e}{c}\hat{A}^i\right) + e\Phi$$

$$= \frac{1}{2m}g^{ij}\left(\hat{p}_i - \frac{e}{c}\hat{A}_i\right)\left(\hat{p}_j - \frac{e}{c}\hat{A}_j\right) + e\Phi, \tag{2.8}$$

where the components of the metric tensor g^{ij} are defined in (**G**2.2.3).

From here on, we may go through the same chain of pseudo-canonical transformations as in chapter 1. The method is straightforward and the formal

Table 2.1

Particle variables in toroidal coordinates: $\hat{q}^i$, $\hat{v}_i$

Definition

$$\hat{q}^i = \hat{q}^i, \qquad \hat{v}_i = \frac{1}{m}\left(\hat{p}_i - \frac{e}{c}\hat{A}_i\right) \tag{1}$$

Fundamental Lie Brackets

$$[\hat{q}^i, \hat{q}^j] = 0, \qquad [\hat{q}^i, \hat{v}_j] = \frac{1}{m}\delta^i_j, \qquad [\hat{v}_i, \hat{v}_j] = \frac{e}{m^2c}\sqrt{g}\,\varepsilon_{ijk}\hat{B}^k \tag{2}$$

Jacobian $(\boldsymbol{q}, \boldsymbol{p}) \to (\hat{q}^i, \hat{v}_i)$

$$J = m^3\sqrt{g} \tag{3}$$

Hamiltonian

$$H = \tfrac{1}{2}m\hat{v}_i\hat{v}^i + e\Phi \tag{4}$$

Equations of Motion

$$\dot{\hat{q}}^i = g^{ij}\hat{v}_j \tag{5}$$

$$\dot{\hat{v}}_i = (e/m)\left(\sqrt{g}\,c^{-1}\varepsilon_{ijk}\hat{v}^j\hat{B}^k - \hat{\nabla}_i\Phi\right) \tag{6}$$

Remarks

All fields A, Φ, B, are evaluated at $x = q$.

$$l_r = 1, \qquad l_\theta = \hat{q}^r, \qquad l_\zeta = R_0 + \hat{q}^r\cos\hat{q}^\theta, \qquad \sqrt{g} = l_r l_\theta l_\zeta.$$

difference with the results of chapter 1 is simply in a careful bookkeeping of covariant and contravariant components of the various vectors.

We first go over from the canonical momenta to the more fundamental (covariant) velocity variables,

$$\hat{v}_i = \frac{1}{m}\left(\hat{p}_i - \frac{e}{c}\hat{A}_i\right). \tag{2.9}$$

The result of this transformation is given in table 2.1.

Table 2 2

Particle variables in toroidal physical coordinates: q_i, v_i.

Definition

$$q_i = \hat{q}^i, \qquad v_i = l_i^{-1}\hat{v}_i \tag{1}$$

Fundamental Lie Brackets

$$[q_i, q_j] = 0, \qquad [q_i, v_j] = \frac{1}{ml_j}\delta_{ij},$$

$$[v_i, v_j] = \frac{e}{m^2c}\varepsilon_{ijk}B_k + \frac{1}{ml_il_j}\left(v_j\frac{\partial}{\partial q_i}l_j - v_i\frac{\partial}{\partial q_j}l_i\right) \tag{2}$$

Jacobian $(\boldsymbol{q}, \boldsymbol{p}) \rightarrow (q_i, v_i)$

$$J = m^3 l_r l_\theta l_\zeta \tag{3}$$

Hamiltonian

$$H = \tfrac{1}{2}mv^2 + e\Phi \tag{4}$$

Equations of Motion

$$\dot{q}_i = \frac{1}{l_i}v_i \tag{5}$$

$$\dot{v}_i = \frac{e}{m}\left(E_i + \frac{1}{c}\varepsilon_{ijk}v_jB_k\right) + \sum_k \frac{1}{l_il_k}v_k\left(v_k\frac{\partial}{\partial q_i}l_k - v_i\frac{\partial}{\partial q_k}l_i\right) \tag{6}$$

Remarks

All fields Φ, $\boldsymbol{B}$, $\boldsymbol{E}$, l_i are evaluated at $\boldsymbol{x} = \boldsymbol{q}$,

$$l_r = 1, \qquad l_\theta = q_r, \qquad l_\zeta = R_0 + q_r\cos q_\theta.$$

In order to illustrate the difference between the covariant velocity components and the physical velocity components, we also consider the pseudo-canonical transformation,

$$q_i = \hat{q}^i, \qquad v_i = \frac{1}{l_i}\hat{v}_i, \tag{2.10}$$

where we use the scale factors l_i from (**G2**.3.4),

$$l_1 = l_r = 1, \qquad l_2 = l_\theta = q_r, \qquad l_3 = l_\zeta = R_0 + q_r \cos q_\theta. \tag{2.11}$$

The result is given in table 2.2. Comparing it with table **1**.4.2, we see a typical geometrical effect in the additional term of the bracket $[v_i, v_j]$. Its physical effect is manifest in the equations of motion: it produces a centrifugal force due to the curvature of the magnetic field. This "inertial" force is of purely geometrical nature.

The pseudo-canonical transformation to the locally "cylindrical" variables $(v_\parallel, v_\perp, \varphi)$ poses no new problem. The covariant velocity components are defined as

$$\hat{v}_j = v_\parallel \hat{b}_j(\boldsymbol{q}) + v_\perp \hat{n}_{1j}(\boldsymbol{q}, \varphi), \tag{2.12}$$

where the covariant components of the "moving basis vectors" are defined, as usual, by (**G2**.1.21),

$$\hat{b}_j = \boldsymbol{b} \cdot \hat{\boldsymbol{e}}_j, \qquad \hat{n}_{rj} = \boldsymbol{n}_r \cdot \hat{\boldsymbol{e}}_j, \quad r = 1, 2, \tag{2.13}$$

where the vectors $\boldsymbol{n}_1$, $\boldsymbol{n}_2$ are defined in (**1**.4.11), and the covariant basis vectors $\hat{\boldsymbol{e}}_j$ are given by

$$\hat{\boldsymbol{e}}_j = l_j \boldsymbol{e}_j, \tag{2.14}$$

with $\boldsymbol{e}_j$ being the physical basis vectors introduced in section **1**.4. Note that the variables $v_\parallel$, $v_\perp$ are scalars, hence they do not need any hat:

$$v_\parallel = \hat{v}_j \hat{b}^j = v_j b_j = \boldsymbol{v} \cdot \boldsymbol{b}, \qquad v_\perp = \hat{v}_j \hat{n}_1^j = v_j n_{1j} = \boldsymbol{v} \cdot \boldsymbol{n}_1. \tag{2.15}$$

The result of the pseudo-canonical transformation is given in table 2.3. We have included in this table the scaling factor ϵ: hence, the result is comparable to table **1**.6.1. The similarity is obvious, but the geometrical effects should not be overlooked in the appearance of the *contravariant* components of the basis vectors $\boldsymbol{b}$, $\boldsymbol{n}_1$, $\boldsymbol{n}_2$ in the Lie brackets involving $\hat{q}^i$, and in the definition of the curl of $\boldsymbol{b}$ and of $\boldsymbol{n}_1$ contained in the vector $\boldsymbol{D}$.

Table 2.3

Scaled particle variables in toroidal coordinates: $\hat{q}^i$, $v_\parallel$, $v_\perp$, φ.

Definition

$$\hat{q}^i = \hat{q}^i, \qquad \hat{v}_i = v_\parallel \hat{b}_i + v_\perp \hat{n}_{1i}(\boldsymbol{q}, \varphi) \tag{1}$$

Fundamental Lie Brackets

$$[\hat{q}^i, \hat{q}^j] = 0, \qquad [\hat{q}^i, v_\parallel] = m^{-1}\hat{b}^i, \qquad [\hat{q}^i, v_\perp] = m^{-1}\hat{n}_1^i$$

$$[\hat{q}^i, \varphi] = -\frac{1}{mv_\perp}\hat{n}_2^i, \qquad [v_\parallel, v_\perp] = \frac{1}{m}\boldsymbol{n}_2\cdot\boldsymbol{D}, \qquad [v_\parallel, \varphi] = \frac{1}{mv_\perp}\boldsymbol{n}_1\cdot\boldsymbol{D}$$

$$[v_\perp, \varphi] = -\frac{1}{\epsilon}\frac{\Omega}{mv_\perp} - \frac{1}{mv_\perp}\boldsymbol{b}\cdot\boldsymbol{D} \tag{2}$$

Jacobian $(\boldsymbol{q}, \boldsymbol{v}) \to (\hat{q}^i, v_\parallel, v_\perp, \varphi)$

$$J = v_\perp l_r l_\theta l_\zeta \tag{3}$$

Hamiltonian

$$H = \tfrac{1}{2}m\left(v_\parallel^2 + v_\perp^2\right) + e\Phi \tag{4}$$

Equations of Motion

$$\dot{\hat{q}}^i = v_\parallel \hat{b}^i + v_\perp \hat{n}_i^i \tag{5}$$

$$\dot{v}_\parallel = v_\perp \boldsymbol{n}_2\cdot\boldsymbol{D} + \frac{e}{m}\boldsymbol{b}\cdot\boldsymbol{E} \tag{6}$$

$$\dot{v}_\perp = -v_\parallel \boldsymbol{n}_2\cdot\boldsymbol{D} + \frac{e}{m}\boldsymbol{n}_1\cdot\boldsymbol{E} \tag{7}$$

$$\dot{\varphi} = \frac{1}{\epsilon}\Omega + \boldsymbol{b}\cdot\boldsymbol{D} - \frac{v_\parallel}{v_\perp}\boldsymbol{n}_1\cdot\boldsymbol{D} - \frac{e}{mv_\perp}\boldsymbol{n}_2\cdot\boldsymbol{D} \tag{8}$$

Remarks

All fields Φ, $\boldsymbol{E}$, B, Ω, $\boldsymbol{b}$, $\boldsymbol{n}_1$, $\boldsymbol{n}_2$ are evaluated at $\boldsymbol{x} = \boldsymbol{q}$;

$$\nabla \equiv \frac{\partial}{\partial \boldsymbol{q}}$$

$$\Omega(\boldsymbol{q}) = \frac{eB(\boldsymbol{q})}{mc}, \qquad \boldsymbol{D} = v_\parallel(\nabla \wedge \boldsymbol{b}) + v_\perp(\nabla \wedge \boldsymbol{n}_1)$$

$$l_r = 1, \qquad l_\theta = \hat{q}_r, \qquad l_\zeta = R_0 + \hat{q}_r \cos\hat{q}_\theta$$

9.3. Equations of the guiding centre motion in toroidal geometry

We now proceed, in analogy with section **1**.8, to the construction of the *averaging pseudo-canonical transformation* which leads to the definition of a set of *natural guiding centre (NGC) variables*. The equations of motion of these variables must be free from rapidly oscillating terms. We assume, of course, that the characteristic drift parameter is small, $\epsilon \ll 1$, and construct the NGC variables as power series in ϵ.

Given the great analogy of tables 2.3 and **1**.6.1, the pseudo-canonical transformation in toroidal coordinates is immediately obtained from the results of section **1**.8. We define the toroidal coordinates of the guiding centre as

$$\hat{Y}^1 = r, \qquad \hat{Y}^2 = \theta, \qquad \hat{Y}^3 = \zeta. \tag{3.1}$$

These are simply the contravariant components of the vector $\boldsymbol{Y}$, defined by (**1**.8.2) (through order ϵ)

$$\hat{Y}^i = \hat{q}^i - \frac{\epsilon}{\Omega} v_\perp \hat{n}_2^i. \tag{3.2}$$

The definition of the scalar quantities U, W, ϕ is the same as in eqs. (**1**.8.3)–(**1**.8.5). The geometrical effects are concealed only in the explicit expressions of the various tensorial operations involved in these definitions. Moreover, no new fact is involved in going over to the other sets of NGC variables. We may thus use any one of the more important sets of NGC variables: $(\boldsymbol{Y}, U, W, \phi)$, $(\boldsymbol{Y}, U, M, \phi)$ or $(\boldsymbol{Y}, \mathscr{E}, M, \phi)$ (in forthcoming chapters, other useful sets of NGC variables will also be used). We only give here the results for the latter set of variables in table 3.1; by comparing this table with table **1**.8.3, the reader will easily see how the results for the other sets of NGC variables must be modified in toroidal coordinates.

In order to make the results completely explicit, we must specify the form of the magnetic and electric fields. In the standard model, we adopt the form given in eqs. (**8**.9.5) for the magnetic field,

$$\boldsymbol{B}(r, \theta) = B_0 \left(\frac{1}{q(r)} \frac{r}{R_0} \boldsymbol{e}_\theta - \frac{1}{1 + (r/R_0) \cos \theta} \boldsymbol{e}_\zeta \right), \tag{3.3}$$

As for the electric field, we assume that it is determined by the gradient of the self-consistent potential $\Phi(r)$. It will be shown later that the electric potential is (to dominant order) a surface quantity; it then follows that the electric field points in the radial direction. This discussion implies that we

Table 3.1

Natural guiding centre variables in toroidal coordinates $\hat{Y}^i$, $\mathscr{E}$, M, ϕ.

Definition

$$\mathscr{E} = (m/2)(U^2 + W^2) + e\Phi, \qquad M = \left[mW^2/2B(Y) \right] \tag{1}$$

Fundamental Lie Brackets

$$[\hat{Y}^i, \hat{Y}^j] = -\epsilon \left(m\Omega\sqrt{g} \right)^{-1} \varepsilon^{ijk} \hat{b}_k, \qquad [\hat{Y}^i, \mathscr{E}] = U\hat{b}^{**i}, \qquad [\hat{Y}^i, M] = 0$$

$$[\hat{Y}^i, \phi] = \epsilon \left(m\Omega\sqrt{g} \right)^{-1} \varepsilon^{ijk} \hat{b}_j \hat{R}_k, \qquad [\mathscr{E}, M] = 0, \tag{2}$$

$$[\mathscr{E}, \phi] = -\epsilon^{-1}\Omega - U\boldsymbol{b}\cdot\boldsymbol{R} + (1/2)U\boldsymbol{b}\cdot(\nabla \wedge \boldsymbol{b}), \qquad [M, \phi] = -\epsilon^{-1}(e/mc) \tag{2}$$

Jacobian $(\boldsymbol{q}, \boldsymbol{v}) \to (\hat{Y}^i, \mathscr{E}, M, \phi)$

$$|J| = \frac{B_{\|}^*}{m^2 |U|}\sqrt{g} \tag{3}$$

Hamiltonian

$$H = \mathscr{E} \tag{4}$$

Equations of Motion:

$$\dot{\hat{Y}}^i = U\hat{b}^{**i} \tag{5}$$

$$\dot{\mathscr{E}} = 0 \tag{6}$$

$$\dot{M} = 0 \tag{7}$$

$$\dot{\phi} = \epsilon^{-1}\Omega + U\boldsymbol{b}\cdot\boldsymbol{R} - \tfrac{1}{2}U\boldsymbol{b}\cdot(\boldsymbol{b} \wedge \boldsymbol{R}) \tag{8}$$

Remarks

All fields Ω, B, $B_{\|}^*$, $\boldsymbol{b}$, $\boldsymbol{b}^{**}$, U, $\boldsymbol{R}$ are evaluated at $\boldsymbol{x} = \boldsymbol{Y}$;

$$\nabla \equiv \frac{\partial}{\partial \boldsymbol{Y}}$$

$$U = \sigma\left[(2/m)(\mathscr{E} - e\Phi - MB)\right]^{1/2}, \quad \sigma = \pm$$

$$B_{\|}^* = B\left(1 + \frac{\epsilon}{\Omega} U\boldsymbol{b}\cdot(\nabla \wedge \boldsymbol{b})\right)$$

$$\boldsymbol{b}^{**} = \boldsymbol{b} + \frac{\epsilon}{\Omega}\boldsymbol{b} \wedge \left(U(\boldsymbol{b}\cdot\nabla)\boldsymbol{b} + \frac{1}{mU}(M\nabla B + e\nabla\Phi) \right)$$

$$\boldsymbol{R} = \nabla \boldsymbol{n}_1 \cdot \boldsymbol{n}_2 = \nabla \boldsymbol{e}_1 \cdot \boldsymbol{e}_2$$

$$l_r = 1, \qquad l_\theta = r, \qquad l_\zeta = R_0 + r\cos\theta, \qquad \sqrt{g} = l_r l_\theta l_\zeta$$

neglect, in the present section, the presence of an *externally induced electric field* $\boldsymbol{E}^{(A)}$, which would be tangent to the magnetic surface.

We must not forget that, in the standard model, the inverse aspect ratio $\bar{\eta} \equiv a/R_0$ is considered as a small parameter (see eq. **8**.9.2). At present, we introduce a *scaling assumption* relating the drift parameter ϵ to the inverse aspect ratio,

$$\bar{\eta} \approx \epsilon \ll 1, \tag{3.4}$$

Hence, from (**8**.9.6) we conclude that

$$\frac{r}{R_0} \leqslant \epsilon \ll 1. \tag{3.5}$$

We now write the equations of motion of the guiding centre in a completely explicit form. We have collected in table 3.2 a set of formulae which are useful for the evaluation of the vector $\boldsymbol{b}^{**}$ defining the drift velocity. Using these formulae, we easily obtain the *equations of motion for the toroidal coordinates* (r, θ, ζ) *of the guiding centre*, in the drift approximation, *through order* ϵ:
(A) Radial coordinate:

$$\dot{r} = -\epsilon \frac{1}{\Omega_0 R_0}\left(U^2 + \frac{1}{m} B_0 M\right) \sin\theta. \tag{3.6}$$

(B) Poloidal angle:

$$r\dot{\theta} = \frac{r}{qR_0} U - \epsilon \frac{1}{\Omega_0 R_0}\left(U^2 + \frac{1}{m} B_0 M\right) \cos\theta + \epsilon \frac{e}{m\Omega_0} \frac{\mathrm{d}\Phi}{\mathrm{d}r}. \tag{3.7}$$

(C) Toroidal angle:

$$(R_0 + r\cos\theta)\,\dot{\zeta} = U. \tag{3.8}$$

It should not be forgotten that we are working here with the dynamical variables $(r, \theta, \zeta, \mathscr{E}, M, \phi)$; hence, the parallel velocity is a function of these variables:

$$U \equiv U(r, \theta, \mathscr{E}, M) = \sigma\left(\frac{2}{m}\left[\mathscr{E} - e\Phi(r) - MB(r, \theta)\right]\right)^{1/2} \tag{3.9}$$

We have also used a natural notation for the characteristic Larmor frequency (evaluated with the constant field B_0),

$$\Omega_0 = \frac{eB_0}{mc}. \tag{3.10}$$

Table 3 2

Useful formulae in toroidal coordinates.

$$\boldsymbol{B} = 0\cdot\boldsymbol{e}_r + \frac{B_0}{q(r)}\frac{r}{R_0}\boldsymbol{e}_\theta + \frac{B_0}{1+(r/R_0)\cos\theta}\boldsymbol{e}_\zeta \tag{1}$$

$$B = \frac{B_0}{1+(r/R_0)\cos\theta} \approx B_0\left(1-\frac{r}{R_0}\cos\theta\right) \tag{2}$$

$$\boldsymbol{b} = \frac{r}{q(r)\,R_0}\boldsymbol{e}_\theta + \boldsymbol{e}_\zeta \tag{3}$$

$$\nabla B = (B_0/R_0)(-\cos\theta\,\boldsymbol{e}_r + \sin\theta\,\boldsymbol{e}_\theta) \tag{4}$$

$$\boldsymbol{b}\wedge\nabla B = (B_0/R_0)\left[-\sin\theta\,\boldsymbol{e}_r - \cos\theta\,\boldsymbol{e}_\theta + (r/qR_0)\boldsymbol{e}_\zeta\right] \tag{5}$$

$$\nabla\wedge\boldsymbol{b} = -\frac{1}{R_0 + r\cos\theta}(\sin\theta\,\boldsymbol{e}_r + \cos\theta\,\boldsymbol{e}_\theta) + \frac{1}{r}\frac{\partial}{\partial r}\frac{r^2}{qR_0}\boldsymbol{e}_\zeta \tag{6}$$

$$(\boldsymbol{b}\cdot\nabla)\boldsymbol{b} = -\boldsymbol{b}\wedge(\nabla\wedge\boldsymbol{b})$$

$$= -\left(\frac{\cos\theta}{R_0 + r\cos\theta} + \frac{1}{qR_0^2}\frac{\partial}{\partial r}\frac{r^2}{q}\right)\boldsymbol{e}_r - \frac{\sin\theta}{R_0 + r\cos\theta}\boldsymbol{e}_\theta - \frac{r\sin\theta}{qR_0^2}\boldsymbol{e}_\zeta \tag{7}$$

$$\boldsymbol{b}\wedge(\boldsymbol{b}\cdot\nabla)\boldsymbol{b} = -\frac{\sin\theta}{R_0 + r\cos\theta}\left(1 + \frac{r^2}{q^2R_0^2}\right)\boldsymbol{e}_r - \left(\frac{\cos\theta}{R_0 + r\cos\theta} + \frac{1}{qR_0}\frac{\partial}{\partial r}\frac{r^2}{qR_0}\right)\boldsymbol{e}_\theta$$

$$-\left(\frac{r\cos\theta}{qR_0(R_0 + r\cos\theta)} + \frac{r}{q^2R_0^2}\frac{\partial}{\partial r}\frac{r^2}{qR_0}\right)\boldsymbol{e}_\zeta \tag{8}$$

$$\boldsymbol{E} = -\nabla\Phi = -\frac{\mathrm{d}\Phi}{\mathrm{d}r}\boldsymbol{e}_r \tag{9}$$

$$\boldsymbol{b}\wedge\nabla\Phi = \frac{\mathrm{d}\Phi}{\mathrm{d}r}\boldsymbol{e}_\theta - \frac{r}{qR_0}\frac{\mathrm{d}\Phi}{\mathrm{d}r}\boldsymbol{e}_\zeta \tag{10}$$

Equations (3.6–3.8) are the basic equations determining the motion of the guiding centre in a toroidal magnetic field configuration of large aspect ratio. They were first derived by Kadomtsev and Pogutse (1967) and Berk and Galeev (1967); they are also considered in the work by Galeev and Sagdeev (1968, 1979).

Let us discuss the various terms of these equations. From eq. (5) of table 3.1 we know that the velocity of the guiding centre consists of two terms of

unequal magnitude: the parallel velocity Ub, which is of order ϵ^0, and the perpendicular drift velocity, which is of order ϵ. As the magnetic field lies in the surface $r = \text{const.}$, the parallel velocity has components in the θ- and in the ζ-direction: these components appear in the first term of the right-hand side of (3.7) and on the right-hand side of (3.8). We note, however, that as the magnetic field points almost in the toroidal direction, i.e.

$$\frac{B_\theta}{B_\zeta} \sim \frac{r}{R_0} \equiv \eta \leqslant \bar{\eta}, \tag{3.11}$$

the θ-component of the parallel velocity is much smaller than its ζ-component. Because of (3.5), the poloidal component of the parallel velocity is of the same order as the drift velocity contributions in eq. (3.7). As a result, the velocity of any guiding centre is much larger in the toroidal direction than in the poloidal or radial directions.

The second term on the right-hand side of (3.7) and the term on the right-hand side of (3.6) are the components of the curvature drift ($\sim U^2$) and of the grad-B drift ($\sim MB_0$). In the present geometry, these two terms always act together: their sum will be briefly called the *inhomogeneity drift*. Finally, the last term of (3.7) is obviously the $\boldsymbol{E} \wedge \boldsymbol{B}$ drift.

9.4. The toroidal invariant of the motion

The most usual method for solving classical equations of motion consists of trying to find *invariants (or constants) of the motion*. Every additional invariant restricts the available phase space for the motion: if enough invariants are known, the *trajectory (or orbit)* of the moving point can be exactly determined.

In our present problem we already know *two* constants of the motion (see table 3.1). The first, trivially obvious one, is the *energy* $\mathscr{E}$. The second invariant is the *magnetic moment* M, discussed in section 1.8. It was shown there that the magnetic moment $\mu = mv_\perp^2/2B(\boldsymbol{q})$ expressed in terms of the particle variables $v_\perp$, $\boldsymbol{q}$, is *not* a constant of the motion. After the pseudo-canonical averaging transformation, the corresponding quantity defined in terms of NGC variables, $M = mW^2/2B(\boldsymbol{Y})$ becomes an approximate constant of the motion. We have here an example of a non-conserved quantity μ, evolving into an *adiabatic invariant* M under the averaging transformation. We now consider a third quantity, for which we find an "opposite" phenomenon: an exact constant of the motion which turns into an *approximate* one under the averaging transformation.

The magnetic field configuration considered here (and which is very nearly realized in many actual confinement systems, such as the tokamak), has the

important property of *axisymmetry*. All the components of the magnetic and electric fields are independent of the toroidal angle ζ. This property is then shared by the vector and scalar potentials $\boldsymbol{A}$ and Φ. On the other hand, the toroidal coordinate system used here is also axisymmetric, i.e. its characteristic scale factors l_i are independent of ζ. As a result, the Hamiltonian (2.8) is independent of ζ: the toroidal angle is a "*cyclic*" (or "*ignorable*") variable (Goldstein 1980, Percival and Richards 1982). We recall a very well known theorem of Hamiltonian mechanics: *The momentum p canonically conjugate to a cyclic variable q is a constant of the motion.*

The theorem is directly applicable to our system, described in section 9.2 in terms of the canonical particle variables $\hat{q}^i$, $\hat{p}_i$, with the fundamental Lie brackets (2.7). We conclude that the *covariant component* $\hat{p}_3$ *of the canonical momentum is an exact invariant of the motion*, which we denote by $\hat{\mathrm{L}}_3$,

$$\hat{L}_3 = \hat{p}_3. \tag{4.1}$$

$\hat{L}_3$ will be called the *toroidal invariant*. Clearly, its property is maintained when any other set of variables is used, provided they are connected to $(\hat{q}^i, \hat{p}_i)$ by an exact pseudo-canonical transformation. Thus, if the variables $(\hat{q}^i, \hat{v}_i)$ are used, the toroidal invariant becomes

$$\hat{L}_e = m\hat{v}_3 + \frac{e}{c}\hat{A}_3(\hat{q}^1, \hat{q}^2). \tag{4.2}$$

Its invariance is easily checked by using the Lie brackets of table 2.1.

A less trivial transformation obtains in going to the variables $(\hat{q}^i, v_\parallel, v_\perp, \phi)$. The important point is that, in order to ensure the comparison of the large terms (of order ϵ^{-1}) coming from $[v_\perp, \varphi]$ in the bracket $[\hat{L}_3, H]$, the toroidal invariant must be scaled as

$$\hat{L}_3 = m\left(v_\parallel \hat{b}_3 + v_\perp \hat{n}_{13}\right) + \frac{1}{\epsilon}\frac{e}{c}\hat{A}_3, \tag{4.3}$$

where $\hat{n}_{13}$ is the covariant component of the vector $\boldsymbol{n}_1(\boldsymbol{q}, \varphi)$ along the local ζ-axis. (Note that this scaling is consistent with Gardner's "recipe" $e \to \epsilon^{-1}e$). We leave to the reader the direct check of the relation $[\hat{L}_3, H] = 0$, by means of the results of table 2.3. This exercise is enlightening, as it shows that the presence of the *gyrophase-dependent* vector $\boldsymbol{n}_1(\boldsymbol{q}, \varphi)$ is essential to the derivation.

We now perform the *averaging pseudo-canonical transformation* leading to the NGC variables $(\hat{Y}^i, \mathscr{E}, M, \phi)$ of table 3.1. A priori, it is not at all obvious that the exact invariant $\hat{L}_3$ can be prolonged into an invariant of the approximate equations of motion of the NGC variables. One may consider as

a candidate the quantity obtained from (4.3) by formally averaging it over the gyrophase, replacing $v_\parallel$ by U and evaluating the fields $\boldsymbol{b}$ and $\boldsymbol{A}$ at the position of the guiding centre:

$$L_3 = mU\hat{b}_3(\hat{Y}^1, \hat{Y}^2) + \frac{1}{\epsilon}\frac{e}{c}\hat{A}_3(\hat{Y}^1, \hat{Y}^2), \tag{4.4}$$

where U is a function of $\hat{Y}^1$, $\hat{Y}^2$, $\mathscr{E}$, M, given in table 3.1. But the Lie bracket of the "averaged" function L_3 with the "averaged" Hamiltonian must not, in general, equal the "averaged" Lie bracket $[L_3, H]$. We must therefore check the conservation law by using the results of table 3.1,

$$[L_3, H] = \left(\frac{1}{\epsilon}\frac{e}{c}\hat{\nabla}_i\hat{A}_3 + m\hat{b}_3\hat{\nabla}_i U + mU\,\hat{\nabla}_i\hat{b}_3\right) \times\left\{U\hat{b}^i + \frac{\epsilon}{\Omega}U\left[\boldsymbol{b}\wedge\left(U(\boldsymbol{b}\cdot\nabla)\boldsymbol{b} + \frac{1}{mU}\nabla(MB + e\Phi)\right)\right]^i + O(\epsilon^2)\right\}, \tag{4.5}$$

where $\hat{\nabla}_i \equiv \partial/\partial\hat{Y}^i$. In order to analyze these contributions, we derive a useful lemma. Let $\boldsymbol{V}$ be an arbitrary vector, and let the vector potential $\boldsymbol{A}$ be independent of $\hat{Y}^3 \equiv \zeta$. Then

$$\begin{aligned}\hat{V}^s\,\hat{\nabla}_s\hat{A}_3 &= \hat{V}^1\,\hat{\nabla}_1\hat{A}_3 + \hat{V}^2\,\hat{\nabla}_2\hat{A}_3\\ &= \hat{V}^1(\hat{\nabla}_1\hat{A}_3 - \hat{\nabla}_3\hat{A}_1) + \hat{V}^2(\hat{\nabla}_2\hat{A}_3 - \hat{\nabla}_3\hat{A}_2)\\ &= \sqrt{g}\,(-\hat{V}^1\hat{B}^2 + \hat{V}^2\hat{B}^1).\end{aligned}$$

Thus

$$\hat{V}^s\,\hat{\nabla}_s\hat{A}_3 = \sqrt{g}\,B\,\varepsilon_{3jk}\hat{b}^j\hat{V}^k, \tag{4.6}$$

or, in ordinary vector notation,

$$\boldsymbol{V}\cdot\nabla\hat{A}_3 = B(\boldsymbol{b}\wedge\boldsymbol{V})_3, \tag{4.7}$$

where the right-hand side must be understood as the covariant component of the vector product.

Returning to (4.5), we consider the contribution of order ϵ^{-1},

$$[L_3, H]_{-1} = \epsilon^{-1} \frac{e}{c} U \hat{b}^i \hat{\nabla}_i \hat{A}_3 = \epsilon^{-1} \frac{e}{c} U \sqrt{g}\, B\, \varepsilon_{3jk} \hat{b}^j \hat{b}^k = 0.$$

The contribution of order ϵ^0 is

$$[L_3, H]_0 = -\hat{b}_3 \boldsymbol{b} \cdot \nabla (e\Phi + MB) + mU^2 (\boldsymbol{b} \cdot \nabla) \hat{b}_3$$

$$+ \frac{1}{B} [(\boldsymbol{b} \wedge \nabla)(e\Phi + MB)] \cdot \nabla \hat{A}_3$$

$$+ \frac{mU^2}{B} [\boldsymbol{b} \wedge (\boldsymbol{b} \cdot \nabla) \boldsymbol{b}] \cdot \nabla \hat{A}_3.$$

It is indeed not difficult to show that the third term cancels the first and the fourth term cancels the second, by using identity (4.7). Thus, we showed that the dynamical function L_3 is conserved at least through the two dominant orders in ϵ:

$$[L_3, H] = 0 \cdot \epsilon^{-1} + 0 \cdot \epsilon^0 + O(\epsilon). \tag{4.8}$$

In order to check the terms of order ϵ, we need to know the fundamental Lie brackets through order ϵ^2 (see eq. 4.5), an approximation which goes beyond the usual one (i.e. it requires the terms of order ϵ^2 in the guiding centre velocity) *. In other words, the equations of motion (3.6–3.8), exact through order ϵ, consistently admit the integral of motion L_3 defined by (4.4).

We now derive a more explicit form of this function for the standard model geometry. For this purpose, we need an expression for the vector potential A. We first recall eqs. (G2.2.7) relating the covariant and contravariant components of any vector to its physical components. Using eq. (3) of table 3.2, we obtain

$$\hat{b}_3 = R_0 + r \cos \theta. \tag{4.9}$$

* The conservation through order ϵ of an appropriately defined toroidal invariant L_3 has been proved by Weyssow (unpublished).

From eq. (1) of the same table, we get the contravariant components of the magnetic field, which are directly related to the vector potential:

$$\hat{B}^1 = \frac{1}{\sqrt{g}}\left(\hat{\nabla}_2\hat{A}_3 - \hat{\nabla}_3\hat{A}_2\right) = 0,$$

$$\hat{B}^2 = \frac{1}{\sqrt{g}}\left(\hat{\nabla}_3\hat{A}_1 - \hat{\nabla}_1\hat{A}_3\right) = \frac{B_0}{R_0}\frac{1}{q(r)},$$

$$\hat{B}^3 = \frac{1}{\sqrt{g}}\left(\hat{\nabla}_1\hat{A}_2 - \hat{\nabla}_2\hat{A}_1\right) = \frac{B_0}{R_0}\left(1 - \frac{r}{R_0}\cos\theta\right). \tag{4.10}$$

Axisymmetry requires

$$\hat{A}_i = \hat{A}_i(r, \theta), \quad i = 1, 2, 3.$$

It then follows from the first equation that $\hat{A}_3$ is a function of r alone. The second equation can then be integrated (to dominant order in η) with the result

$$\hat{A}_3(r) = -B_0\int_0^r \mathrm{d}r'\frac{r'}{q(r')}. \tag{4.11}$$

The last equation reduces to

$$\hat{\nabla}_1\hat{A}_2 - \hat{\nabla}_2\hat{A}_1 = rB_0.$$

It leaves us with the characteristic freedom of choice of a gauge. We choose the simple solution

$$\hat{A}_1 = 0 \tag{4.12}$$

from which follows

$$\hat{A}_2 = \tfrac{1}{2}B_o r^2. \tag{4.13}$$

The *toroidal invariant* (4.4) thus takes the explicit form

$$L_3 = -\frac{1}{\epsilon}m\Omega_0\int_0^r \mathrm{d}r'\frac{r'}{q(r')} + mR_0\left(1 + \frac{r}{R_0}\cos\theta\right)U. \tag{4.14}$$

This expression was first derived (by a non-Hamiltonian method) by Berk and Galeev (1967) and further discussed by Galeev and Sagdeev (1968, 1979).

9.5. *Topological classification of the guiding centre orbits*

We now possess all we need in order to solve the equations of motion (3.6)–(3.8). For simplicity, we first discuss the problem in absence of an electric field,

$$\Phi = 0.$$

The modifications necessary for the inclusion of the electric field will be given in section 9.8.

We first note that, for the present axisymmetric system, eq. (3.8) is decoupled from the others. The two equations (3.6), (3.7) constitute a closed set for the two unknowns $r(t)$, $\theta(t)$. Their solution must be inserted into (3.8) in order to determine $\zeta(t)$.

Next, we note that the knowledge of three constants of the motion is sufficient for the complete determination of the *trajectory* (or *orbit*) in the r–θ plane (i.e. the projection of the orbit onto an arbitrary section ζ = const. of the torus). Indeed, we simply write that the toroidal invariant L_3 (divided, for convenience, by the constant mR_0) has a constant value L,

$$-\frac{1}{\epsilon}\Omega_0 \int_0^r \mathrm{d}r'\, G(r') + \left(1 + \frac{r}{R_0}\cos\theta\right) U(r, \theta, \mathscr{E}, M) = L. \tag{5.1}$$

For given values $\mathscr{E}$, M, L of the three invariants, this equation provides us with a functional relationship between r and θ. In other words, (5.1) is the *equation of the guiding centre orbit* *.

We introduce here function $G(r)$,

$$G(r) = \frac{r}{R_0 q(r)}. \tag{5.2}$$

* In the remainder of the present and of the following section, we shall most frequently call "an orbit" the object which should properly be called "the projection of an orbit onto a section ζ = const. of the torus" The context allows no confusion between this projected orbit and the complete, three-dimensional orbit, which also takes into account the motion in the toroidal direction.

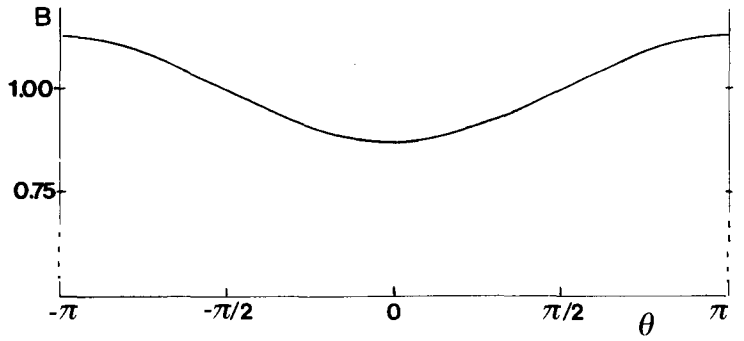

Fig 5.1. Magnetic field intensity as a function of θ for given r.

The parallel velocity (3.9) is conveniently expressed as follows, for the magnetic field intensity taken from eq. (2) of table 3.2 *:

$$U(r, \theta, \mathscr{E}, M)$$

$$= \sigma\left(\frac{2}{m}\right)^{1/2}\left[\mathscr{E} - MB_0\left(1 - \frac{r}{R_0}\right) + MB_0\frac{r}{R_0}(\cos\theta - 1)\right]^{1/2}. \qquad (5.3)$$

The *key point* in understanding the motion of the guiding centres is the fact that, for given r (i.e. on every magnetic surface), *the magnetic field intensity* $B(r, \theta)$ *is a periodic function of the poloidal angle* θ. In the standard model [eq. (2), table 3.2], this function has a single *maximum* for $\theta = \pi$, i.e. on the inner part of the torus. This function is shown in fig. 5.1.

This figure immediately suggests the mechanical problem of the motion of a particle in a potential well. But the analogy is not quite accurate, because the Newton equation with a potential well is a second-order differential equation, whereas (3.7) is a first-order equation. The physical reason for which certain particles can be reflected by this magnetic field configuration lies in the invariance of the magnetic moment. This is the property which explains the magnetic mirror effect, well known from elementary plasma physics.

In order to analyze this effect more quantitatively, we consider eq. (3.7) in which we neglect, provisionally, the inhomogeneity drift. Indeed, this term is small compared to the first one (it has a factor R_0^{-1} in addition to the factor

* In the subsequent formulae, the variables $\mathscr{E}$ M will not always be written explicitly; thus $U(r, \theta; \mathscr{E}, M) \equiv U(r, \theta)$.

ϵ). Its main effect is to produce a small excursion of the guiding centre away from its initial magnetic surface (see the qualitative discussion in section 9.1). If this effect is neglected in first approximation, we reduce eq. (3.7) to

$$\dot{\theta} = \frac{G(r_0)}{r_0}\, U(r_0, \theta), \tag{5.4}$$

in which we treat the radial coordinate r as a constant, equal to its "initial" value r_0,

$$r = r_0, \qquad \theta = 0. \tag{5.5}$$

The first question to be asked is the following: *is it possible for the parallel velocity $U(r_0, \theta)$ to vanish for a value of $\theta = \theta_*$ smaller than π?* *

The answer to this question depends on the values of the constants of the motion $\mathscr{E}$, M, and on the initial value r_0. For a certain range of values of these quantities, the answer is affirmative. There exists then a *turning point* at $\theta = \theta_*$, at which the particle is reflected: U vanishes and then changes its sign. The particle is said to be *trapped*. Its motion is confined to the range of poloidal angles: $-\theta_* \leqslant \theta \leqslant \theta_*$. In the opposite case, the particle is called *passing*: *the sign of the parallel velocity U is then maintained during the motion*; the angle θ increases steadily, and the particle turns around the torus in the poloidal direction.

At this stage, the analogy with the classical *pendulum motion* is quite obvious. The well-known equation of the pendulum is (see e.g. Greiner 1981)

$$\ddot{\varphi} = -\omega^2 \sin \varphi.$$

If this equation is integrated once, one finds

$$\tfrac{1}{2}\dot{\varphi}^2 = \omega^2 \cos \varphi + C,$$

or

$$\dot{\varphi} = \sqrt{2}\,\omega\,\sigma\sqrt{\cos \varphi + k}\,,$$

where k is a constant, depending on the initial condition. If $|k| < 1$, the angular velocity vanishes for $\cos \varphi_* = -k$: the pendulum *oscillates* between

* As $U(r, \theta)$ is an even function of θ, it is sufficient to discuss the range of positive values of θ: $0 \leqslant \theta \leqslant \pi$.

$-\varphi_*$ and $+\varphi_*$ (trapped motion). If $|k| > 1$, the angular velocity can never vanish and the pendulum *rotates* around its axis (passing motion).

Returning to our problem, we define the convenient quantity κ, called the *trapping parameter*,

$$\kappa^2 = \frac{\mathscr{E} - MB_0[1 - (r_0/R_0)]}{2(r_0/R_0)MB_0}. \tag{5.6}$$

The parallel velocity can then be written as

$$U(r_0, \theta) = \sigma\left(\frac{2}{m}MB_0\right)^{1/2}\left(\frac{r_0}{R_0}\right)^{1/2}\sqrt{2\kappa^2 - 1 + \cos\theta}\,. \tag{5.7}$$

It is now clear that if $\kappa^2 < 1$, there exists a real angle θ_* in the range $0 \leqslant \theta_* \leqslant \pi$ (i.e. $+1 \geqslant \cos\theta_* \geqslant -1$), for which $U(r_0, \theta_*) = 0$:

$$\cos\theta_* = 1 - 2\kappa^2. \tag{5.8}$$

Thus, the type of trajectory is defined by the following criterion:

$$\begin{aligned} &\kappa^2 < 1: \quad \text{trapped,} \\ &\kappa^2 > 1: \quad \text{passing.} \end{aligned} \tag{5.9}$$

It is thus the particular combination (5.6) of the constants $\mathscr{E}$, M, r_0 which determines the topological nature of the orbits. This can be vividly pictured by drawing the "phase space" $(\dot{\theta}, \theta)$ of the differential equation (5.4). One sees in fig. 5.2 the region of trapped trajectories, which appear as closed curves centered around the origin (or around its homologous points $\theta = 2k\pi$). The passing orbits are open curves along which θ increases indefinitely. The limit between the two regions is the *separatrix*, corresponding to $\kappa^2 = 1$. The motion for this value of κ has peculiar properties (see section 9.7).

The trapping condition (5.9) can be interpreted in a physically transparent way, if we note that the initial value (for $\theta = 0$) of the parallel velocity is given from eq. (5.3) by

$$U(r_0, 0) \equiv U_0 = \sigma\left(\frac{2}{m}\right)^{1/2}\left[\mathscr{E} - MB_0\left(1 - \frac{r_0}{R_0}\right)\right]^{1/2}. \tag{5.10}$$

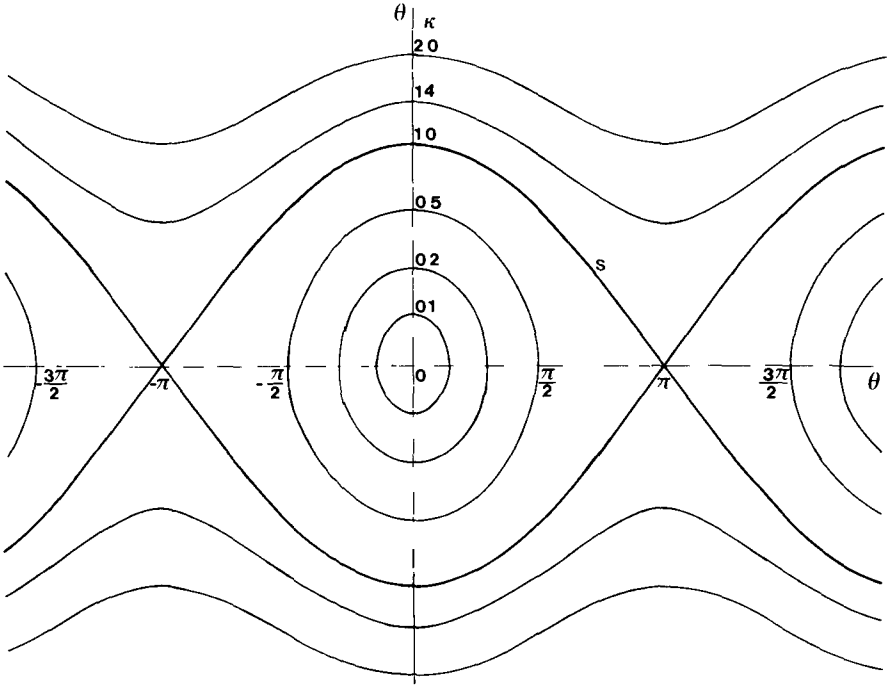

Fig. 5.2. Phase space of the poloidal motion

Hence, the trapping condition can also be written as

$$\frac{(m/2)U_0^2}{MB_0} < 2\frac{r_0}{R_0},$$

or, because of the smallness of the ratio r_0/R_0,

$$\frac{(m/2)U_0^2}{\mathscr{E}} \lesssim 2\frac{r_0}{R_0}. \tag{5.11}$$

Thus, the trapped particles are those for which the (initial) parallel kinetic energy is a small fraction (of order 2η) of their total energy (Kadomtsev and Pogutse 1967, Galeev and Sagdeev 1968).

9.6. Shape of the guiding centre orbits

Having uncovered the existence of two topologically distinct classes of trajectories, we now study their shape more precisely. We no longer neglect the

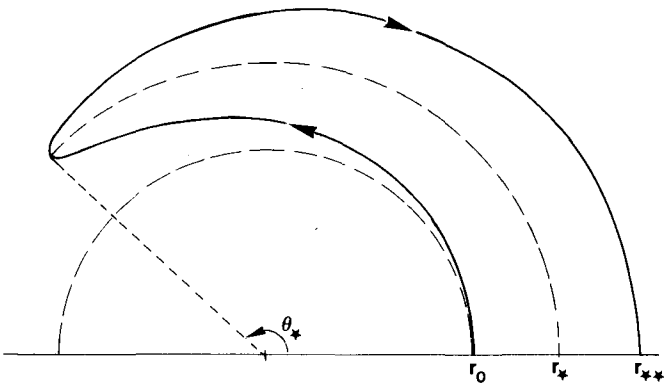

Fig. 6 1 A typical trapped half-orbit.

variation of the radial coordinate, thus taking full account of the inhomogeneity drift,which was neglected in section 9.5. In other words, we now consider the exact solution of eqs. (3.6) and (3.7). As stated at the beginning of section 9.5, this solution is provided by eq. (5.1). We analyze this equation in some detail, for the two classes of orbits.

A. Trapped orbits

Consider a guiding centre starting at the point ($r = r_0$, $\theta = 0$) with values $\mathscr{E}$, M, L of the constants of its motion. The coordinate r_0 is related to these constants by relation (5.1):

$$-\frac{\Omega_0}{\epsilon}\int_0^{r_0} dr'\, G(r') + \left(1 + \frac{r_0}{R_0}\right) U(r_0, 0) = L. \tag{6.1}$$

Assume that the guiding centre moves towards increasing θ and r (see fig. 6.1) and that the values of $\mathscr{E}$, M, r_0 are such that $\kappa^2 < 1$. As it reaches the *turning point* (r_*, θ_*) its parallel velocity vanishes. At this point, eq. (5.1) reduces to

$$-\frac{\Omega_0}{\epsilon}\int_0^{r_*} dr'\, G(r') = L. \tag{6.2}$$

The constant L can be eliminated between these two equations and an expression for r_* as a function of r_0 can be obtained by simply subtracting the two previous equations, side by side, from one another:

$$-\frac{\Omega_0}{\epsilon}\left[\int_0^{r_*} dr'\, G(r') - \int_0^{r_0} dr'\, G(r')\right] - \left(1 + \frac{r_0}{R_0}\right) U(r_0, 0) = 0.$$

If the radial displacement is small (as will be checked a posteriori), the left-hand side can be expanded around r_0, with the result

$$-\frac{\Omega_0}{\epsilon}(r_* - r_0)G(r_0) = \left(1 + \frac{r_0}{R_0}\right)U(r_0, 0),$$

or

$$r_* - r_0 \simeq -\frac{\epsilon}{\Omega_0}\frac{U(r_0, 0)}{G(r_0)}. \tag{6.3}$$

At the turning point, the guiding centre is reflected, and continues its motion back towards $\theta = 0$; but its radial coordinate continues to increase, because of the inhomogeneity drift (see the discussion in section 9.1). It therefore reaches the point $\theta = 0$ with a parallel velocity of opposite sign and at a value r_{**} of the radial coordinate, determined by

$$-\frac{\Omega_0}{\epsilon}\int_0^{r_{**}} dr'\, G(r') - \left(1 + \frac{r_{**}}{R_0}\right)U(r_{**}, 0) = L. \tag{6.4}$$

Subtracting again this equation from (6.1) and expanding the functions through first order in $r_{**} - r_0$, we find

$$\Delta r \equiv r_{**} - r_0 \simeq -2\frac{\epsilon}{\Omega_0}\frac{U(r_0, 0)}{G(r_0)}. \tag{6.5}$$

This is the maximum displacement of the guiding centre away from its initial magnetic surface. Indeed, as it pursues its motion in the lower half-plane, it follows an orbit which is symmetrical to its upper part. This follows immediately from the evenness of $U(r, \theta)$ and thus of $L(r, \theta)$ as functions of θ. The physical origin of this property was discussed in section 9.1.

We now note that the displacement Δr may be positive or negative, the sign being fixed by the sign σ of the initial parallel velocity and by the sign of the charge of the particle (through Ω_0). Thus

$$\Delta r > 0 \begin{cases} \text{for electrons with} & U_0 > 0, \\ \text{for ions with} & U_0 < 0, \end{cases}$$

$$\Delta r < 0 \begin{cases} \text{for electrons with} & U_0 < 0, \\ \text{for ions with} & U_0 > 0. \end{cases} \tag{6.6}$$

Next, we note that, combining eqs. (5.10) and (5.6), the initial parallel velocity may be written in the form

$$U(r_0, 0) = \sigma\sqrt{\frac{2}{m}}\left(2MB_0\frac{r_0}{R_0}\right)^{1/2}\kappa. \tag{6.7}$$

Hence, the magnitude of the radial displacement, for given r_0, increases linearly with κ, and reaches its maximum value Δr_{MT} for a "*barely trapped*" guiding centre, for which $\kappa = 1 - \delta$ (where δ is an infinitesimal positive number):

$$\Delta r = \kappa\, \Delta r_{MT}, \tag{6.8}$$

with

$$\Delta r_{MT} = -\frac{2\epsilon}{\Omega_0 G(r_0)}\sigma\left(\frac{4MB_0}{m}\frac{r_0}{R_0}\right)^{1/2}. \tag{6.9}$$

An estimate of the *relative value of the maximum radial displacement of a barely trapped guiding centre* can be obtained as follows. Using (5.2), we have

$$\frac{|\Delta r_{MT}|}{r_0} = 2^{3/2}\frac{\epsilon}{(r_0/R_0)^{1/2}}\frac{(2MB_0/m)^{1/2}}{(r_0/q_0)\Omega_0}.$$

But, for a trapped guiding centre,

$$MB_0 \simeq \mathscr{E} + O(\eta) \simeq \frac{m}{2}V_T^2,$$

where V_T is the thermal velocity, hence,

$$\frac{\epsilon}{\Omega_0}\left(\frac{2MB_0}{m}\right)^{1/2} \simeq \frac{\epsilon}{\Omega_0}V_T = \rho_L,$$

where ρ_L is the thermal Larmor radius. We thus have

$$\frac{|\Delta r_{MT}|}{r_0} \simeq 2^{3/2}\frac{1}{(r_0/R_0)^{1/2}}\frac{\rho_L}{r_0/q_0}. \tag{6.10}$$

As the ratio of the Larmor radius to the macroscopic length (r_0/q_0) is of order

ϵ in the drift approximation, and as $(r_0/R_0) \sim \eta \sim \epsilon$ in the standard model, we have

$$\frac{|\Delta r_{MT}|}{r_0} \sim \epsilon^{1/2}. \tag{6.11}$$

This result will be of importance in transport theory.

B. Passing orbits

Consider a guiding centre starting at the point $r = r_0$, $\theta = 0$. At this point we can write again eq. (6.1). But now the constants of the motion $\mathscr{E}$, M, L have values corresponding to $\kappa > 1$. Hence, the moving point reaches $\theta = \pi$, at a radial distance r_+, without reflection (fig. 6.2). Afterwards, it continues its motion in the lower half-plane along a trajectory symmetrical to its upper half. The sign of the parallel velocity is now maintained during the whole motion. It is thus clear that a passing particle suffers no radial displacement at $\theta = 0$ (contrary to the trapped particles). Its maximum radial displacement occurs on the *inner side* of the torus, at $\theta = \pi$. The magnitude of this displacement can be obtained as in the previous case, by noting that, for $\theta = \pi$,

$$-\frac{\Omega_0}{\epsilon}\int_0^{r_+} dr'\, G(r') + \left(1 + \frac{r_+}{R_0}\right) U(r_+, \pi) = L. \tag{6.12}$$

Subtracting this equation from (6.1) and expanding the functions of r_+ around r_0, we obtain

$$r_+ - r_0 \simeq \frac{\epsilon}{\Omega_0 G(r_0)} \sigma \left(\frac{4MB_0}{m}\frac{r_0}{R_0}\right)^{1/2} \left[(\kappa^2 - 1)^{1/2} - \kappa\right],$$

or, using (6.7),

$$\Delta r \equiv r_+ - r_0 = -\frac{\epsilon}{\Omega_0 G(r_0)} U(r_0, 0)\left[1 - \frac{(\kappa^2 - 1)^{1/2}}{\kappa}\right]. \tag{6.13}$$

The sign of this displacement is the same as for the trapped particles (6.5), but its magnitude is a *decreasing function of* κ, tending to zero for $\kappa \gg 1$. Its maximum value is obtained for a "*barely passing*" guiding centre, with $\kappa = 1 + \delta$ (δ being an infinitesimal positive number):

$$\Delta r_{MP} = -\frac{\epsilon}{\Omega_0 G(r_0)} \sigma \left(\frac{4MB_0}{m}\frac{r_0}{R_0}\right)^{1/2}. \tag{6.14}$$

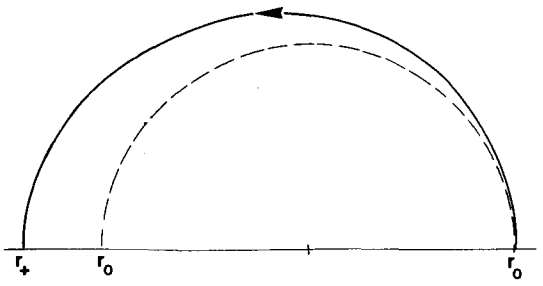

Fig. 6.2 A typical passing half-orbit

Comparing this result with (6.9), we see that the maximum radial displacement (at $\theta = \pi$) for a barely passing guiding centre equals exactly *one half* of the maximum radial displacement (at $\theta = 0$) of a barely trapped guiding centre:

$$|\Delta r_{MP}| = \tfrac{1}{2}|\Delta r_{MT}|. \tag{6.15}$$

The maximum radial displacement is pictured in fig. 6.3 for the whole range of the trapping parameter κ. The discontinuity at $\kappa = 1$ vividly represents the abrupt change of topology upon crossing the separatrix.

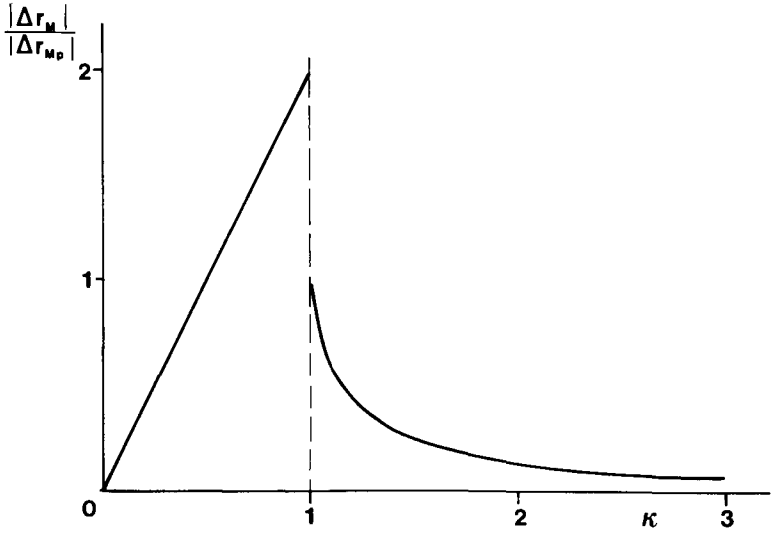

Fig. 6 3 Variation of the maximum radial displacement as a function of the trapping parameter.

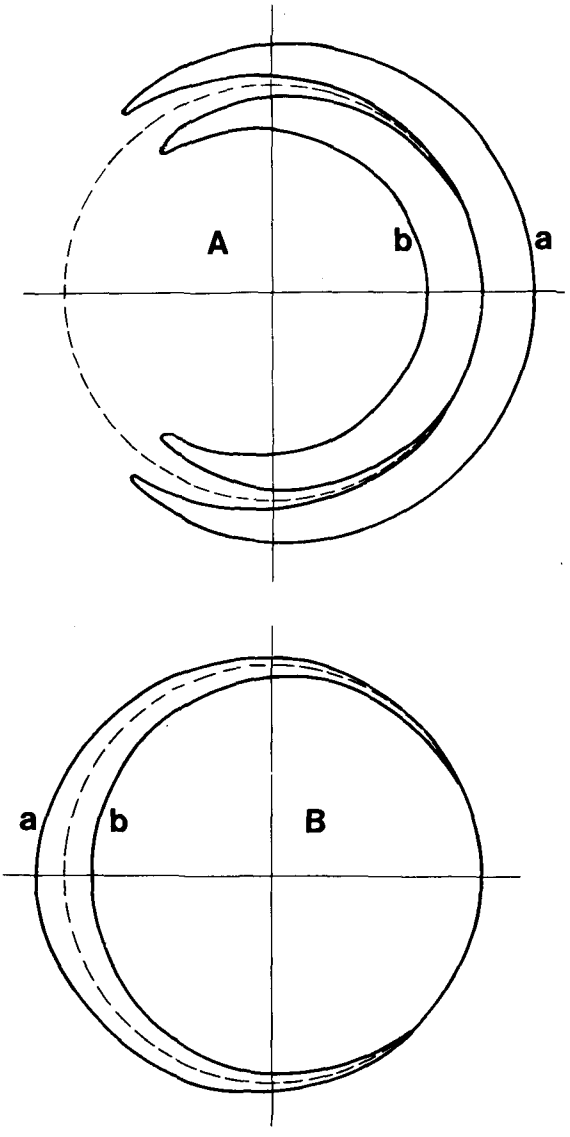

Fig 6 4. Typical guiding centre orbits in the $r-\theta$ plane. For $U > 0$· Aa and Ba are electron orbits, Ab and Bb are ion orbits. Aa, Ab are trapped orbits, Ba, Bb are passing orbits.

Finally, we show in fig. 6.4 several typical guiding centre orbits, both trapped and passing. We have pictured in both cases an example of an electron orbit and of an ion orbit, with radial displacements of opposite sign.

The shape of the trapped orbits clearly suggests the name of "*banana orbits*" which has entered the technical language since 1970.

9.7. Solution of the equations of motion

As a last stage of our study of the motion of charged particles in a toroidal geometry, we need to determine the coordinates $r(t)$, $\theta(t)$, $\zeta(t)$ as functions of time. This amounts to an integration of the set of equations of motion (3.6–3.8), a difficult problem! But, if we make again the approximation of section 9.5, i.e. we neglect the inhomogeneity drift, and thus the radial displacement, the integration of eq. (3.7) can be done analytically. Indeed, treating r as a constant, this equation reduces to (5.4), which can be combined with (5.7) to yield

$$\dot{\theta} = \sqrt{2}\, A\, (2\kappa^2 - 1 + \cos\theta)^{1/2}, \tag{7.1}$$

where we used the abbreviation

$$A = \sigma \frac{G(r_0)}{r_0} \left(\frac{MB_0}{m} \frac{r_0}{R_0} \right)^{1/2}. \tag{7.2}$$

Using an elementary trigonometrical identity, we rewrite the equation in the form

$$\dot{\theta} = 2A \left[\kappa^2 - \sin^2(\theta/2) \right]^{1/2}. \tag{7.3}$$

Assuming the initial value $\theta = 0$ for $t = 0$, this equation is immediately integrated:

$$\int_0^{\theta/2} \mathrm{d}\alpha \, \frac{1}{\kappa(1 - \kappa^{-2} \sin^2\alpha)^{1/2}} = At. \tag{7.4}$$

This result is conveniently expressed in terms of an *elliptic integral of the first kind*, $F(\varphi;\, k)$ of argument φ and modulus k *, defined as follows

$$F(\varphi;\, k) = \int_0^{\varphi} \mathrm{d}\alpha \, \frac{1}{(1 - k^2 \sin^2\alpha)^{1/2}}. \tag{7.5}$$

* The reader is warned against the fact that, in many tables of elliptic functions, the "parameter" $m \equiv k^2$ is used instead of the modulus k.

The properties of the elliptic integrals and of the related functions are found in many textbooks on the theory of functions, see for instance Abramowitz and Stegun (1965), Gradshtein and Ryzhik (1980), Erdelyi et al. (1953), Lavrentiev and Shabat (1972). Equation (7.4) thus reads

$$At = \frac{1}{\kappa} F\left(\frac{\theta}{2}; \frac{1}{\kappa}\right). \tag{7.6}$$

This is the solution of eq. (7.1) in implicit form. We now discuss separately the two types of motion.

A. Trapped particles

In this case, the modulus $\kappa^{-1} > 1$. In order to discuss the properties of the elliptic integral, we must use the important relation between an elliptic integral of modulus k and the corresponding function of modulus $1/k$:

$$F(\varphi_1; k_1) = k F(\varphi; k), \tag{7.7}$$

where:

$$k_1 = 1/k, \qquad \sin \varphi_1 = k \sin \varphi, \quad k \leqslant 1. \tag{7.8}$$

Hence, for trapped particles, we rewrite (7.6) in the form

$$At = F\left(\sin^{-1}[(1/\kappa) \sin(\theta/2)]; \kappa\right), \quad \kappa \leqslant 1. \tag{7.9}$$

A different form ("Legendre's form") of the elliptic integral of the first kind is

$$F(\varphi; \kappa) = \int_0^{\sin \varphi} du\left[(1-u^2)(1-\kappa^2 u^2)\right]^{-1/2} \equiv \alpha. \tag{7.10}$$

The inversion of this function is, by definition, the *Jacobian elliptic function* $\mathrm{sn}(\alpha; \kappa)$. Thus, in our case (7.9), we express the angle θ as a function of time through

$$\sin(\theta/2) = \kappa \, \mathrm{sn}(At; \kappa), \quad \kappa \leqslant 1. \tag{7.11}$$

It is well known that the Jacobian function $\mathrm{sn}(\alpha;\ \kappa)$ is a generalization of the trigonometric function $\sin(\alpha)$, to which it reduces for $\kappa \to 0$. It is a periodic function of α, the period and shape of which depends on the value of κ.

The *period* of the function $\mathrm{sn}(\alpha;\ \kappa)$ is expressed in terms of the complete elliptic integral $K(\kappa)$:

$$K(\kappa) = \int_0^1 \mathrm{d}u\left[(1-u^2)(1-\kappa^2 u^2)\right]^{-1/2}. \tag{7.12}$$

Hence, the period of the guiding centre motion is given by

$$T = \frac{4}{A}K(\kappa) = \frac{4}{(r_0/R_0)^{1/2}}\left(\frac{m}{2MB_0}\right)^{1/2}\frac{r_0}{G(r_0)}K(\kappa). \tag{7.13}$$

A peculiar property of the complete elliptic integral is that it tends to infinity as $\kappa \to 1$. We can therefore describe the motion qualitatively as follows.

For $\kappa = 0$, $\mathrm{sn}(At;\ 0) = \sin(At)$ but, because of the factor κ in (7.11), we have simply

$$\theta = 0, \quad \forall t, \quad \kappa = 0. \tag{7.14}$$

The particle does not move at all: its orbit reduces to a point. This is consistent with the phase space picture of fig. 5.2.

As κ increases from 0 towards 1, the graph of function $\theta(t)$ is periodic, with an increasing period and a more and more flattened shape.

When $\kappa = 1$, the Jacobian function $\mathrm{sn}(At;\ 1)$ can be expressed in terms of hyperbolic functions, and (7.11) reduces to

$$\sin(\theta/2) = \tanh At, \quad \kappa = 1. \tag{7.15}$$

Thus, the motion of a guiding centre along the separatrix is no longer periodic. The angle increases from 0 to π, but the latter value is only reached for $t = \infty$.

B. Passing particles

For $\kappa > 1$, we use the initial form of solution (7.6), which is inverted in the form

$$\sin(\theta/2) = \mathrm{sn}(\kappa At;\ 1/\kappa), \quad \kappa > 1. \tag{7.16}$$

But now, θ is no longer a periodic function of t. Indeed, when $\mathrm{sn}(\kappa At;\ 1/\kappa)$

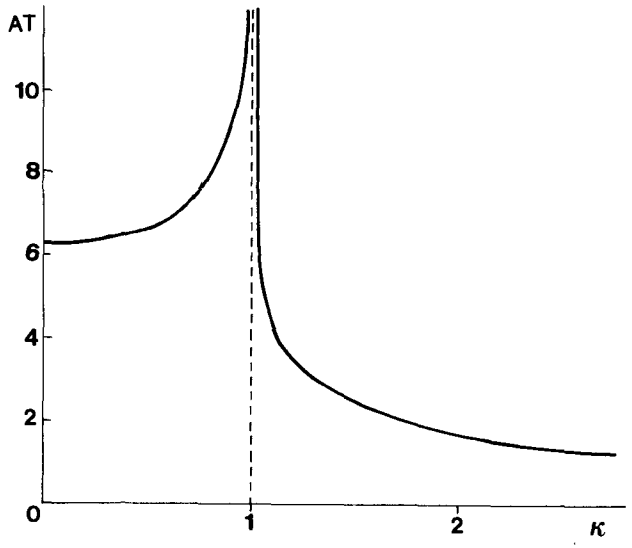

Fig. 7.1. Period of trapped motion and quasi-period of passing motion.

reaches 1, θ reaches π; as t increases beyond that point, $\mathrm{sn}(\kappa At; 1/\kappa)$ decreases towards 0, but the corresponding angle θ increases from π to 2π. Thus, the angle $\theta(t)$ is a monotonically increasing function of t along a passing trajectory.

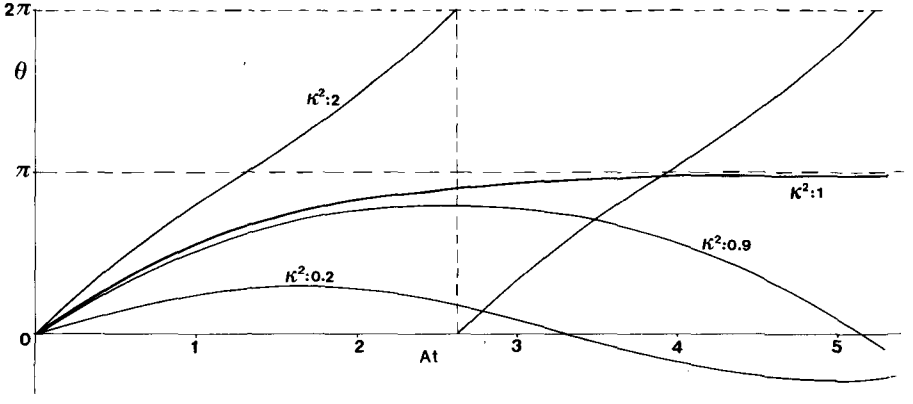

Fig. 7.2. The poloidal angle θ as a function of time.

One may, however, still define a *quasi-period* as the time necessary for the guiding centre to complete a turn in θ from 0 to 2π. This time corresponds to half a period of the Jacobian function, thus

$$T = \frac{2}{A\kappa} K\left(\frac{1}{\kappa}\right), \quad \kappa > 1. \tag{7.17}$$

The quasi-period is a decreasing function of κ for passing particles. The period and quasi-period of the guiding centre motion is shown in fig. 7.1, both for trapped and for passing particles.

In fig. 7.2, the graphs of $\theta(t)$ are shown for several typical values of κ, in order to clearly visualize the discussions in the present section. For drawing such graphs, the tables of Jacobian elliptic functions of Milne-Thomson (1956) are invaluable.

C. Motion in the toroidal direction

Having reached this stage, we are able, in principle, to study the motion in the third, toroidal direction. We should substitute functions $\theta(t)$ [and $r(t)$!] into (3.8) and integrate the resulting function of time:

$$\dot{\zeta} = \frac{U(r(t), \theta(t))}{R_0 + r(t)\cos\theta(t)}. \tag{7.18}$$

Although the solution of this equation is obtained by a mere quadrature, the integral cannot be evaluated analytically. For this reason, we only give here a qualitative discussion.

There is no difficulty in picturing the motion of the passing particles. Their parallel velocity always keeps the same sign, therefore ζ (like θ) is a steadily increasing function of time. The three-dimensional trajectory of such a guiding centre is a "helix" wrapped around the toroidal magnetic surface (with small bulges on the inner side, due to the radial displacement).

The motion of the trapped particles is more complex. In first approximation, comparing (3.8) and (5.4), we see that whenever there is a turning point in the θ-direction [$U(r, \theta) = 0$], there is also a turning point in the ζ-direction. Hence, when the guiding centre comes back to $\theta = 0$, it also comes back to $\zeta = 0$. Thus, not only the projection, but also the three-dimensional orbit has the characteristic banana shape.

If, however, the problem is treated more accurately, by using the complete equation (3.7) rather than (5.4), we see that the right-hand sides of θ and of ζ are no longer proportional to each other. Because of the additional (small) inhomogeneity drift, the turning points in θ and in ζ no longer coincide.

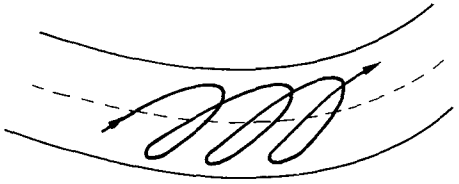

Fig 7 3 Toroidal precession of a banana orbit.

Although the exact treatment of this problem is not simple, one can visualize the consequence of this fact. When the guiding centre has completed a banana orbit in the θ-direction, its toroidal coordinate no longer comes back precisely to zero. Hence, the bouncing motion in the poloidal direction is accompanied by a slow *precession* of the banana in the toroidal direction (fig. 7.3). This precession is due to the inhomogeneity drift.

9.8. Effects of electric drift and of non-axisymmetry

After the detailed discussions of the simple case of the previous sections, we now briefly consider several generalizations.

We first determine the modifications of the results for the situations in which a *radial electric field* is present. *

We must now study the complete equation (3.7), together with (3.9), for $\Phi(r) \neq 0$. We note that its exact solution is still provided by (5.1), but the analysis of its properties must be slightly modified.

In order to derive the *trapping criterion*, we follow the argument of section 9.5, and neglect the inhomogeneity drift. We thus write the differential equation for θ (assuming again $r = r_0 = \text{const.}$) in the form

$$r_0 \dot{\theta} = G(r_0)\, \tilde{U}(r_0, \theta), \tag{8.1}$$

where we define

$$\tilde{U}(r, \theta) = U(r, \theta) + \epsilon\, G^{-1}(r)\, V_{\mathrm{E}}(r), \tag{8.2}$$

with

$$V_{\mathrm{E}}(r) = \frac{c}{B_0} \frac{\mathrm{d}\Phi(r)}{\mathrm{d}r}. \tag{8.3}$$

* The effect of an external induced electric field $E^{(A)}$ lying in the magnetic surface will be discussed later, in connection with the "Ware effect", see chapter 15.

The reader should be careful in noting that *the last term in* (8.2) *is not necessarily small* compared to the others, in spite of the factor ϵ. Indeed, $G(r)$ is of order η (see eq. 5.2), hence of order ϵ, by (3.4). From (3.9), the following form of $\tilde{U}(r, \theta)$ can be derived:

$$\tilde{U}(r, \theta) = \sigma\left(U^2(r, 0) + \frac{r}{R_0}\frac{2MB_0}{m}(\cos\theta - 1)\right)^{1/2} + \epsilon\frac{V_E}{G(r)}. \tag{8.4}$$

The discussion in section 9.5 is not directly applicable, because $\tilde{U}(r, \theta)$ is not of the standard ("pendulum-like") form (5.7). It is, however, not difficult to cast it in that form, through order ϵ. Indeed, given three functions of r: $a(r)$, $\alpha(r)$, $b(r)$, it is always possible to determine a function $A(r, \theta)$, such that

$$[a + \eta\alpha(\cos\theta - 1)]^{1/2} + b = [A(\theta) + \eta\alpha(\cos\theta - 1)]^{1/2}. \tag{8.5}$$

By squaring both sides, we find

$$\begin{aligned} A &= a + b^2 + 2ba^{1/2}[1 + \eta(\alpha/a)(\cos\theta - 1)]^{1/2} \\ &= (a^{1/2} + b)^2 + \eta\alpha ba^{1/2}(\cos\theta - 1) + O(\eta^2). \end{aligned} \tag{8.6}$$

Applying this result to (8.4), we find the following representation:

$$\tilde{U}(r, \theta) = \tilde{\sigma}\left(\tilde{U}^2(r, 0) + \frac{r}{R_0}\frac{2\tilde{M}(r)B_0}{m}(\cos\theta - 1)\right)^{1/2}, \tag{8.7}$$

where $\tilde{\sigma}$ is the sign of $\tilde{U}(r, \theta)$, and

$$\tilde{M}(r) = M\left(1 + \epsilon\frac{V_E(r)}{G(r)\,U(r, 0)}\right). \tag{8.8}$$

$\tilde{U}(r, \theta)$ is now of the standard form, and the argument of section 9.5 is trivially extended. We define a new *trapping parameter* κ_F by *

$$\kappa_F^2 = \frac{(m/2)\tilde{U}^2(r_0, 0)}{2(r_0/R_0)\tilde{M}B_0}. \tag{8.9}$$

* This result differs from the result of Berk and Galeev (1967) and of Galeev and Sagdeev (1968). These authors missed the correction (8.8) of the magnetic moment, which is not small (as pointed out above)

We then write

$$\tilde{U}(r_0, \theta) = \tilde{\sigma}\left(\frac{r_0}{R_0}\frac{2\tilde{M}B_0}{m}\right)^{1/2}\left(2\kappa_E^2 - 1 + \cos\theta\right)^{1/2} \tag{8.10}$$

and find the new *trapping criterion*:

$$\kappa_E^2 < 1: \quad \text{trapped},$$

$$\kappa_E^2 > 1: \quad \text{passing}. \tag{8.11}$$

The analysis of section 9.6 can also be easily generalized. Consider, for instance, the determination of the coordinates of the turning point of a banana orbit. For $r = r_0$, $\theta = 0$, we have

$$-\frac{\Omega_0}{\epsilon}\int_0^{r_0} dr'\, G(r') + \left(1 + \frac{r_0}{R_0}\right)U(r_0, 0) = L. \tag{8.12}$$

But at the turning point ($r = r_*$, $\theta = \theta_*$) we no longer have relation (6.2), because $U(r_*, \theta_*) \neq 0$; thus

$$-\frac{\Omega_0}{\epsilon}\int_0^{r_*} dr'\, G(r') + \left(1 + \frac{r_*}{R_0}\right)U(r_*, \theta_*) = L. \tag{8.13}$$

At the turning point, we have

$$\tilde{U}(r_*, \theta_*) = 0.$$

Hence, using (8.2) and neglecting terms of order ϵ,

$$-\frac{\Omega_0}{\epsilon}\int_0^{r_*} dr'\, G(r') - \epsilon\frac{V_E(r_*)}{G(r_*)} = L. \tag{8.14}$$

Subtracting (8.12) from (8.14) and proceeding as in section 9.6, we easily find

$$r_* - r_0 \cong -\frac{\epsilon}{\Omega_0}\frac{\tilde{U}(r_0, 0)}{G(r_0)}. \tag{8.15}$$

All the other results of sections 9.6 and 9.7 can be generalized with the following simple conclusion. *The formulae of sections* 9.6 *and* 9.7 *remain valid in the presence of a radial electric field, provided one makes the following replacements*:

$$\kappa \to \kappa_E, \qquad U(r, \theta) \to \tilde{U}(r, \theta), \qquad M \to \tilde{M}(r). \tag{8.16}$$

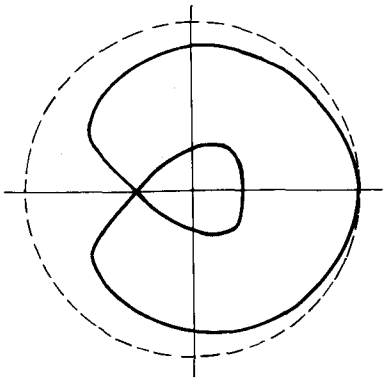

Fig. 8.1. A pinch orbit.

Further generalizations can be considered, but their study becomes more and more difficult and must rely on numerical computations. An interesting analytical study can still be performed for systems which possess the axisymmetric tokamak geometry, with $|B_\zeta| \gg |B_\parallel|$, but for which one does not assume the precise functional forms of the standard model. Mercier et al. (1981) obtained classification criteria for the trajectories in this more general situation. The only new qualitative feature is the possible appearance of a different type of trapped orbit, called a "*pinch orbit*" (fig. 8.1), which has a double point. It results from the deformation of an ordinary banana orbit. Such trajectories are, however, rather exceptional.

A very extensive numerical study and classification of the guiding centre orbits in realistic tokamak geometries, in which the magnetic field is determined self-consistently by solving the Grad–Shafranov equation, can be found in the article by Rome and Peng (1979) (see also Rome et al. 1976).

When we go over to *non-axisymmetric toroidal geometries*, the complication of the problem of particle motions increases by an order of magnitude. To the periodicity of the magnetic field intensity as a function of θ, studied in detail in the previous sections, an additional inhomogeneity is added in the ζ-direction in such devices as the stellarators or the bumpy tori. The magnetic field intensity, plotted along a field line, looks schematically like in fig. 8.2. Because of the presence of two periods, the exact analytic solution is no longer possible. One may, however, construct a qualitative picture as follows. According to the value of the energy and of the parallel velocity, there will again be two types of guiding centre trajectories: passing and trapped. Among the latter, one may now distinguish between *toroidally trapped* orbits which pass over several local wells, and *locally trapped* orbits which are trapped in a local minimum. The latter are the more important ones for the transport problem.

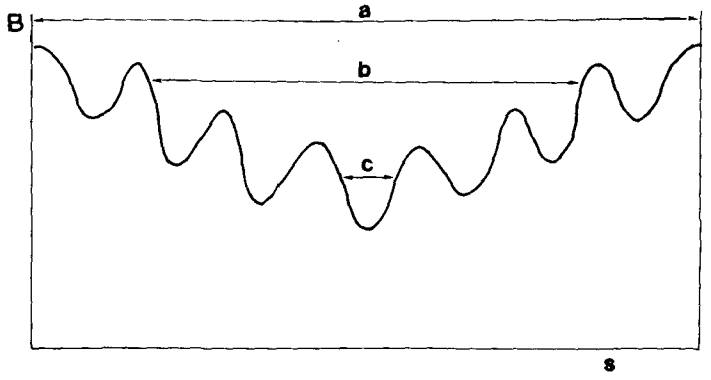

Fig. 8.2. Magnetic field intensity along a field line: (a) passing particles (b) toroidally trapped particles, (c) locally trapped particles.

The motion of the corresponding guiding centres can be visualized as a bouncing motion, analogous to the banana motion of the previous sections. But now, the banana orbit suffers a precession along the field line. This can lead to either one of two situations. The precessing guiding centre may go all the way around the torus (fig. 8.3a) or else, it may be reflected by the large scale inhomogeneity (fig. 8.3b), in which case we obtain a *super-banana orbit*. The latter exhibits a particularly large deviation from the initial magnetic surface, and therefore plays an important role in transport theory.

The first serious discussion of the motion of particles in non-axisymmetric toroidal geometries can be found in the work by Morozov and Soloviev

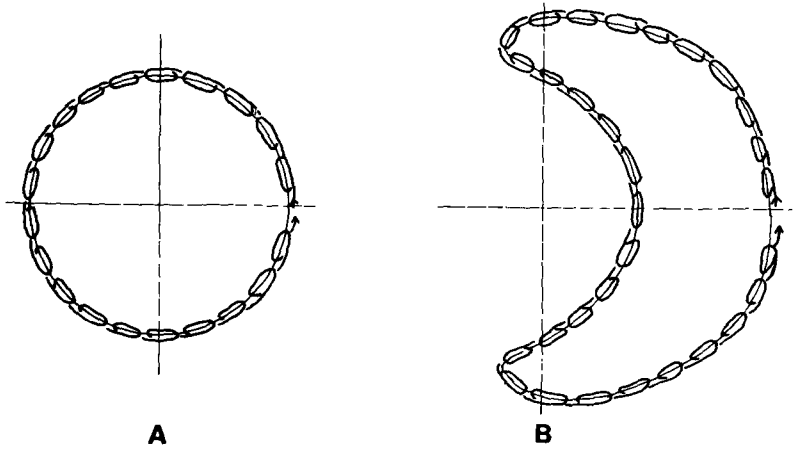

Fig. 8.3. Locally trapped guiding centre orbits: (a) "super-passing", (b) "super-banana".

(1960a, b). Later, a detailed analysis of all the types of motions, based both on analytical approximations and on numerical computations, was given by Gibson and Taylor (1967). A short, but clear account is found in the article by Galeev et al. (1969) [see also the review paper by Galeev and Sagdeev (1979)]. A rather detailed discussion is given by Kovrizhnykh (1970).

A possible approximate treatment of the problem suggests itself from a consideration of fig. 8.3. It appears that the relation of a super-banana orbit to the precessing banana motion of the guiding centre is analogous to the relation between the simple banana orbit and the Larmor gyration of the real particle. One may therefore consider the possibility of an *averaging procedure* over the "rapid" bounce time scale, in order to obtain equations of motion free from bounce oscillations. A first (rather crude) approach in this direction was given by Galeev and Sagdeev (1979). A quite elegant Hamiltonian treatment of the problem was given by Littlejohn (1982).

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10

Kinetic equation and local equilibrium of a magnetically confined plasma

10.1. Characteristic parameters

We now start the study of the kinetic equation for magnetically confined plasmas. In the situations considered hitherto, it was always assumed that the magnetic field was homogeneous (or very slowly varying) and (at least locally) straight. But in a magnetically confined plasma, the field is necessarily inhomogeneous and curved, as was amply discussed in chapter 8. It will be seen in forthcoming chapters that the global topological characteristics of the magnetic field have a pronounced influence on the transport phenomena in these systems. The classical transport theory needs a thorough generalization in order to cover these situations.

As before, the discussion must start with a dimensional consideration of the various length scales and time scales present in the problem, in order to define various regimes within which various approximations are valid. In a confined plasma we can define, just as before, the three basic characteristic lengths (see sections **2.6** and **4.1**):

(a) λ_D the *Debye length*, measuring the effective range of the correlations between the particles;

(b) λ_{mfp}, the *mean free path*, measuring the average distance travelled by a particle between two successive collisions;

(c) L_H, the *hydrodynamic length*, related to the spatial variation of the macroscopic quantities and of the magnetic and electric fields. Note that in a hot, toroidally confined plasma, L_H may also be taken as a measure of some geometrical characteristics of the particle trajectories (e.g. the average size of the trapped particle "banana" orbits; see chapter 9): these quantities may reach macroscopic dimensions under some circumstances.

Associated with these are three characteristic times:

(A) τ_c, the *correlation time*, which is of the order of the inverse plasma frequency;

(B) τ_R, the *relaxation time*, or the inverse collision frequency;

(C) τ_H, the *hydrodynamical time*.

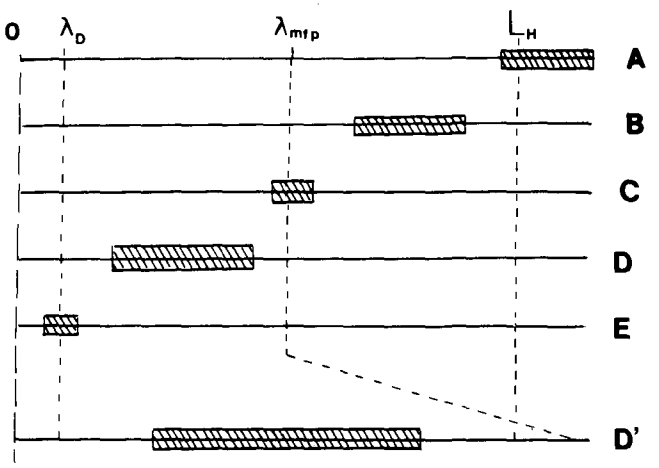

Fig 1.1 Various transport regimes. The hatched rectangle represents the range of ρ_L. A: Weak magnetic field; B, C, D: drift regime + Landau collisions; E: drift regime + modified Landau collisions; D′: long mean free path regime.

We have previously defined a *collision-dominated plasma regime* as a range of temperatures and densities leading to the inequalities

$$\lambda_D \ll \lambda_{mfp} \ll L_H, \tag{1.1}$$

or, equivalently,

$$\tau_c \ll \tau_R \ll \tau_H. \tag{1.2}$$

The new feature in a magnetized plasma is the essential role played by the quantities measuring the magnetic field, i.e.

(d) ρ_L, the *Larmor radius*, and

(D) Ω, the *Larmor frequency*.

The new length may be inserted anywhere between the three previous ones, according to the strength of the magnetic field. The various characteristic situations are depicted schematically in fig. 1.1.

In case (A), the magnetic field is so weak that the Larmor radius is larger than, or of the order of L_H:

$$\frac{\rho_L}{L_H} > 1, \quad \text{or} \quad \frac{\rho_L}{L_H} \approx 1. \tag{1.3}$$

In this case, the derivation of the Landau collision term given in section 2.6 is valid, because

$$\frac{\rho_L}{\lambda_D} \gg 1. \tag{1.4}$$

On the other hand, the individual particle motion cannot be described by the drift approximation, because (1.3) violates the validity condition of the latter, (1.5.26). Hence, the system is described by a Vlassov–Landau equation (using particle variables), in which the magnetic field appears as a small perturbation. The treatment of the transport given in chapter 5 is then valid as it stands for this case.

In case (B), the Larmor radius is such that

$$\lambda_{\mathrm{mfp}} \ll \rho_{\mathrm{L}} \ll L_{\mathrm{H}}. \tag{1.5}$$

In case (C), we have

$$\lambda_{\mathrm{mfp}} \approx \rho_{\mathrm{L}}. \tag{1.6}$$

In case (D), we have

$$\lambda_{\mathrm{D}} \ll \rho_{\mathrm{L}} \ll \lambda_{\mathrm{mfp}}. \tag{1.7}$$

In all three cases, we have

$$\frac{\rho_{\mathrm{L}}}{\lambda_{\mathrm{D}}} \gg 1, \tag{1.8}$$

which ensures the validity of the Landau collision term. Moreover, we also have, in all three cases,

$$\epsilon \equiv \frac{\rho_{\mathrm{L}}}{L_{\mathrm{H}}} \ll 1. \tag{1.9}$$

It follows that the motion of the particles under the action of the magnetic field can be validly treated in the *drift approximation*.

Finally, we consider case (E), in which the Larmor radius is of the order of the Debye length,

$$\rho_{\mathrm{L}} \approx \lambda_{\mathrm{D}}. \tag{1.10}$$

In this case, the final part of the argument of section 2.6 [the part following eq. (2.6.22)] breaks down. The collision term no longer has the simple Landau form (2.6.24), but must be evaluated more completely by using the propagators in the presence of the magnetic field. We are faced with an interference between collisions and free motion. As a result, the collision term depends on the magnetic field, unlike the Landau collision term.

We want to stress the fact that none of the situations described above are particularly "exotic": they can all be realized in present-day fusion devices. Hence, they all deserve careful study. In the present chapter, we shall be interested in the regime defined by eq. (1.7). It describes a *strongly magnetized plasma, in which the Larmor radius is much smaller than the mean free path.* This regime is of special interest in tokamaks and other magnetic fusion devices.

Let us stress that we have not particularly insisted on the position of the hydrodynamic length in relation to the other characteristic lengths of the problem. Indeed, the relation does not play an important role in the arguments developed in the present chapter. In other words, the results derived below apply equally well to two distinct situations. The first one may be called the *short mean free path regime* and is defined by diagram D in fig. 1.1, or by

$$\lambda_D \ll \rho_L \ll \lambda_{mfp} \ll L_H. \tag{1.11}$$

Because of the relative ordering of the three lengths λ_D, λ_{mfp} and L_H this situation is very similar to the one treated in chapters 4 and 5; the toroidal magnetic field produces, however, a number of quite important effects on the transport phenomena. These will be studied in forthcoming chapters.

Equation (1.7) is also compatible with a *long mean free path regime* depicted in diagram D′ in fig. 1.1:

$$\lambda_D \ll \rho_L \ll L_H < \lambda_{mfp}. \tag{1.12}$$

This regime, which can be realized in large, high-temperature tokamaks, has peculiarities of its own, due to the presence of the toroidal field. These features make it very different from the usual Knudsen regime of ordinary gases. In the long mean free path regime, the theory of transport needs a thorough generalization, as will appear in forthcoming chapters.

In the current literature the two regimes (1.11) and (1.12) are often called "collisional" and collisionless" regimes. We believe that these names are very misleading, and should be banned from the plasma literature. Indeed, as will appear clearly in forthcoming chapters, the collisions play an important, though different, role in *both* regimes. On the other hand, it has become customary to study these two regimes together under the name of *neoclassical theory of transport.*

10.2. Kinetic equation and natural guiding centre variables

We now consider a strongly magnetized plasma, defined by eq. (1.7), and derive the appropriate form of the kinetic equation. The general form of the

latter is given by (2.4.15), a form which can be made explicit by using any set of phase space variables. An exact form is obtained by using the particle variables $(\boldsymbol{q}, \boldsymbol{v})$ of table 1.4.2:

$$\partial_t f^\alpha(\boldsymbol{q}, \boldsymbol{v}; t) = -\boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{q}} f^\alpha(\boldsymbol{q}, \boldsymbol{v}; t) - \frac{e_\alpha}{m_\alpha}\left(\boldsymbol{E}(\boldsymbol{q}; t) + \frac{1}{c}\boldsymbol{v}\wedge\boldsymbol{B}(\boldsymbol{q})\right)$$

$$\cdot\frac{\partial}{\partial \boldsymbol{v}} f^\alpha(\boldsymbol{q}, \boldsymbol{v}; t) + \mathscr{K}^\alpha\{f, f\}, \tag{2.1}$$

where the collision term has been given an abbreviated notation standing for:

$$\mathscr{K}^\alpha\{f, f\} \equiv \sum_{\beta=e,i} \mathscr{K}^{\alpha\beta}\{f^\alpha, f^\beta\}. \tag{2.2}$$

The Landau collision term (2.6.24) will be used here. $\boldsymbol{E}$ and $\boldsymbol{B}$ contain both the external fields and the self-consistent Vlassov fields; it is however assumed that the external magnetic field largely dominates the self-consistent fields.

The peculiarity of the strongly magnetized plasma is not immediately apparent in the present variables. In order to exhibit it explicitly, we go over to the scaled particle variables introduced in section 1.6 (table 1.6.1) and write explicitly the drift parameter ϵ appearing in (1.9): *

$$\begin{aligned}
&\partial_t f^\alpha(\boldsymbol{q}, v_\parallel, v_\perp, \varphi; t) \\
&= \Bigg[-\left[v_\parallel \boldsymbol{b} + v_\perp \boldsymbol{n}_1(\varphi)\right]\cdot\frac{\partial}{\partial \boldsymbol{q}} + \left(-v_\perp \boldsymbol{n}_2(\varphi)\cdot\boldsymbol{D}(\varphi) - \frac{e_\alpha}{m_\alpha}\boldsymbol{b}\cdot\boldsymbol{E}\right)\frac{\partial}{\partial v_\parallel} \\
&\quad + \left(v_\parallel \boldsymbol{n}_2(\varphi)\cdot\boldsymbol{D}(\varphi) - \frac{e_\alpha}{m_\alpha}\boldsymbol{n}_1(\varphi)\cdot\boldsymbol{E}\right)\frac{\partial}{\partial v_\perp} \\
&\quad + \Bigg(-\frac{1}{\epsilon}\Omega_\alpha + \boldsymbol{b}\cdot\boldsymbol{D}(\varphi) - \frac{v_\parallel}{v_\perp}\boldsymbol{n}_1(\varphi)\cdot\boldsymbol{D}(\varphi) \\
&\qquad + \frac{e_\alpha}{m_\alpha v_\perp}\boldsymbol{n}_2(\varphi)\cdot\boldsymbol{E}\Bigg)\frac{\partial}{\partial\varphi}\Bigg] f^\alpha + \mathscr{K}^\alpha\{f, f\}.
\end{aligned} \tag{2.3}$$

* In writing this equation, whose purpose is illustrative, we disregard the effect of the slow time-dependence of the electromagnetic fields. This feature will be introduced explicitly in eq. (2.4).

The characteristic feature appearing here is the "odious" oscillatory dependence on the gyrophase variable φ of most coefficients in this partial differential equation. As was discussed in detail in chapter 1, this feature introduces a very serious complication. However, when the *drift approximation* is applicable, i.e. when $\epsilon \ll 1$, the equation can be simplified. Traditionally, eq. (2.3) was treated by an extension of the method of the average, described in section **1.6**. This treatment leads to very heavy algebra and is not physically very transparent. Typical examples of work in this direction are the papers by Hastie et al. (1967) and by Frieman (1970).

Since the development of the modern Hamiltonian methods described in section **1.7–1.9**, the derivation of the basic equations of the neoclassical theory can be done much more easily and clearly.

We recall the results of chapter 1, where we proved the existence of a pseudo-canonical transformation such that the Lie brackets of the new variables are independent of the (transformed) gyrophase ϕ. Hence, the "odious" gyrophase dependence of the coefficients in eq. (2.3) can be eliminated in a single stroke by going over to any one of the *Natural Guiding Centre (NGC) variables* defined in section **1.8** and **1.9**. This transformation was illustrated in detail in section **2.2** and **2.3** for the case of a set of independent charged particles. The procedure is exactly the same for eq. (2.3) because, except for $\mathscr{K}^\alpha$, all the terms on the right-hand side formally derive from a Hamiltonian. The only difference with the one-particle Liouvillian equation (**2.2.21**) is in the fact that the electric field $\boldsymbol{E}$ of (2.3) contains, in addition to a (time-dependent) external field, the self-consistent Vlassov field. Hence, the kinetic equation is coupled to the Poisson equation (see section **2.5**). This, however, does not affect its Hamiltonian form.

We thus choose a set of NGC variables and perform the pseudo-canonical transformation as shown in sections **2.2** and **2.3**. It is clear that the kinetic equation will take its simplest form when we use the variables $(\boldsymbol{Y}, \mathscr{E}, M, \phi)$ defined in table **1.8.3**. (Actually, in the presence of a slowly time-dependent electric field, we must use the extended Hamiltonian method of section **1.9**, followed by a reduction to the physical phase-space, as explained in section **2.3**.) The kinetic equation thus reduces to a form derived from (**2.3.22**):

$$\begin{aligned}
&\partial_t f^\alpha(\boldsymbol{Y}, \mathscr{E}, M, \phi; t) \\
&\quad = -(U_\alpha \boldsymbol{b} + \epsilon \boldsymbol{V}_{\mathrm{D}}^\alpha) \cdot \frac{\partial}{\partial \boldsymbol{Y}} f^\alpha - e_\alpha U_\alpha \boldsymbol{b} \cdot \boldsymbol{E}^{(A)} \frac{\partial}{\partial \mathscr{E}} f^\alpha \\
&\quad\quad - \left[\epsilon^{-1}\Omega_\alpha + U_\alpha \boldsymbol{b} \cdot \boldsymbol{R} - \tfrac{1}{2} U_\alpha \boldsymbol{b} \cdot (\nabla \wedge \boldsymbol{b})\right] \frac{\partial}{\partial \phi} f^\alpha + \mathscr{K}^\alpha\{f, f\}. \qquad (2.4)
\end{aligned}$$

We introduced here the notation $\boldsymbol{E}^{(A)}$ for the external time-dependent, "non-potential" part of the electric field,

$$\boldsymbol{E}^{(A)} = -c^{-1}\,\partial_t \boldsymbol{A}(\boldsymbol{Y}, t). \tag{2.5}$$

We recall definition (2.3.23) of the total drift velocity of the particles of species α:

$$\boldsymbol{V}_{\mathrm{D}}^{\alpha} = \frac{1}{m_\alpha \Omega_\alpha}\,\boldsymbol{b} \wedge \left[m_\alpha U_\alpha^2 (\boldsymbol{b}\cdot\nabla)\boldsymbol{b} + M\,\nabla B + e_\alpha \nabla \Phi \right]. \tag{2.6}$$

We also recall that the quantity U_α, representing the transformed parallel velocity, is a function of the independent variables $\boldsymbol{Y}$, $\mathscr{E}$ and M (not of ϕ!):

$$U_\alpha = \sigma \left[\frac{2}{m_\alpha} \left(\mathscr{E} - e_\alpha \Phi(\boldsymbol{Y}) - MB(\boldsymbol{Y}) \right) \right]^{1/2}. \tag{2.7}$$

Equation (2.4) is the *complete kinetic equation, expressed in natural guiding centre variables*. Note that, in contrast to (2.3), it is an *approximate* equation, valid only within the *drift approximation* $\epsilon \ll 1$: indeed, the coefficients are written only through first order in ϵ. The main achievement is, of course, the disappearance of the gyrophase dependence in the coefficients of the first three terms on the right-hand side of the equation.

We now arrive at the discussion of the *collision term* $\mathscr{K}^\alpha$, which has not been written explicitly. Clearly, this term must also be expressed as a function of the NGC variables. But the collision term does not have the simple Hamiltonian form of a Lie bracket, hence its transformation must be done "the hard way". In other words, the differential and integral operators involved in expression (2.6.24) of the collision term must be transformed to the new variables defined by (1.8.2)–(1.8.5).

The most important consequence of the non-Hamiltonian form of the collision term is that the averaging pseudo-canonical transformation does not suppress the dependence on the gyrophase ϕ in this term. Hence, the *coefficients of the collision term are still functions of the gyrophase, even in the new variables*.

For the general purpose of the present chapter we shall not need the explicit form of the collision term in NGC variables. This explicit form is quite complicated and constitutes one of the difficulties of the neoclassical theory. It will be discussed in chapter 11, along with the additional approximations needed in order to make the theory operational.

10.3. The multiple time-scale perturbation expansion

The problem of the kinetic theory of magnetically confined plasmas boils down to the solution of eq. (2.4). Quite naturally, in solving this equation, we should make an optimum use of the presence of the small parameter ϵ. In such problems, the method of solution which immediately comes to the mind is a *perturbation expansion*. We therefore seek a solution for f^α in the form

$$f^\alpha = f_0^\alpha + \epsilon f_1^\alpha + \epsilon^2 f_2^\alpha + \cdots . \tag{3.1}$$

From here on, one is tempted to start the standard perturbation theory. But one would quickly be stopped by a very common difficulty arising in many problems of statistical physics: the appearance of *secularities*. Indeed, some of the contributions to the correction terms may be proportional to positive powers of the time, such as t^n. It follows that, in spite of the smallness of the parameter, ϵ^n, for long enough times, these terms become very large and invalidate the classification of (3.1): the whole perturbation expansion breaks down. This fact is so much more troublesome that we are mostly interested in the long-time behaviour of the solution.

Several mathematical methods have been developed for curing this difficulty (see the excellent book by Nayfeh 1973). Among them, a particularly simple and clear one is the so-called *multiple time-scale perturbation expansion*. This method, originally used by Frieman (1963) and by Sandri (1963) in the context of kinetic theory, is very clearly explained in Davidson's (1972) book. The idea is based on the observation that, with the ordering (1.7), there exist widely separated time-scales in the system (a situation which is not new to us, see section 2.6). In the multiple time-scale method, this fact is exploited by artificially "extending" the unique time variable t into several variables t_{-1}, t_0, $t_1, \ldots$, defined by the differential equations

$$\frac{\mathrm{d}t_{-1}}{\mathrm{d}t} = \frac{1}{\epsilon}, \qquad \frac{\mathrm{d}t_0}{\mathrm{d}t} = 1, \qquad \frac{\mathrm{d}t_1}{\mathrm{d}t} = \epsilon. \tag{3.2}$$

In other words, any function of the time t is transformed into a function of many times, which are considered formally as *independent variables*:

$$g(t) \to g(t_{-1}, t_0, t_1, \ldots). \tag{3.3}$$

The time derivative of this function in the initial representation becomes

$$\frac{\partial g(t)}{\partial t} \to \frac{1}{\epsilon}\frac{\partial g(t_{-1}, t_0, t_1, \ldots)}{\partial t_{-1}} + \frac{\partial g(t_{-1}, t_0, t_1, \ldots)}{\partial t_0} + \epsilon \frac{\partial g(t_{-1}, t_0, t_1, \ldots)}{\partial t_1} + \cdots . \tag{3.4}$$

Obviously, the dependence on t_{-1} represents the fast evolution on the shortest time-scale (typically, the inverse Larmor frequency in our problem), the dependence on t_0 represents the slower evolution on a longer time-scale (e.g. the relaxation time-scale), etc. Clearly, the introduction of (3.4) into the perturbation expansion will modify the standard ordering, because it takes into account not only the size of the terms expressed by (3.1), but also their rate of change in time. The solutions $f_n^\alpha(t_{-1}, t_0, t_1, \ldots)$ obtained in this way go beyond the initial problem, because they depend on arbitrarily many times. At this point, this added flexibility is used by imposing *constraints* on the variables. These constraints are chosen in such a way as to eliminate the secular terms in each order in ϵ. Hence, a perturbation expansion of type (3.1) can be constructed, which is *uniformly valid* for all times. In other words, all the terms f_n^α remain bounded in time. Once the multiple time-scale solution has been obtained in this form to the desired degree of accuracy, we go back to the original time variable by the replacements

$$t_{-1} \to \frac{1}{\epsilon} t, \qquad t_0 \to t, \qquad t_1 \to \epsilon t, \ldots . \tag{3.5}$$

Further details on the general method can be found in the references quoted above. We now apply this method to our specific problem, i.e. eq. (2.4).

We first write expansion (3.1) more explicitly, in agreement with our previous discussion. Introducing the provisional abbreviation: $\eta \equiv (\boldsymbol{Y}, \mathscr{E}, M, \phi)$, we write

$$f^\alpha(\eta; t) = f_0^\alpha(\eta; t_{-1}, t_0, t_1, \ldots) + \epsilon f_1^\alpha(\eta; t_{-1}, t_0, t_1, \ldots)$$

$$+ \epsilon^2 f_2^\alpha(\eta; t_{-1}, t_0, t_1, \ldots) + \cdots . \tag{3.6}$$

In order to define the problem completely, we also need an assumption about the (slow) time-dependence of the electric and magnetic fields. Specifically, we assume, in agreement with (1.9.11),

$$\boldsymbol{A}(\boldsymbol{Y}, t) = \boldsymbol{A}(\boldsymbol{Y}, t_2). \tag{3.7}$$

Note that this assumption implies, by using (1.9.9) and (2.5), that the term $e_\alpha \boldsymbol{E}^{(A)}$ in (2.4) is of order ϵ.

We now introduce (3.6) and (3.7) into (2.4), taking account of (3.4), and collect the terms of the same order in ϵ. We thus obtain the following equations.

Order ϵ^{-1}:

$$\frac{\partial f_0^\alpha}{\partial t_{-1}} = -\Omega_\alpha \frac{\partial}{\partial \phi} f_0^\alpha. \tag{3.8}$$

Order ϵ^0:

$$\begin{aligned}
&\frac{\partial}{\partial t_{-1}} f_1^\alpha + \frac{\partial}{\partial t_0} f_0^\alpha + U_\alpha \boldsymbol{b} \cdot \frac{\partial}{\partial \boldsymbol{Y}} f_0^\alpha \\
&\quad + \left[U_\alpha \boldsymbol{b} \cdot \boldsymbol{R} - \tfrac{1}{2} U_\alpha \boldsymbol{b} \cdot (\nabla \wedge \boldsymbol{b}) \right] \frac{\partial}{\partial \phi} f_0^\alpha - \mathscr{K}_0^\alpha \{ f_0, f_0 \} \\
&\quad = -\Omega_\alpha \frac{\partial}{\partial \phi} f_1^\alpha.
\end{aligned} \tag{3.9}$$

Order ϵ:

$$\begin{aligned}
&\frac{\partial}{\partial t_{-1}} f_2^\alpha + \frac{\partial}{\partial t_0} f_1^\alpha + \frac{\partial}{\partial t_1} f_0^\alpha \\
&\quad + U_\alpha \boldsymbol{b} \cdot \frac{\partial}{\partial \boldsymbol{Y}} f_1^\alpha + \boldsymbol{V}_{\mathrm{D}}^\alpha \cdot \frac{\partial}{\partial \boldsymbol{Y}} f_0^\alpha + e_\alpha U_\alpha \boldsymbol{b} \cdot \boldsymbol{E}^{(A)} \cdot \frac{\partial}{\partial \mathscr{E}} f_0^\alpha \\
&\quad - \mathscr{K}_0^\alpha \{ f_0, f_1 \} - \mathscr{K}_1^\alpha \{ f_0, f_0 \} \\
&\quad + \left[U_\alpha \boldsymbol{b} \cdot \boldsymbol{R} - \tfrac{1}{2} U_\alpha \boldsymbol{b} \cdot (\nabla \wedge \boldsymbol{b}) \right] \frac{\partial}{\partial \phi} f_1^\alpha + \Omega_\alpha^{[1]} \frac{\partial}{\partial \phi} f_0^\alpha \\
&\quad = -\Omega_\alpha \frac{\partial}{\partial \phi} f_2^\alpha,
\end{aligned} \tag{3.10}$$

where $\Omega_\alpha^{[1]}$ is the contribution of order ϵ to the equation of motion $\dot{\phi}$ [eq. (7), table 1.8.3]: its explicit form is not needed for our purpose.

We terminate our discussion at this order in the perturbation expansion. The three equations obtained here will be analyzed in detail in the next section.

10.4. The drift kinetic equation

It is physically clear that, among the variables $\boldsymbol{Y}$, $\mathscr{E}$, M, ϕ characterizing the phase space, the gyrophase ϕ plays a specific role. This variable moves at a

much faster rate than the others [see eq. (7), table **1**.8.3]; on the other hand, the important macroscopic observables are slowly varying functions. It is therefore convenient to find a way of studying separately the evolution on the various time-scales. This can be obtained by a widely used method. We observe that any meaningful physical quantity may depend on the angle ϕ only as a *periodic function*. Bearing this in mind, we note that any periodic function can be represented uniquely as a sum of two terms:

$$A(\phi) = \bar{A} + \tilde{A}(\phi). \tag{4.1}$$

The *average part* $\bar{A}$ of A is defined as

$$\bar{A} = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi\, A(\phi). \tag{4.2}$$

It is, by definition, independent of ϕ. The *oscillating part* $\tilde{A}$ of A is defined by (4.1) as the remainder of A after subtraction of the average. It has the obvious property

$$\bar{\tilde{A}} \equiv \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi\, \tilde{A}(\phi) = 0. \tag{4.3}$$

This operation is the same as in section **1**.6 (see eqs. **1**.6.12, **1**.6.13), but it is now applied to the distribution function f^α rather than to the dynamical variables. We thus define, for each order in the perturbation expansion (3.6),

$$f_{\mathrm{P}}^\alpha(\boldsymbol{Y}, \mathscr{E}, M, \phi; t_{-1}, t_0, t_1, \ldots) = \bar{f}_{\mathrm{P}}^\alpha(\boldsymbol{Y}, \mathscr{E}, M; t_{-1}, t_0, t_1, \ldots) + \tilde{f}_{\mathrm{P}}^\alpha(\boldsymbol{Y}, \mathscr{E}, M, \phi; t_{-1}, t_0, t_1, \ldots). \tag{4.4}$$

We note that every *moment* of the distribution function involves a contribution from each of the two terms,

$$B(\boldsymbol{Y}; t_{-1}, t_0, t_1, \ldots) = \sum_\alpha \int \mathrm{d}\mathscr{E}\, \mathrm{d}M\, \mathrm{d}\phi\, |J_\alpha|\, b_\alpha(\boldsymbol{Y}, \mathscr{E}, M, \phi)(\bar{f}^\alpha + \tilde{f}^\alpha)$$

$$\equiv \bar{B}(\boldsymbol{Y}; t_{-1}, t_0, t_1, \ldots) + \tilde{B}(\boldsymbol{Y}; t_{-1}, t_0, t_1, \ldots), \tag{4.5}$$

where J_α is the Jacobian of the pseudo-canonical transformation (table **1**.8.3).

With these facts in mind, we go over to the analysis of the kinetic equation in successive orders.

Order ϵ^{-1}

Substituting (4.4) into (3.8), the latter equation splits into two independent equations:

$$\frac{\partial}{\partial t_{-1}} \bar{f}_0^\alpha(\boldsymbol{Y}, \mathscr{E}, M; t_{-1}, t_0, t_1) = 0 \tag{4.6}$$

and

$$\frac{\partial}{\partial t_{-1}} \tilde{f}_0^\alpha(\boldsymbol{Y}, \mathscr{E}, M, \phi; t_{-1}, t_0, t_1)$$

$$= -\Omega_\alpha \frac{\partial}{\partial \phi} \tilde{f}_0^\alpha(\boldsymbol{Y}, \mathscr{E}, M, \phi; t_{-1}, t_0, t_1). \tag{4.7}$$

The first equation tells us that the average distribution function in zeroth order is independent of the fast time t_{-1},

$$\bar{f}_0^\alpha = \bar{f}_0^\alpha(\boldsymbol{Y}, \mathscr{E}, M; t_0, t_1). \tag{4.8}$$

Apart from this important property, the functional form of $\bar{f}_0^\alpha$ is left completely unspecified at this stage.

Next, eq. (4.7) is easily solved:

$$\tilde{f}_0^\alpha(\boldsymbol{Y}, \mathscr{E}, M, \phi; t_{-1}, t_0, t_1)$$

$$= \tilde{f}_0^\alpha(\boldsymbol{Y}, \mathscr{E}, M, \phi - \Omega_\alpha t_{-1}; t_{-1} = 0, t_0, t_1). \tag{4.9}$$

In order to choose a physically reasonable initial condition, we invoke the following argument. We know from the microscopic theory of chapter 1 that, upon averaging, the rapidly oscillating perpendicular velocity $v_\perp \boldsymbol{n}_1(\boldsymbol{q}, \varphi)$ of a particle gives rise to a small perpendicular drift velocity of order ϵ. If we now calculate the statistical ensemble average $\boldsymbol{u}_\perp^\alpha$ of the perpendicular velocity $v_\perp \boldsymbol{n}_1$, it is clear from (4.5) that only the oscillating part $\tilde{f}^\alpha$ contributes to this macroscopic quantity. If $\bar{f}_0^\alpha$ is non-zero, the average perpendicular velocity will be of order ϵ^0, in contradiction with the microscopic result. In order to avoid this dilemma, the only reasonable choice for $\bar{f}_0^\alpha$ at time $t_{-1} = 0$ is $\bar{f}_0^\alpha = 0$; it then follows from (4.9) that $\tilde{f}_0^\alpha$ is zero for all times t_{-1}, hence for all times t:

$$\tilde{f}_0^\alpha(\boldsymbol{Y}, \mathscr{E}, M, \phi; t) = 0. \tag{4.10}$$

To sum up our discussion, we have found that *the zeroth order distribution function must be a slowly varying function, independent of the gyrophase*:

$$f_0^\alpha(\boldsymbol{Y}, \mathscr{E}, M, \phi; t_{-1}, t_0, t_1) = \bar{f}_0^\alpha(\boldsymbol{Y}, \mathscr{E}, M; t_0, t_1). \tag{4.11}$$

Order ϵ^0

Taking account of result (4.11), eq. (3.9) reduces to

$$\left(\frac{\partial}{\partial t_{-1}} + \Omega_\alpha \frac{\partial}{\partial \phi}\right) f_1^\alpha = -\left(\frac{\partial}{\partial t_0} + U_\alpha \boldsymbol{b} \cdot \frac{\partial}{\partial \boldsymbol{Y}}\right) \bar{f}_0^\alpha + \mathscr{K}_0^\alpha\{\bar{f}_0, \bar{f}_0\}. \tag{4.12}$$

We introduce decomposition (4.4) for f^α and consider first the equation for the oscillating part:

$$\left(\frac{\partial}{\partial t_{-1}} + \Omega_\alpha \frac{\partial}{\partial \phi}\right) \tilde{f}_1^\alpha = \widetilde{\mathscr{K}_0^\alpha}\{\bar{f}_0, \bar{f}_0\}. \tag{4.13}$$

We now anticipate a property which will be checked explicitly in a forthcoming chapter,

$$\widetilde{\mathscr{K}_0^\alpha}\{\bar{f}_0, \bar{f}_0\} = 0. \tag{4.14}$$

The oscillating part of the zeroth order collision term, evaluated with average distribution functions, vanishes. This property can be inferred from the isotropic nature of the operator $\mathscr{K}_0^\alpha$. Equation (4.14) is the only specific property of the collision operator (expressed in NGC variables) that will be used in the present chapter.

Equation (4.13) for $\tilde{f}_1^\alpha$ is of the same form as (4.7), thus its solution is

$$\tilde{f}_1^\alpha(\boldsymbol{Y}, \mathscr{E}, M, \phi; t_{-1}, t_0, t_1) = \tilde{f}_1^\alpha(\boldsymbol{Y}, \mathscr{E}, M, \phi - \Omega_\alpha t_{-1}; t_{-1} = 0, t_0, t_1). \tag{4.15}$$

But now there is no argument for the vanishing of the initial condition. Hence, in general, *there is an oscillating part of the form* (4.15) *in the distribution function to order* ϵ.

Next, we average both sides of eq. (4.12) over the gyrophase, thus obtaining

$$-\frac{\partial}{\partial t_{-1}} \bar{f}_1^\alpha = \left(\frac{\partial}{\partial t_0} + U_\alpha \boldsymbol{b} \cdot \frac{\partial}{\partial \boldsymbol{Y}}\right) \bar{f}_0^\alpha - \overline{\mathscr{K}_0^\alpha}\{\bar{f}_0, \bar{f}_0\}$$

$$\equiv G_0^\alpha, \tag{4.16}$$

where G_0^α is simply an abbreviation for the right-hand side. [Note that the overbar on the collision term can be omitted because of eq. (4.14).] The important fact is that the function G_0^α is independent of the fast time t_{-1} (because of eq. (4.11). Hence, eq. (4.16) is immediately integrated:

$$\bar{f}_1^\alpha(\boldsymbol{Y}, \mathscr{E}, M; t_{-1}, t_0, t_1) = \bar{f}_1^\alpha(\boldsymbol{Y}, \mathscr{E}, M; t_{-1} = 0, t_0, t_1) - t_{-1} G_0^\alpha. \tag{4.17}$$

The main feature of this result is the appearance of the *secular term* $t_{-1}G_0^\alpha$, which grows linearly in the fast time. At this point, we take advantage of the flexibility offered by the multiple time-scale perturbation theory. We impose a condition on $\bar{f}_0^\alpha$ which ensures the vanishing of the secular term, hence the *uniform* validity of the perturbation expansion. The condition is simply: $G_0^\alpha = 0$, i.e.

$$\left(\frac{\partial}{\partial t_0} + U_\alpha \boldsymbol{b} \cdot \frac{\partial}{\partial \boldsymbol{Y}}\right) \bar{f}_0^\alpha(Y, \mathscr{E}, M; t_0, t_1) = \mathscr{K}_0^\alpha\{\bar{f}_0, \bar{f}_0\}. \tag{4.18}$$

We thus obtained two results from this analysis:

(a) The first-order contribution to the average distribution function is independent of the fast time variable:

$$\bar{f}_1^\alpha = \bar{f}_1^\alpha(\boldsymbol{Y}, \mathscr{E}, M; t_0, t_1). \tag{4.19}$$

(b) *The form of the zeroth-order average distribution function*, which was left unspecified in the previous stage, eq. (4.11), *is now made specific: it must be a solution of* Eq. (4.18). The appropriate solution will be determined in section 10.5.

Order ϵ

The procedure of the previous subsection is now applied to eq. (3.10). Substituting (4.4) and averaging both sides of the equation, we obtain

$$\begin{aligned} -\frac{\partial}{\partial t_{-1}} \bar{f}_2^\alpha = {} & \frac{\partial}{\partial t_0} \bar{f}_1^\alpha + \frac{\partial}{\partial t_1} \bar{f}_0^\alpha + U_\alpha \boldsymbol{b} \cdot \frac{\partial}{\partial \boldsymbol{Y}} \bar{f}_1^\alpha + \boldsymbol{V}_{\mathrm{D}}^\alpha \cdot \frac{\partial}{\partial \boldsymbol{Y}} \bar{f}_0^\alpha \\ & + e_\alpha U_\alpha \boldsymbol{b} \cdot \boldsymbol{E}^{(A)} \frac{\partial}{\partial \mathscr{E}} \bar{f}_0^\alpha - \overline{\mathscr{K}_0^\alpha\{f_0, f_1\}} - \overline{\mathscr{K}_1^\alpha\{f_0, f_0\}} \\ \equiv {} & G_1^\alpha, \end{aligned} \tag{4.20}$$

where G_1^α is again an abbreviation for the right-hand side. By the same argument as before, G_1^α is independent of t_{-1}. Hence, in order to avoid a secular behaviour, we impose the constraint $G_1^\alpha = 0$ and obtain two results:

(a) The second-order average distribution function is independent of t_{-1}.

$$\bar{f}_2^\alpha = \bar{f}_2^\alpha(\boldsymbol{Y}, \mathscr{E}, M; t_0, t_1). \tag{4.21}$$

(b) *The first-order average distribution function must obey the following equation*:

$$\frac{\partial}{\partial t_0}\bar{f}_1^\alpha + \frac{\partial}{\partial t_1}\bar{f}_0^\alpha + U_\alpha \boldsymbol{b}\cdot\frac{\partial}{\partial \boldsymbol{Y}}\bar{f}_1^\alpha + \boldsymbol{V}_{\mathrm{D}}^\alpha\cdot\frac{\partial}{\partial \boldsymbol{Y}}\bar{f}_0^\alpha + e_\alpha U_\alpha \boldsymbol{b}\cdot\boldsymbol{E}^{(A)}\frac{\partial}{\partial \mathscr{E}}\bar{f}_0^\alpha$$

$$= \overline{\mathscr{K}_0^\alpha\{f_0, f_1\}} + \overline{\mathscr{K}_1^\alpha\{f_0, f_0\}}. \tag{4.22}$$

This is the so-called *drift kinetic equation*, which is a basic tool of transport theory for toroidally confined plasmas. It has been derived by many authors (Hastie et al. 1967, Rutherford 1970, Frieman 1970). One may contrast the simplicity of the present method with the older derivations, involving very painful averaging calculations. Here we benefit from the great advantage of the pseudo-canonical transformation: by performing this transformation in the first place, we obtain a form (2.4) of the kinetic equation, in which all the coefficients (except for the collision term) are independent of the gyrofrequency ϕ. As a result, the averaging becomes a trivial operation.

Finally, we consider the second-order oscillating distribution function; it must obey the following equation:

$$-\left(\frac{\partial}{\partial t_{-1}} + \Omega_\alpha\frac{\partial}{\partial\phi}\right)\tilde{f}_2^\alpha = \frac{\partial}{\partial t_0}\tilde{f}_1^\alpha + U_\alpha \boldsymbol{b}\cdot\frac{\partial}{\partial \boldsymbol{Y}}\tilde{f}_1^\alpha$$

$$+ U_\alpha\left[\boldsymbol{b}\cdot\boldsymbol{R} - \tfrac{1}{2}\boldsymbol{b}\cdot(\nabla\wedge\boldsymbol{b})\right]\frac{\partial}{\partial\phi}f_1^\alpha$$

$$-\widetilde{\mathscr{K}_0^\alpha\{f_0, f_1\}} - \widetilde{\mathscr{K}_1^\alpha\{f_0, f_0\}}. \tag{4.23}$$

This equation must be analyzed in order to detect possible secularities, thus imposing additional constraints on $\tilde{f}_1^\alpha$. This step will however not be needed in our book.

10.5. The local equilibrium state for a magnetically confined plasma

We now return to the zeroth-order average distribution function, obeying eq. (4.18), and seek a solution appropriate for transport theory. In analogy with

the discussion in sec. **4**.2, we shall look for a quasi-stationary solution of this equation [note that the overbar on $\bar{f}_0^\alpha$ is superfluous, because of eq. (4.10)]:

$$U_\alpha \boldsymbol{b} \cdot \frac{\partial}{\partial \boldsymbol{Y}} f_0^\alpha - \mathscr{K}_0^\alpha\{f_0, f_0\} = 0. \tag{5.1}$$

We follow here the main line of an argument due to Glasser and Thompson (1973) and to Hinton and Hazeltine (1976). We multiply both sides of (5.1) by $|J_\alpha| \ln f_0^\alpha$ and integrate over $\mathscr{E}$, M, ϕ (J_α is the Jacobian of the pseudo-canonical averaging transformation):

$$\int \mathrm{d}\mathscr{E}\, \mathrm{d}M\, \mathrm{d}\phi\, |J_\alpha| \ln f_0^\alpha \left(U_\alpha \boldsymbol{b} \cdot \frac{\partial}{\partial \boldsymbol{Y}} f_0^\alpha - \mathscr{K}_0^\alpha\{f_0, f_0\} \right) = 0. \tag{5.2}$$

We now note that, to order ϵ^0, $\boldsymbol{Y} = \boldsymbol{q}$, $U_\alpha = v_\parallel$, $W = v_\perp$, $\phi = \varphi$; hence, we may go back to the particle variables $\boldsymbol{q}$, $\boldsymbol{v}$:

$$\int \mathrm{d}^3\boldsymbol{v} \ln f_0^\alpha \left(v_\parallel \boldsymbol{b} \cdot \frac{\partial}{\partial \boldsymbol{q}} f_0^\alpha - \mathscr{K}_0^\alpha\{f_0, f_0\} \right) = 0. \tag{5.3}$$

Noting that:

$$\ln f_0^\alpha \frac{\partial}{\partial \boldsymbol{q}} f_0^\alpha = \frac{\partial}{\partial \boldsymbol{q}} (f_0^\alpha \ln f_0^\alpha - f_0^\alpha),$$

we rewrite (5.3) in the form

$$\boldsymbol{B} \cdot \frac{\partial}{\partial \boldsymbol{q}} \frac{1}{B} \int \mathrm{d}^3\boldsymbol{v}\, v_\parallel (\ln f_0^\alpha - 1) f_0^\alpha = \int \mathrm{d}^3\boldsymbol{v} \ln f_0^\alpha\, \mathscr{K}_0^\alpha\{f_0, f_0\}. \tag{5.4}$$

We now *average both sides over a magnetic surface*, as in section **8**.6. Recalling the general property (**8**.6.3), we see that the average of the left-hand side is zero, hence we obtain

$$\left\langle \int \mathrm{d}^3\boldsymbol{v} \ln f_0^\alpha\, \mathscr{K}_0^\alpha\{f_0, f_0\} \right\rangle = 0. \tag{5.5}$$

But we know from the H-theorem for the Landau equation (section **6**.1) that the argument of the average has a definite sign,

$$\int \mathrm{d}^3\boldsymbol{v} \ln f_0^\alpha\, \mathscr{K}_0^\alpha\{f_0, f_0\} \leqslant 0.$$

It follows that eq. (5.5) can only be satisfied if

$$\mathscr{K}_0^\alpha\{f_0, f_0\} = 0. \tag{5.6}$$

We know from section 4.2 that the general solution of this equation for an electron–ion plasma is the local plasma equilibrium distribution function,

$$f^{\alpha}(\boldsymbol{q}, \boldsymbol{v}; t_1) = \left(\frac{m_\alpha}{2\pi T_{\alpha 0}(\boldsymbol{q}; t_1)}\right)^{3/2} n_{\alpha 0}(\boldsymbol{q}; t_1)$$

$$\times \exp\left(-\frac{m_\alpha}{2T_{\alpha 0}(\boldsymbol{q}; t_1)} |\boldsymbol{v} - \boldsymbol{u}_0^{\alpha}(\boldsymbol{q}; t_1)|^2\right), \tag{5.7}$$

where the density $n_{\alpha 0}$, the local velocity $\boldsymbol{u}_0^{\alpha}$ and the temperature $T_{\alpha 0}$ may depend on the position and on the (slow) time t_1.

However, eq. (5.7) is not yet an acceptable solution to our initial problem: it must satisfy some additional constraints. In order to determine the latter, we go back to the guiding centre variables. The zeroth-order distribution function thus becomes

$$f_0^{\alpha} = \left(\frac{m_\alpha}{2\pi T_{\alpha 0}}\right)^{3/2} n_{\alpha 0} \exp\left(-\frac{m_\alpha}{2T_{\alpha 0}} |U_\alpha \boldsymbol{b} + W_\alpha \boldsymbol{n}_1 - \boldsymbol{u}_0^{\alpha}|^2\right)$$

$$= \left(\frac{m_\alpha}{2\pi T_{\alpha 0}}\right)^{3/2} n_{\alpha 0}$$

$$\times \exp\left\{-\frac{m_\alpha}{2T_{\alpha 0}}\left[(U_\alpha - \boldsymbol{b}\cdot\boldsymbol{u}_0^{\alpha})^2 + (W_\alpha - \boldsymbol{n}_1\cdot\boldsymbol{u}_0^{\alpha})^2 + (\boldsymbol{n}_2\cdot\boldsymbol{u}_0^{\alpha})^2\right]\right\}, \tag{5.8}$$

where U_α and W_α are considered as functions of $\boldsymbol{Y}$, $\mathscr{E}$, and M. In the last line, the vector $\boldsymbol{u}_0^{\alpha}$ has been resolved along the rotating local reference frame $\boldsymbol{b}(\boldsymbol{Y})$, $\boldsymbol{n}_1(\boldsymbol{Y}, \phi)$, $\boldsymbol{n}_2(\boldsymbol{Y}, \phi)$, defined in (1.4.11) and (1.4.13).

We now note that, because of (4.11), f_0^{α} cannot depend on the gyrophase ϕ. Equation (5.8) is compatible with this requirement only if the local velocity $\boldsymbol{u}_0^{\alpha}$ satisfies

$$\boldsymbol{n}_1(\boldsymbol{Y}, \phi)\cdot\boldsymbol{u}_0^{\alpha} = \boldsymbol{n}_2(\boldsymbol{Y}, \phi)\cdot\boldsymbol{u}_0^{\alpha} = 0, \tag{5.9}$$

i.e. the local velocity is parallel to the magnetic field. Equation (5.8) thus reduces to

$$f_0^{\alpha} = \left(\frac{m_\alpha}{2\pi T_{\alpha 0}}\right)^{3/2} n_{\alpha 0} \exp\left[-\frac{m_\alpha}{2T_{\alpha 0}}\left(U_\alpha^2 + W_\alpha^2 - 2U_\alpha \boldsymbol{b}\cdot\boldsymbol{u}_0^{\alpha} + u_0^{\alpha 2}\right)\right]$$

$$= \left(\frac{m_\alpha}{2\pi T_{\alpha 0}}\right)^{3/2} n_{\alpha 0} \exp\left[-\frac{1}{T_{\alpha 0}}\left(\mathscr{E} - e_\alpha \Phi_0 - m_\alpha U_\alpha \boldsymbol{b}\cdot\boldsymbol{u}_0^{\alpha} + \tfrac{1}{2} m_\alpha u_0^{\alpha 2}\right)\right]. \tag{5.10}$$

We now note that (5.10) satisfies (5.6); but function f_0^α must also satisfy the *complete* zeroth-order equation (5.1), which now reduces to

$$\boldsymbol{B}\cdot\frac{\partial}{\partial \boldsymbol{Y}} f_0^\alpha(\boldsymbol{Y},\ \mathscr{E},\ M;\ t_1) = 0. \tag{5.11}$$

We recall eq. (**8**.1.2) and see that (5.11) requires the zeroth-order distribution function to be a *surface quantity*. In other words, if the guiding centre position $\boldsymbol{Y}$ is expressed in terms of the HH system of toroidal coordinates $(\rho,\ \theta,\ \zeta)$ defined in section **8**.8, function f_0^α *can only depend on the radial coordinate* ρ. This requirement can only be satisfied by taking all the functions of $\boldsymbol{Y}$ in (5.10), i.e. $n_{\alpha 0}$, $\boldsymbol{u}_0^\alpha$, $T_{\alpha 0}$ and Φ_0, as functions of ρ alone. But even this is not yet sufficient, because the exponential in (5.10) contains the term

$$U_\alpha\, \boldsymbol{b}\cdot\boldsymbol{u}_0^\alpha = \sigma\left[(2/m_\alpha)(\mathscr{E} - e_\alpha\Phi_0(\rho) - MB(\boldsymbol{Y}))\right]^{1/2}\boldsymbol{b}\cdot\boldsymbol{u}_0^\alpha(\rho).$$

The magnetic field amplitude $B(\boldsymbol{Y})$ is certainly *not* a surface quantity. Thus (5.11) can only be satisfied if this term vanishes, which in turn requires the condition

$$\boldsymbol{b}\cdot\boldsymbol{u}_0^\alpha = 0. \tag{5.12}$$

But this condition, combined with (5.10), simply gives

$$\boldsymbol{u}_0^\alpha(\boldsymbol{Y},\ t_1) = 0. \tag{5.13}$$

We have now completed the solution of the kinetic equation (5.1) to zeroth order in the drift parameter ϵ. This solution will play the role of a *reference state* in the transport theory of toroidally confined plasmas, just as the local equilibrium (**4**.2.6) did in the theory of unconfined plasmas. The new reference state is defined as follows, in terms of the natural guiding centre variables $\boldsymbol{Y}$, $\mathscr{E}$, M, ϕ:

$$\begin{aligned} f_0^\alpha(\boldsymbol{Y},\ \mathscr{E};\ t_1) = {} & \left(\frac{m_\alpha}{2\pi T_{\alpha 0}(\rho;\ t_1)}\right)^{3/2} n_{\alpha 0}(\rho;\ t_1) \\ & \times \exp\left(\frac{e_\alpha}{T_{\alpha 0}(\rho;\ t_1)}\,\Phi_0(\rho;\ t_1)\right) \\ & \times \exp\left(-\frac{1}{T_{\alpha 0}(\rho;\ t_1)}\,\mathscr{E}\right), \end{aligned} \tag{5.14}$$

where ρ is the radial coordinate labelling the magnetic surfaces. Alternatively, this distribution function can be expressed (to zeroth order in ϵ) in terms of the particle variables $\boldsymbol{q}$, $\boldsymbol{v}$,

$$f_0^\alpha(\boldsymbol{q}, \boldsymbol{v}; t_1) = \left(\frac{m_\alpha}{2\pi T_{\alpha 0}(\rho; t_1)}\right)^{3/2} n_{\alpha 0}(\rho; t_1) \exp\left(-\frac{m_\alpha}{2T_{\alpha 0}(\rho; t_1)} v^2\right). \tag{5.15}$$

The plasmadynamical functions entering the reference state: $n_{\alpha 0}$, $\boldsymbol{u}_0^\alpha$, $T_{\alpha 0}$, Φ_0 must satisfy the following conditions (to zeroth order in ϵ):
– they are slowly varying functions of time (i.e. they depend at most on the slow time t_1);
– the local velocity $\boldsymbol{u}_0^\alpha$ must vanish identically to order ϵ^0;
– the scalar plasmadynamical quantities must be constant on all magnetic surfaces:

$$\boldsymbol{b} \cdot \nabla n_{\alpha 0} = 0, \qquad \boldsymbol{b} \cdot \nabla T_{\alpha 0} = 0, \qquad \boldsymbol{b} \cdot \nabla \Phi_0 = 0; \tag{5.16}$$

– any deviation from conditions (5.13) and (5.16) is at most of order ϵ.

Constraints (5.13) *and* (5.16) *represent the main difference between the reference state of a toroidally confined plasma and an unconfined plasma.* We shall see in the forthcoming chapters how these constraints introduce important differences in the transport properties of the confined plasmas.

10.6. Ordering of the hydrodynamical quantities in a toroidally confined plasma

In order to complete the picture of a toroidally confined plasma, which will be the reference state for the study of transport processes, we need some further details.

(A) We have seen that the plasmadynamical quantities n_α, Φ, T_α, determining the reference state (5.14), (5.15) must be slowly varying in time: they must depend on t_1. They are determined by a set of plasmadynamical differential equations of the general form

$$\partial p / \partial t = A,$$

where p is any plasmadynamical quantity and A is some function of these

quantities. Typical examples of such equations are eqs. (5.2.1). The discussion of those equations in section 5.2 is particularly revealing. It shows that, in order to obtain expressions of the fluxes relevant for transport theory, the time derivatives in these equations could be neglected: (5.2.1) could be simply replaced by the algebraic equations (5.2.19). The reason for this fact is that the temporal variation represents a "rapid" transient process (on the relaxation time scale), leading to an asymptotic state which varies much more slowly (on the hydrodynamical time scale).

Translating this discussion in our present language, we may say the following. If we are interested only in the quantities relevant for transport theory (and thus disregard the fast transient adjustment to the quasi-steady state), we may omit the variation of the plasmadynamical quantities on the time-sale t_1, which can be identified with the relaxation time-scale. We assume that these quantities have already reached their quasisteady state, hence they depend only on the next-slower time-scale t_2. In particular,

$$n_\alpha = n_\alpha(x;\, t_2), \qquad T_\alpha = T_\alpha(x;\, t_2), \qquad \Phi = \Phi(x;\, t_2). \tag{6.1}$$

Another way of expressing these conditions (Hinton & Hazeltine 1976) is the scaling assumption

$$\tau_{II}\left|\frac{\partial \ln p}{\partial t}\right| = \mathrm{O}(\epsilon^2), \quad p = n_\alpha,\ T_\alpha,\ \Phi. \tag{6.2}$$

In particular, the local equilibrium distribution function (5.15), which depends on the zeroth order quantities $n_{\alpha 0}$, $T_{\alpha 0}$, should be replaced by the stronger version

$$f_0^\alpha(\boldsymbol{q},\, \boldsymbol{v};\, t_2) = \left(\frac{m_\alpha}{2\pi T_{\alpha 0}(\rho;\, t_2)}\right)^{3/2} n_{\alpha 0}(\rho;\, t_2)\, \exp\left(-\frac{m_\alpha}{2T_{\alpha 0}(\rho;\, t_2)}\, v^2\right). \tag{6.3}$$

(B) Having discussed the time dependence of the scalar plasmadynamical quantities, we now consider their space dependence. If p denotes again any one of these quantities, we may always write, in the spirit of the drift approximation, an expansion of the type

$$p(x;\, t) = p_0(x;\, t) + \epsilon p_1(x;\, t) + \cdots. \tag{6.4}$$

In the toroidal confinement geometry, it is natural to use for the space coordinates a set of toroidal coordinates ρ, θ, ζ, such as those discussed in

chapter 8. In terms of these variables, we know from section 10.5 that the zeroth-order moments p_0 can only depend on the radial coordinate ρ. Hence, the dependence on the poloidal and the toroidal angles θ, ζ can only come from the small term ϵp_1. As a result, the *gradient of p has a large component in the radial direction* and a small component in the tangential direction. More precisely,

$$L_H | \boldsymbol{e}_\rho \cdot \nabla \ln p | \equiv L_H | \nabla_\rho \ln p | = O(\epsilon^0), \tag{6.5}$$

$$L_H | \boldsymbol{e}_\theta \cdot \nabla \ln p | \equiv L_H | \nabla_\theta \ln p | = O(\epsilon), \tag{6.6}$$

$$L_H | \boldsymbol{e}_\zeta \cdot \nabla \ln p | \equiv L_H | \nabla_\zeta \ln p | = O(\epsilon). \tag{6.7}$$

If moreover, the plasma is in an *axisymmetric confinement configuration*, eq. (6.7) is replaced by

$$\boldsymbol{e}_\zeta \cdot \nabla \ln p \equiv \nabla_\zeta \ln p = 0 \qquad [AXISYM]. \tag{6.8}$$

In these formulae, $\boldsymbol{e}_\rho$, $\boldsymbol{e}_\theta$, $\boldsymbol{e}_\zeta$, are the basis vectors of the coordinate system (see General Appendix **G2**.2) and the notation ∇_ρ, ∇_θ, ∇_ζ are defined by the left-hand sides.

(C) Consider now the various *fluxes*, i.e. the *odd moments* of the distribution function. We know already that they all vanish, to order ϵ^0, in the local equilibrium state. We now show that it is possible to derive very simply some important properties of these fluxes, to *order* ϵ. We consider a state deviating slightly from the reference state and determine the moments by using the plasmadynamical equations.

Consider the momentum balance equation (**3**.4.10): it contains the peculiar Lorentz force term $e_\alpha n_\alpha c^{-1} B(\boldsymbol{u}^\alpha \wedge \boldsymbol{b})$. Defining, as usual, the perpendicular component of the flux of particles of species α [see table **3**.2.1] as

$$\Gamma_\perp^\alpha = \boldsymbol{b} \wedge (\Gamma^\alpha \wedge \boldsymbol{b}),$$

we obtain an exact expression of this quantity by taking the vector product of both sides of (**3**.4.10) with $\boldsymbol{b}$ (Hinton and Hazeltine 1976):

$$\Gamma_\perp^\alpha = \frac{1}{m_\alpha \Omega_\alpha} \boldsymbol{b} \wedge (\nabla P_\alpha + \nabla \cdot \pi^\alpha + \nabla \cdot m_\alpha \boldsymbol{u}^\alpha \Gamma^\alpha - e_\alpha n_\alpha \boldsymbol{E}$$

$$- \boldsymbol{R}^\alpha + m_\alpha \, \partial_t \Gamma^\alpha). \tag{6.9}$$

We know from section 1.6 that a factor Ω_α^{-1} introduces an order ϵ, hence Γ^α is, as it should be, of order ϵ at most. In order to isolate the dominant contribution, we discard the dissipative terms $\boldsymbol{R}^\alpha$, π^α, as well as the non-linear terms $\boldsymbol{u}^\alpha \Gamma^\alpha$; we also neglect the time derivative, because of (6.2), and finally note that the dominant (order ϵ^0) contribution to the electric field derives from the local equilibrium potential Φ_0. Hence, the first-order perpendicular particle flux density is

$$\Gamma^\alpha_{\perp 1} = \frac{1}{m_\alpha \Omega_\alpha} \boldsymbol{b} \wedge (\nabla P_{\alpha 0} + e_\alpha n_{\alpha 0} \nabla \Phi_0). \tag{6.10}$$

A similar analysis can be performed on the heat flux equation (4.5.22) (rewritten for the dimensional moment $\boldsymbol{q}^\alpha$). It leads to the conclusion that the first-order perpendicular flux is

$$\boldsymbol{q}^\alpha_{\perp 1} = \frac{5}{2} \frac{1}{m_\alpha \Omega_\alpha} P_{\alpha 0} \boldsymbol{b} \wedge \nabla T_{\alpha 0}. \tag{6.11}$$

These first-order fluxes turn out to have peculiar properties, which will be progressively unveiled in the forthcoming discussions. At present we note that they are entirely determined by the gradients of the local equilibrium moments $n_{\alpha 0}$, $T_{\alpha 0}$, Φ_0.

Our knowledge that the equilibrium scalar moments are necessarily *surface quantities* implies that their gradients are vectors directed perpendicularly to the magnetic surfaces. Hence, *the vectors* $\Gamma^\alpha_{\perp 1}$, $\boldsymbol{q}^\alpha_{\perp 1}$, which must be perpendicular both to $\nabla \rho$ and to $\boldsymbol{b}$, *lie in the tangent plane of the magnetic surfaces.*

Equations (6.10), (6.11) do not yield the complete first-order flux vectors: nothing precludes the possibility of a non-vanishing component *parallel to* $\boldsymbol{b}$, which cannot be found from (6.9). Additional information about this parallel component can be extracted from the even-order moment equations. Thus, the continuity equations (3.4.5), combined with (6.2), tell us that

$$\nabla \cdot \Gamma^\alpha_1 = 0 + \mathrm{O}(\epsilon^2). \tag{6.12}$$

As $\Gamma^\alpha_1 = n_{\alpha 0} \boldsymbol{u}^\alpha_1$ is perpendicular to $\nabla n_{\alpha 0}$, this relation also implies

$$\nabla \cdot \boldsymbol{u}^\alpha_1 = 0 + \mathrm{O}(\epsilon^2). \tag{6.13}$$

We also note that, because of the assumed quasi-neutrality (see eq. 4.1.10), the electric current density must be divergenceless,

$$\nabla \cdot \boldsymbol{j} = 0. \tag{6.14}$$

Similarly, the temperature equation (3.4.11), in which we neglect, as usual, the heat exchange term $Q^{(2)}$, as well as the dissipative pressure tensor π^α and the second-order term $\partial_t T_\alpha$, yields, to first order in ϵ,

$$-\tfrac{2}{3}\nabla \cdot \boldsymbol{q}_1^\alpha - n_{\alpha 0}\boldsymbol{u}_1^\alpha \cdot \nabla T_{\alpha 0} - \tfrac{2}{3} n_{\alpha 0} T_{\alpha 0}\ \nabla \cdot \boldsymbol{u}_1^\alpha = 0.$$

Using (6.13) and noting that $\boldsymbol{u}_1^\alpha$ is perpendicular to $\nabla T_{\alpha 0}$, we conclude that

$$\nabla \cdot \boldsymbol{q}_1^\alpha = 0 + O(\epsilon^2). \tag{6.15}$$

In conclusion, we found that the particle flux Γ^α, the electric current $\boldsymbol{j}$ and the heat flux $\boldsymbol{q}^\alpha$ in a toroidally confined plasma have a *dominant first-order part, which is tangent to the magnetic surface and which has zero divergence.* In some texts, these first-order fluxes are called *equilibrium rotation.* It will be seen later that these fluxes do not produce any entropy, hence they may also be called *non-dissipative fluxes.*

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11

Approximation methods for the kinetic equation

11.1. Test-particle collisions and field particle collisions

In deriving the drift kinetic equation in section **10**.4, the left-hand side, describing the rate of change of the gyro-averaged distribution function under the action of the parallel motion and of the perpendicular drift, was very simply obtained by the method of pseudo-canonical transformations. This method, however, does not help in transforming the collision term to NGC variables, because this term is not in the form of a Lie bracket (see the discussion in chapter 2).

One could, of course, apply brute force to the problem. Starting with the Landau equation (**2**.6.24), he needs to express both velocity variables v_1, v_2 in terms of NGC variables by means of (**1**.8.3)–(**1**.8.5); he then needs to transform the derivatives $\partial/\partial v_r$ into derivatives with respect to the NGC variables. After having covered several pages with mathematical symbols, he will soon give up the task in front of the formidable complexity of the expressions, even to the leading order in ϵ.

The alternative attack on the problem consists of doing a number of clever approximations and simplifications *before* the transformation to NGC variables and the gyro-averaging process. The *Lorentz process* described in section **2**.8 proceeds from this philosophy. Its main result consists of isolating in the electron–ion collision term the Lorentz operator as a leading term when the mass ratio of the two collision partners is small. This Lorentz operator will play a leading role in the neoclassical theory, as will be presently seen. But the Lorentz process of section **2**.8 is insufficient for solving our present difficulties. In particular, it does nothing to the *like-particle collision operator*. This was no serious difficulty in the classical theory of chapter 4, but in the drift kinetic equation it leads to unsurmountable difficulties, as described above. The problem of the like-particle collisions has plagued the neoclassical theory for a long time; it was treated in a purely ad hoc manner until a satisfactory rigorous treatment was found by Hirshman and Sigmar (1976).

The collision term appearing in the drift kinetic equation is actually not the complete Landau term (**2**.6.24), but only its *linearized version*, which we now proceed to write explicitly. We expand the distribution function in powers of ϵ, as in (**10**.3.1),

$$f^\alpha(\boldsymbol{v}) = f_0^\alpha(v) + \epsilon f_1^\alpha(\boldsymbol{v}) + \cdots . \tag{1.1}$$

In the first stages we use ordinary particle variables ($\boldsymbol{q}$, $\boldsymbol{v}$); the transformation to NGC variables and the gyro-averaging will be done at the end of the treatment. In order to unburden the notations, we shall only write (when necessary) the velocity argument $\boldsymbol{v}$, omitting the variables $\boldsymbol{q}$, t in the distribution functions. The reference function, as we know, is a local Maxwellian given by (**10**.5.15) (we now drop the subscript $_0$ on $n_{\alpha 0}$, $T_{\alpha 0}$, Φ_0):

$$f_0^\alpha(v) = n_\alpha \left(\frac{m_\alpha}{2\pi T_\alpha} \right)^{3/2} \exp\left(- \frac{m_\alpha}{2T_\alpha} v^2 \right). \tag{1.2}$$

When expansion (1.1) is substituted into the collision term and the terms of order higher than ϵ are discarded, two types of contributions are found:

(A) There is a *source term*, formally of zeroth order in ϵ. Its presence is due to the fact that (1.2) does not annul the collision term, unless the temperatures of the two species are identical. This point was discussed in section **4**.2. The source term is mainly responsible for the heat exchange between the two species (a process that is very slow in the case of electrons and ions, as we know). Its form is easily obtained from (1.1) and (**2**.6.24):

$$\mathscr{K}^{\alpha\beta}\{f_0, f_0\} = \frac{2\pi e_\alpha^2 e_\beta^2 \ln \Lambda}{m_\alpha} \cdot \frac{1}{2} \left(\frac{1}{T_\alpha} - \frac{1}{T_\beta} \right)$$

$$\times \frac{\partial}{\partial v_{1r}} v_{1s} \int \mathrm{d}\boldsymbol{v}_2 \, \frac{g^2 \delta_{rs} - g_r g_s}{g^3} f_0^\alpha(v_1) \, f_0^\beta(v_2). \tag{1.3}$$

We recall the notation $\boldsymbol{g} \equiv \boldsymbol{v}_1 - \boldsymbol{v}_2$ for the relative velocity.

Clearly, only unlike-particle collisions ($\alpha \neq \beta$) contribute to the source term. This is the term appearing in the last position on the right-hand side in the drift kinetic equation (**10**.4.22). It is easily seen that this term does not contribute to the determination of any vector quantity, such as the particle or heat fluxes. Indeed, multiplying it by any function of the form $v_{1n}\psi(\boldsymbol{v}_1)$ and integrating over $\boldsymbol{v}_1$ gives a vanishing result, as can be seen simply from the

parity of the integrand. This term will therefore be omitted in the forthcoming treatment.

(B) The *linearized collision term* is obtained in the form

$$\mathscr{K}_{\mathrm{L}}^{\alpha\beta} = \frac{2\pi e_\alpha^2 e_\beta^2 \ln \Lambda}{m_\alpha} \int \mathrm{d}\boldsymbol{v}_2 \frac{\partial}{\partial v_{1r}} G_{rs}(\boldsymbol{v}_1 - \boldsymbol{v}_2)$$

$$\times \left(f_0^\beta(v_2) \frac{1}{m_\alpha} \frac{\partial}{\partial v_{1s}} f_1^\alpha(\boldsymbol{v}_1) + f_1^\beta(\boldsymbol{v}_2) \frac{1}{m_\alpha} \frac{\partial}{\partial v_{1s}} f_0^\alpha(v_1) \right.$$

$$\left. - f_0^\alpha(v_1) \frac{1}{m_\beta} \frac{\partial}{\partial v_{2s}} f_1^\beta(\boldsymbol{v}_2) - f_1^\alpha(\boldsymbol{v}_1) \frac{1}{m_\beta} \frac{\partial}{\partial v_{2s}} f_0^\beta(v_2) \right). \tag{1.4}$$

In further analyzing the linearized collision term, it is often useful to make a distinction between two groups of terms:

$$\mathscr{K}_{\mathrm{L}}^{\alpha\beta} = \mathscr{K}_{\mathrm{T}}^{\alpha\beta} + \mathscr{K}_{\mathrm{F}}^{\alpha\beta}, \tag{1.5}$$

where we define the *test-particle collision term* as the sum of the first and fourth bracketted terms in (1.4):

$$\mathscr{K}_{\mathrm{T}}^{\alpha\beta} = A_{\alpha\beta} \frac{\partial}{\partial v_r} \int \mathrm{d}\boldsymbol{v}' \; G_{rs}(\boldsymbol{v} - \boldsymbol{v}') \; f_0^\beta(v') \left(\frac{\partial}{\partial v_s} f_1^\alpha(\boldsymbol{v}) + \frac{m_\alpha}{T_\beta} v_s f_1^\alpha(\boldsymbol{v}) \right). \tag{1.6}$$

We switched now to the more convenient notation

$$\boldsymbol{v}_1 \equiv \boldsymbol{v}, \quad \boldsymbol{v}_2 \equiv \boldsymbol{v}'$$

and introduced the abbreviation

$$A_{\alpha\beta} = \frac{2\pi e_\alpha^2 e_\beta^2 \ln \Lambda}{m_\alpha^2}. \tag{1.7}$$

The *field-particle collision term* is provided by the remaining two terms in (1.4), after an integration by parts,

$$\mathscr{K}_{\mathrm{F}}^{\alpha\beta} = -A_{\alpha\beta} \frac{\partial}{\partial v_r} f_0^\alpha(v) \left(\frac{m_\alpha}{T_\alpha} v_s + \frac{m_\alpha}{m_\beta} \frac{\partial}{\partial v_s} \right) \int \mathrm{d}\boldsymbol{v}' \; G_{rs}(\boldsymbol{v} - \boldsymbol{v}') \; f_1^\beta(\boldsymbol{v}') \tag{1.8}$$

This terminology goes back to the classical works of Chandrasekhar (1943) on the connection between the problem of Brownian motion and the dynamics of a stellar cluster (a system in many ways analogous to a plasma) (see also Spitzer 1967, Balescu 1963).

The *test-particle collision term* describes the motion of the particles α ("test-particles") through a fictitious medium of particles β which are (and remain) in thermal equilibrium $[f_1^\beta = 0]$. The motion of the test-particles is then a "pure" Brownian motion, described by a proper *Fokker–Planck equation* *:

$$\mathscr{K}_T^{\alpha\beta} = A_{\alpha\beta} \frac{\partial}{\partial v_r} \left(-\frac{m_\alpha}{T_\beta} F_r^\beta(\boldsymbol{v}) + D_{rs}^\beta(\boldsymbol{v}) \frac{\partial}{\partial v_s} \right) f_1^\alpha(\boldsymbol{v}). \tag{1.9}$$

This is a partial differential equation, whose coefficients are functions of $\boldsymbol{v}$. They are given the following traditional names:

– The *dynamical friction force*:

$$F_r^\beta(\boldsymbol{v}) = -v_s \int \mathrm{d}\boldsymbol{v}'\, G_{rs}(\boldsymbol{v}-\boldsymbol{v}')\, f_0^\beta(v'). \tag{1.10}$$

– The *diffusion tensor* (in velocity space):

$$D_{rs}^\beta(\boldsymbol{v}) = \int \mathrm{d}\boldsymbol{v}'\, G_{rs}(\boldsymbol{v}-\boldsymbol{v}')\, f_0^\beta(v'). \tag{1.11}$$

Some simple calculations (similar to those of Appendix **4A**.3) show that these two coefficients are of the form

$$F_r^\beta(\boldsymbol{v}) = -\frac{v_r}{v}\, \alpha_1^\beta(v),$$

$$D_{rs}^\beta(\boldsymbol{v}) = \left(\delta_{rs} - \frac{v_r v_s}{v^2} \right) \alpha_0^\beta(v) + \frac{v_r v_s}{v^3}\, \alpha_1^\beta(v), \tag{1.12}$$

where α_0^β and α_1^β are functions of the absolute value of the velocity

$$\alpha_0^\beta(v) = \frac{1}{2v^2} \int \mathrm{d}\boldsymbol{v}' \left(v^2 \delta_{rs} - v_r v_s \right) G_{rs}(\boldsymbol{v}-\boldsymbol{v}')\, f_0^\beta(v'),$$

$$\alpha_1^\beta(v) = \frac{1}{v} \int \mathrm{d}\boldsymbol{v}'\, v_r v_s\, G_{rs}(\boldsymbol{v}-\boldsymbol{v}')\, f_0^\beta(v'). \tag{1.13}$$

* We clearly see here why the often-used name "Fokker–Planck equation" given to the Landau equation is improper, *even in the linearized version* Only the test-particle part is a Fokker–Planck equation, the presence of a field-particle collision term introduces an important difference with ordinary Brownian motion theory

We will come back to these functions later [section 11.4 and Appendix 11A.1]. Let us make the following remark, in passing. As $F_r^\beta(\boldsymbol{v})$ and $D_{rs}^\beta(\boldsymbol{v})$ are non-linear functions of the velocity, eq. (1.9) is called a "non-linear Fokker–Planck equation" according to the standard terminology (van Kampen 1981).

We now substitute (1.12) into the test-particle collision term (1.9) and find

$$\mathscr{K}_{\mathrm{T}}^{\alpha\beta} = A_{\alpha\beta}\, v\, \alpha_0^\beta(v) \frac{\partial}{\partial v_r} G_{rs}(\boldsymbol{v}) \frac{\partial}{\partial v_s} f_1^\alpha(\boldsymbol{v})$$

$$+ A_{\alpha\beta} \frac{\partial}{\partial v_r} \frac{\alpha_1^\beta(v)}{v^3} v_r \left(\frac{m_\alpha}{T_\beta} v^2 + v_s \frac{\partial}{\partial v_s} \right) f_1^\alpha(\boldsymbol{v}). \tag{1.14}$$

This is a very remarkable result. It shows that one can isolate in all the Landau collision terms (i.e. for all pairs α, β) a first part which has exactly the *Lorentz form*, like the dominant term of the electron–ion collision term (2.8.11). In other words, this term describes a collision process in which a particle α collides with an infinitely heavy partner. This process is frequently called *pitch-angle scattering*. We insist on the fact that the form (1.14) holds for both like-particle and unlike-particle collisions, and that no approximation was made in its derivation.

The Lorentz operator has a particularly simple form; moreover, it will play a central role in neoclassical theory. However, it is not alone! The second, much more complicated term in (1.14) is, in general, not small. In addition, the complete collision term includes also the *field-particle collision term*, which is even more complicated. It accounts for the fact that the collision process cannot be described in terms of a set of test-particles moving in a permanent medium. As the test-particles evolve, they also change the state of the medium, i.e. of the "field-particles" which, after all, are of the same nature as the former. The field-particle term thus ensures the internal self-consistency of the theory. This term, however, is of a mathematical nature very different from a Fokker–Planck equation: the unknown function f_1^α appears here under an integral.

We now make the connection with our previous results (section 2.8) for the case of the unlike-particle collisions between electrons and ions. In this case, we know from (2.8.6) that there is a small parameter η in the problem,

$$\eta = \frac{m_e}{m_i} \frac{T_i}{T_e} \ll 1. \tag{1.15}$$

Considering first the electron–ion collisions ($\alpha = \mathrm{e}$, $\beta = \mathrm{i}$), we evaluate the coefficient $v\alpha_0^{\mathrm{i}}(v)$ appearing in the pitch-angle scattering operator. As we know from (2.8.8),

$$G_{rs}(\boldsymbol{v} - \boldsymbol{v}') = G_{rs}(\boldsymbol{v}) + \mathrm{O}(\eta).$$

Hence, (1.2) yields

$$\begin{aligned} v\alpha_0^{\mathrm{i}}(v) &= \frac{1}{2v}\left(v^2\delta_{rs} - v_r v_s\right)\int \mathrm{d}\boldsymbol{v}'\; G_{rs}(\boldsymbol{v} - \boldsymbol{v}')\; f_0^{\mathrm{i}}(v') \\ &\approx \frac{1}{2v}\left(v^2\delta_{rs} - v_r v_s\right) G_{rs}(\boldsymbol{v}) \int \mathrm{d}\boldsymbol{v}'\; f_0^{\mathrm{i}}(v') \\ &= \tfrac{1}{2} v\,\delta_{rs}\,\frac{\delta_{rs} - v_r v_s}{v^3}\, n_{\mathrm{i}} = n_{\mathrm{i}}. \end{aligned} \tag{1.16}$$

Hence, the first term of (1.14) agrees exactly with the first term of (2.8.11); the remaining terms of (1.14) are of order η.

Considering next the ion–electron collision term ($\alpha = \mathrm{i}$, $\beta = \mathrm{e}$), G_{rs} must be expanded as in (2.8.15); it is easily seen that the function $v\alpha_0^{\mathrm{e}}(v)$ is of order η. The leading term is actually the field-particle collision term (1.8), in agreement with (2.8.17). Thus, *the pitch-angle scattering term is negligibly small for the ion–electron collisions.*

The collision term of the kinetic equation possesses the basic *conservation properties* demonstrated in section 2.6: the conservation of the *number of particles* of each species, of the *total momentum* and of the *total energy.*

When discussing the *linearized* version of the collision operator, we may also say something about its eigenstates and eigenvalues. Their properties are not known, in general, except for a simple and important fact. Any physically correct collision operator must possess a fivefold degenerate eigenvalue equal to zero. The corresponding eigenfunctions are $f_0(v)$, $v_r f(v)$, $v^2 f_0(v)$. These eigenfunctions constitute the *null space* of the linearized collision operator (see Balescu 1975, Cercignani 1969).

The conservation properties and the null space are fundamental properties of any collision operator. Hence, these properties must be possessed by any acceptable approximate form of the collision operator.

11.2. Expansion of the linearized collision term

It was shown in section 11.1 that the collision term can always be written in the form of a *pitch-angle scattering (PAS) term*, plus a remainder. For

simplicity, we first discuss in detail the *like-particle collision term* (which is also the one that poses problems in neoclassical theory):

$$\mathscr{K}^{\alpha\alpha} = \mathscr{K}^{\alpha\alpha}_{\mathrm{PAS}} + \hat{\mathscr{K}}^{\alpha\alpha}, \tag{2.1}$$

where $\mathscr{K}^{\alpha\alpha}_{\mathrm{PAS}}$ is the first term of (1.14) and $\hat{\mathscr{K}}^{\alpha\alpha}$ contains the remainder of the linearized collision term. The PAS term has many simple properties; moreover, it is readily transformed into NGC variables (section 11.5). One would therefore be tempted to simply neglect the complicated $\hat{\mathscr{K}}^{\alpha\alpha}$. But, independently of any considerations of orders of magnitude, *the PAS term cannot be considered, by itself, as an acceptable collision term.* Indeed, it lacks one basic property: the *conservation of momentum.* A simple integration by parts, using (1.14), shows that

$$\int \mathrm{d}\boldsymbol{v}\, v_m \mathscr{K}^{\alpha\alpha}_{\mathrm{PAS}} = -2A_{\alpha\alpha}\int \mathrm{d}\boldsymbol{v}\, \frac{\alpha_0(v)}{v^2}\, v_m f_1^{\alpha}(\boldsymbol{v}) \neq 0. \tag{2.2}$$

In the first years of the neoclassical theory, the prevalent philosophy for curing this disease was the following. It was tried to add to the PAS operator a "simple" term $k^{\alpha\alpha}$, constructed in a more or less ad hoc manner, in such a way as to compensate the non-conservation of momentum. Thus

$$\mathscr{K}^{\alpha\alpha}_{\mathrm{mod}} = \mathscr{K}^{\alpha\alpha}_{\mathrm{PAS}} + k^{\alpha\alpha}, \tag{2.3}$$

and

$$\int \mathrm{d}\boldsymbol{v}\, v_m (\mathscr{K}^{\alpha\alpha}_{\mathrm{PAS}} + k^{\alpha\alpha}) = 0. \tag{2.4}$$

Several such models have been proposed in the neoclassical literature (Kovrizhnykh 1970, Rosenbluth et al. 1972, Furth et al. 1970, Hinton and Hazeltine 1976); an independent, and more general discussion of such models can be found in ch. 4 of Cercignani (1969).

The use of such models (even though they may give good practical approximations) was a basic weakness of the theory, for two reasons:

(1) The correction $k^{\alpha\alpha}$ is constructed in a completely ad hoc manner, and does not reflect at all the mechanism of the real collision process.

(2) The conservation of the momentum does not assure us at all that other moments are correctly, or even approximately well evaluated by eq. (2.3).

A much superior method for handling this problem was introduced by Hirshman and Sigmar (1976). It is not an ad hoc construct, but rather a systematic approximation procedure, based on an algorithm that can be carried out, in principle, as far as one wishes. At each step, all basic conditions for an approximate collision operator are satisfied (conservation properties, null space, ...); other advantages will appear below. *

We first do some simple transformations, going over to the dimensionless variables $\boldsymbol{c} = (m_\alpha/T_\alpha)^{1/2}\boldsymbol{v}$, use representation (4.3.8), and write $\alpha_1(v) \equiv n_\alpha \tilde{\alpha}_1(c)$. We then obtain

$$f_1^\alpha(\boldsymbol{v}) = n_\alpha \left(\frac{m_\alpha}{T_\alpha}\right)^{3/2} \phi^0(c)\ \chi(\boldsymbol{c}),$$

$$\hat{\mathscr{K}}^{\alpha\alpha} = n_\alpha^2 \left(\frac{m_\alpha}{T_\alpha}\right)^3 A_\alpha \hat{\mathbb{K}}\{\phi^0\chi\},$$

where the *operator* $\hat{\mathbb{K}}\{\phi^0\chi\}$, i.e. a *functional* of the function χ, is defined as

$$\hat{\mathbb{K}}\{\phi^0\chi\} = \frac{\partial}{\partial c_p}\left[\frac{\tilde{\alpha}_1(c)}{c^3} c_p \left(c^2 + c_s \frac{\partial}{\partial c_s}\right)\phi^0(c)\ \chi(\boldsymbol{c}) - \phi^0(c)\left(c_s + \frac{\partial}{\partial c_s}\right)\int d\boldsymbol{c}'\ G_{ps}(\boldsymbol{c}-\boldsymbol{c}')\ \phi^0(c')\ \chi(\boldsymbol{c}')\right]. \tag{2.5}$$

For simplicity, we temporarily suppress the superscripts α.

In the remainder of this section we deal with various types of functions and functionals; it is essential to have a clear understanding of these concepts. We therefore use in this chapter a conventional system of notations in order to characterize these various objects:

$f(\boldsymbol{c})$: a set of parentheses $(\ldots)$ denotes a *function* of $\boldsymbol{c}$ (i.e. a *number* whose value depends on the variable $\boldsymbol{c}$);

$\psi[f]$: a set of square brackets $[\ldots]$ denotes a *functional* of f (i.e. a *number* whose value depends on the choice of a function f);

$K\{f\}$: a set of curly brackets $\{\ldots\}$ denotes a *functional* of f that is also a *function* of $\boldsymbol{c}$ (i.e. a function of $\boldsymbol{c}$ whose form depends on the choice of the function f).

* The exposition of the method in the present section departs in detail from Hirshman and Sigmar. In particular, we use here irreducible Hermite polynomials, which we believe to result in a less heavy notation system than the Burnett functions used in the original paper.

Let us give two trivial examples of the last two objects,

$$\psi[f] = \int \mathrm{d}\boldsymbol{c}\, \mathrm{e}^{-c^2} f(c),$$

$$K\{f\} = \left(c^2 + \boldsymbol{c} \cdot \frac{\partial}{\partial \boldsymbol{c}}\right) f(\boldsymbol{c}).$$

When a *specific function* is chosen for f, a functional $K\{f\}$ becomes an ordinary function of $\boldsymbol{c}$,

$$K\{\mathrm{e}^{-c^2}\} = \left(c^2 + \boldsymbol{c} \cdot \frac{\partial}{\partial \boldsymbol{c}}\right) \mathrm{e}^{-c^2} = -c^2\, \mathrm{e}^{-c^2} \equiv g(\boldsymbol{c}).$$

When we try to approximate the operator $\hat{\mathscr{K}}^{\alpha\alpha}$, we immediately think of using some kind of expansion on an orthogonal basis. In the present case, the irreducible Hermite polynomials are naturally our choice. The application of such an expansion is, however, ambiguous in the present case. Two extreme possibilities suggest themselves.

In the first case, we *expand the distribution function* in a series of Hermite polynomials as in (4.3.11). For simplicity, we only consider the term involving *vector moments* in this equation. This is because we shall be mainly interested in calculating particle and heat fluxes, and because the isotropic linear operator $\hat{\mathscr{K}}^{\alpha\alpha}$ does not couple the vector fluxes to any other flux of different tensorial character. We thus write

$$\chi(\boldsymbol{c}) = \sum_{m=0}^{N} h_r^{(2m+1)}[\chi]\, H_r^{(2m+1)}(\boldsymbol{c}), \tag{2.6}$$

the expansion being truncated at some level $m = N \geqslant 0$ ($N = 1$ would correspond to the 13M approximation). Substituting into (2.5), we obtain

$$\hat{\mathbb{K}}\{\phi^0 \chi\} = \sum_{m=0}^{N} h_r^{(2m+1)}[\chi]\, \nu_r^{(2m+1)}(\boldsymbol{c})\, \phi^0(c). \tag{2.7}$$

This is an expansion of the collision operator in which:
– the constant coefficients $h_r^{(2m+1)}[\chi]$ are the (physically transparent) Hermitian moments, i.e. some functionals of the distribution function (see eq. 4.3.12);

– the functions of the velocity $\nu_r^{(2m+1)}(\boldsymbol{c})$ are defined as the value of the functional $\hat{\mathbb{K}}\{\phi^0\chi\}$ for $\chi = H_r^{(2m+1)}(\boldsymbol{c})$:

$$\phi^0(c)\nu_r^{(2m+1)}(\boldsymbol{c}) = \hat{\mathbb{K}}\left\{\phi^0 H_r^{(2m+1)}\right\}, \tag{2.8}$$

or, explicitly,

$$\begin{aligned}\phi^0(c)\ \nu_r^{(2m+1)}(\boldsymbol{c}) &= \frac{\partial}{\partial c_p}\left[\frac{\tilde{\alpha}_1(c)}{c^3}c_p\left(c^2 + c_s\frac{\partial}{\partial c_s}\right)\phi^0(c)\ H_r^{(2m+1)}(\boldsymbol{c})\right. \\ &\qquad \left. -\phi^0(c)\left(c_s + \frac{\partial}{\partial c_s}\right)\int d\boldsymbol{c}'\ G_{ps}(\boldsymbol{c}-\boldsymbol{c}')\ \phi^0(c')\ H_r^{(2m+1)}(\boldsymbol{c}')\right].\end{aligned} \tag{2.9}$$

We obtained an expansion (2.7) of the collision term in terms of a set of functions $\nu_r^{(2m+1)}(\boldsymbol{c})$ which not only are complicated, but are not mutually orthogonal. Moreover, it can be shown that the truncated expansion does not guarantee the satisfaction of the basic conservation laws. This is a major drawback.

Let us understand the difference between the present treatment and the treatment of chapter 4. One could say that eq. (2.7) is an "incomplete moment expansion". In chapter 4 we derived equations for the coefficients $h_r^{(2m+1)}$. The collisional contributions $Q_r^{(2n+1)}$ to these equations are given in table 4.6.2 in the form of a linear combination, $c_{2n+1,2m+1}h_r^{(2m+1)}$ (+nonlinear terms). This form is obtained from (2.7) by projection on $H_s^{(2n+1)}$,

$$c_{2n+1,2m+1} \sim \int d\boldsymbol{c}\ \phi^0(c)\ H_s^{(2n+1)}(\boldsymbol{c})\ \nu_r^{(2m+1)}(\boldsymbol{c}).$$

In the present chapter we do not take this extra step, because we need a modelization of the collision term as a function of $\boldsymbol{c}$. The Hermitian moments *per se* are not directly useful for the solution of the kinetic equation in the long mean free path regime, as will be seen in chapter 14.

Another, complementary way of implementing the idea is to *expand the collision operator* in a series of Hermite polynomials. We then write

$$\hat{\mathbb{K}}\{\phi^0\chi\} = \sum_{m=0}^{N} \eta_r^{(2m+1)}[\chi]\ \phi^0(c)\ H_r^{(2m+1)}(\boldsymbol{c}). \tag{2.10}$$

In this case the expansion basis is a set of properly orthogonal functions, whose coefficients are

$$\eta_r^{(2m+1)}[\chi] = \int \mathrm{d}\boldsymbol{c}\, H_r^{(2m+1)}(\boldsymbol{c})\, \hat{\mathbb{K}}\{\phi^0\chi\}. \tag{2.11}$$

It can be checked that the truncated expansion (2.10) satisfies the conservation laws, but does not preserve the correct null space of the exact linearized collision operator. Thus, (2.10) cannot be accepted as a collision model.

We now come to the method of Hirshman and Sigmar (1976), which is, in a sense, intermediate between the two previous ones. We shall introduce it "slowly", in order to let the reader appreciate its logic.

We start by defining a *sequence of functionals* $\Delta^{(n)}\{\phi^0\chi\}$ of an arbitrary function χ: this we do recursively. $\Delta^{(1)}\{\phi^0\chi\}$ is defined simply by the action of the collision operator $\hat{\mathbb{K}}$ on $\phi^0\chi$,

$$\Delta^{(1)}\{\phi^0\chi\} = \hat{\mathbb{K}}\{\phi^0\chi\}. \tag{2.12}$$

The next functional in the sequence is

$$\Delta^{(3)}\{\phi^0\chi\} = -\psi_r^{(1)}[\chi]\, \Delta^{(1)}\{\phi^0 H_r^{(1)}\} + \Delta^{(1)}\{\phi^0\chi\}, \tag{2.13}$$

where $\psi_r^{(1)}[\chi]$ is a functional of χ, to be defined below. The symbol $\Delta^{(1)}\{\phi^0 H_r^{(1)}\}$ denotes the *function* of $\boldsymbol{c}$ obtained by choosing specifically $\chi = H_r^{(1)}$ in (2.12); in other words, it is the result of the action of the collision operator on the Hermite polynomial $H_r^{(1)}(\boldsymbol{c})$: this is precisely the function $\nu_r^{(1)}(\boldsymbol{c})$ given in (2.8). It should be noted that $\Delta^{(1)}\{\phi^0 H_r^{(1)}\}$ is a *vector function* (because of the isotropy of $\hat{\mathbb{K}}$). [As usual, the summation over the repeated vector index r is understood in eq. (2.13).]

We now determine the functional $\psi_r^{(1)}[\chi]$ by requiring that *the function* $\Delta^{(3)}\{\phi^0\chi\}$, *for arbitrary* χ, *be orthogonal to the Hermite function* $\phi^0 H_s^{(1)}$. We note that the scalar product must be defined here in a slightly different way, compared to chapter 4 or General Appendix G1:

$$(A, B) = \int \mathrm{d}\boldsymbol{c}\, \frac{1}{\phi^0} A(\boldsymbol{c})\, B(\boldsymbol{c}). \tag{2.14}$$

This is because the functions $A(\boldsymbol{c})$, $B(\boldsymbol{c})$ considered here are of the form $\phi^0\chi$ rather than χ. This definition is, of course, consistent with the previous one:

$$(\phi^0\chi, \phi^0\chi') = \int \mathrm{d}\boldsymbol{c}\, \frac{1}{\phi^0}\phi^0\chi\phi^0\chi' = \int \mathrm{d}\boldsymbol{c}\, \phi^0\chi\chi' = (\chi, \chi').$$

The orthogonality requirement is obtained by using eqs. (2.12)–(2.14),

$$\begin{aligned}\left(\phi^0 H_s^{(1)},\ \Delta^{(3)}\{\phi^0\chi\}\right) &\equiv \int \mathrm{d}\boldsymbol{c}\,\frac{1}{\phi^0}\,\phi^0 H_s^{(1)}\,\Delta^{(3)}\{\phi^0\chi\}\\ &= -\psi_r^{(1)}[\chi]\int \mathrm{d}\boldsymbol{c}\,H_s^{(1)}\,\hat{\mathbb{K}}\{\phi^0 H_r^{(1)}\}\\ &\quad + \int \mathrm{d}\boldsymbol{c}\,H_s^{(1)}\,\hat{\mathbb{K}}\{\phi^0\chi\} = 0. \end{aligned} \tag{2.15}$$

Because of the isotropy of the collision operator, the tensor appearing as the coefficient of $\psi_r^{(1)}[\chi]$ is diagonal,

$$\int \mathrm{d}\boldsymbol{c}\,H_s^{(1)}\,\hat{\mathbb{K}}\{\phi^0 H_r^{(1)}\} = \delta_{rs}\,\tfrac{1}{3}\int \mathrm{d}\boldsymbol{c}\,H_m^{(1)}\,\hat{\mathbb{K}}\{\phi^0 H_m^{(1)}\}. \tag{2.16}$$

Hence, the orthogonality condition is ensured by the choice

$$\psi_s^{(1)}[\chi] = \frac{\int \mathrm{d}\boldsymbol{c}\,H_s^{(1)}\,\hat{\mathbb{K}}\{\phi^0\chi\}}{\tfrac{1}{3}\int \mathrm{d}\boldsymbol{c}\,H_m^{(1)}\,\hat{\mathbb{K}}\{\phi^0 H_m^{(1)}\}}. \tag{2.17}$$

The next functional in the sequence is defined by analogy with (2.13),

$$\Delta^{(5)}\{\phi^0\chi\} = -\psi_r^{(3)}[\chi]\ \Delta^{(3)}\{\phi^0 H_r^{(3)}\} + \Delta^{(3)}\{\phi^0\chi\}, \tag{2.18}$$

where $\psi_r^{(3)}[\chi]$ is determined by requiring the orthogonality of the function $\Delta^{(5)}\{\phi^0\chi\}$ to $\phi^0 H_s^{(3)}$. It is then checked that $\Delta^{(5)}\{\phi^0\chi\}$ is automatically orthogonal to $\phi^0 H_s^{(1)}$ as well. We now come to the general definition of the functionals Δ,

$$\Delta^{(2n+1)}\{\phi^0\chi\} = -\psi_r^{(2n-1)}[\chi]\ \Delta^{(2n-1)}\{\phi^0 H_r^{(2n-1)}\} + \Delta^{(2n-1)}\{\phi^0\chi\}. \tag{2.19}$$

The coefficients are given by

$$\psi_r^{(m)}[\chi] = \frac{\int \mathrm{d}\boldsymbol{c}\,H_r^{(m)}(\boldsymbol{c})\ \Delta^{(m)}\{\phi^0\chi\}}{\tfrac{1}{3}\int \mathrm{d}\boldsymbol{c}\,H_s^{(m)}(\boldsymbol{c})\ \Delta^{(m)}\{\phi^0 H_s^{(m)}\}}. \tag{2.20}$$

With this choice, we have the fundamental orthogonality property

$$\left(\phi^0 H_r^{(2j+1)},\ \Delta^{(2n+1)}\{\phi^0\chi\}\right) = 0, \quad j < n. \tag{2.21}$$

The recursive proof of this property is left to the reader.

Having defined the set of functionals $\Delta^{(n)}\{\phi^0\chi\}$, we define the set of functions $G_r^{(n)}(\boldsymbol{c})$ as the values of the functionals $\Delta^{(n)}$ for $\chi = H_r^{(n)}(\boldsymbol{c})$,

$$G_r^{(n)}(\boldsymbol{c}) = \frac{1}{\phi^0}\,\Delta^{(n)}\{\phi^0 H_r^{(n)}\}. \tag{2.22}$$

These will be the basis functions for the expansion of the collision term. We postulate the following form for an approximation of order $2N+1$ to the collision operator $\mathbb{K}\{\phi^0\chi\}$:

$$\hat{\mathbb{K}}^{(2N+1)}\{\phi^0\chi\} = \sum_{n=0}^{N} \psi_r^{(2n+1)}[\chi]\ \phi^0\, G_r^{(2n+1)}(\boldsymbol{c}). \tag{2.23}$$

We shall explore the properties of this approximate collision operator in the next section.

11.3. Properties of the basis functions

The construction of the functions $G_r^{(2n+1)}(\boldsymbol{c})$ incorporates two essential features in a non-trivial way: the collision operator $\hat{\mathbb{K}}$ and the irreducible Hermite polynomials $H_r^{(2n+1)}$. The functions $G_r^{(2n+1)}(\boldsymbol{c})$ are *adapted to the specific collision operator*. This appears implicitly in the definition of the functionals $\Delta^{(n)}\{f\}$ which are constructed recursively from an initial building stone, which is none other than the collision operator (2.12). At the end of this section we shall make the connection between $G_r^{(2n+1)}$ and the collision term more explicit. Presently, we wish to explore the properties of the functions $G_r^{(2n+1)}$, thus showing why they are valuable in modelling the collision term.

We first note a direct consequence of relation (2.21), which follows from setting $\chi = H_r^{(2n+1)}$ and using (2.22),

$$\left(\phi^0 H_r^{(2j+1)},\ \phi^0 G_r^{(2n+1)}\right) = 0, \quad j < n, \tag{3.1}$$

or, more explicitly,

$$\int \mathrm{d}\boldsymbol{c}\ \phi^0(c)\ H_r^{(2j+1)}(\boldsymbol{c})\ G_r^{(2n+1)}(\boldsymbol{c}) = 0, \quad j < n. \tag{3.2}$$

This is a very interesting *one-sided orthogonality* property: the functions $G_r^{(2n+1)}$ are orthogonal to all Hermite polynomials *of rank lower than their own*, but not to those of higher rank. This property has a quite remarkable consequence for the value of the modelization (2.23). It concerns the *generalized friction coefficients* Q_r, i.e. the collisional contributions to the moment equations (**4**.5.22)–(**4**.5.24), defined by (**4**.5.21). We thus formulate:

Corollary 1. *The generalized friction coefficients of rank* $2j+1 \leqslant 2N+1$ *are evaluated exactly by the truncated collision term* (2.23),

$$nQ_r^{(2j+1)} \equiv \int \mathrm{d}\boldsymbol{c}\, H_r^{(2j+1)}\, \hat{\mathbb{K}}\{\phi^0 \chi\}$$

$$= \int \mathrm{d}\boldsymbol{c}\, H_r^{(2j+1)}\, \hat{\mathbb{K}}^{(2N+1)}\{\phi^0 \chi\}, \quad j \leqslant N. \tag{3.3}$$

The proof is very simple:

$$nQ_r^{(2j+1)} = \sum_{n=0}^{\infty} \int \mathrm{d}\boldsymbol{c}\, H_r^{(2j+1)}\, \psi_s^{(2n+1)}\, G_s^{(2n+1)}\, \phi^0$$

$$= \sum_{n=0}^{j} \int \mathrm{d}\boldsymbol{c}\, H_r^{(2j+1)}\, \psi_s^{(2n+1)}\, G_s^{(2n+1)}\, \phi^0.$$

Because of the one-sided orthogonality (3.2), only the terms up to j in (2.23) contribute to the friction coefficient $Q_r^{(2n+1)}$. An immediate consequence of corollary 1 is

Corollary 2. *Momentum conservation is automatically ensured by all truncated collision models* (2.23), *for* $N \geqslant 0$.

Indeed, momentum conservation is a property of the friction coefficient $Q_r^{(1)}$, which is already exactly evaluated with $N = 0$ *.

* The thoughtful reader has not forgotten that the properties under discussion here are those of the collision operator $\hat{\mathbb{K}}\{\phi^0\chi\}$, which is the remainder of the total collision operator after subtraction of the PAS term (see eq. 2.1) The coefficient $Q_r^{(1)}$ is thus different from zero, and cancels exactly the contribution of $Q_r^{(1)}$ from the PAS term, evaluated in (2.2)

We now look at a different type of properties. Consider the sequence of functionals $\Delta^{(n)}\{\phi^0\chi\}$, evaluated for $\chi = H_r^{(1)}$. Combining definitions (2.12) and (2.22), we find, for $n = 1$,

$$\Delta^{(1)}\{\phi^0 H_r^{(1)}\} = \hat{\mathbb{K}}\{\phi^0 H_r^{(1)}\} \equiv G_r^{(1)}. \tag{3.4}$$

For $n = 3$, the result is calculated for (2.13) and (2.17),

$$\Delta^{(3)}\{\phi^0 H_r^{(1)}\} = -\frac{\int \mathrm{d}\boldsymbol{c}\, H_s^{(1)}\hat{\mathbb{K}}\{\phi^0 H_r^{(1)}\}}{\frac{1}{3}\int \mathrm{d}\boldsymbol{c}\, H_m^{(1)}\hat{\mathbb{K}}\{\phi^0 H_m^{(1)}\}}\hat{\mathbb{K}}\{\phi^0 H_s^{(1)}\} + \hat{\mathbb{K}}\{\phi^0 H_r^{(1)}\}.$$

But, because of (2.16), the right-hand side is strictly null. This argument is very simply generalized, step by step, with the result

$$\Delta^{(2m+1)}\{\phi^0 H_r^{(2j-1)}\} = 0, \quad j \leqslant m. \tag{3.5}$$

As an immediate consequence, we derive from (2.19) the following property of the expansion coefficients in the collision operator (2.23):

$$\psi_r^{(2m+1)}[H_s^{(2j-1)}] = 0, \quad j \leqslant m. \tag{3.6}$$

This leads us to

Corollary 3. *Consider a distribution function of the form*

$$f^{[2N+1]} = \phi^0 \sum_{r=0}^{N} h_r^{(2n+1)} H_r^{(2n+1)}(\boldsymbol{c}) \equiv \phi^0 \chi^{[2N+1]}. \tag{3.7}$$

Such an Ansatz is basic to the moment method of chapter 4; $N = 1$ corresponds to the 13M approximation, $N = 2$ to the 21M approximation, etc.) *The ollisional evolution of such a function is described exactly by a collision term of he form (2.23), truncated at the level N*:

$$\hat{\mathbb{K}}^{[2N+1]}\{\phi^0\chi^{[2N+1]}\} = \hat{\mathbb{K}}\{\phi^0\chi^{[2N+1]}\}. \tag{3.8}$$

Indeed, the complete collision term can be written as

$$\hat{\mathbb{K}}\{\phi^0\chi^{[2N+1]}\} = \sum_{n=0}^{N} \psi_r^{(2n+1)}[\chi^{[2N+1]}]\,\phi^0 G_r^{(2n+1)} + \sum_{n=1}^{\infty} \psi_r^{(2(N+n)+1)}[\chi^{[2N+1]}]\,\phi^0 G_r^{(2(N+n)+1)}. \tag{3.9}$$

Because of the form (3.7) and property (3.6), all the expansion coefficients ψ in the second term are strictly null. We finally quote the following additional property

Corollary 4. *For any self-adjoint collision operator* $\hat{\mathbb{K}}$, *the approximate representation* (2.23) *is a self-adjoint operator for every* $N \geqslant 0$.

The proof of this statement is left to the reader.

As announced at the beginning of this section, we now derive a more explicit form for the functions $G_r^{(n)}$. From (2.22) and (2.13) we find

$$G_r^{(3)} = -\psi_n^{(1)}\left[\phi^0 H_r^{(3)}\right] G_n^{(1)} + \left(\phi^0\right)^{-1} \hat{\mathbb{K}}\left\{\phi^0 H_r^{(3)}\right\}, \tag{3.10}$$

with

$$\psi_n^{(1)}\left[\phi^0 H_r^{(3)}\right] = \frac{\int \mathrm{d}\boldsymbol{c}\ \phi^0\, H_r^{(3)}\, G_n^{(1)}}{\frac{1}{3}\int \mathrm{d}\boldsymbol{c}\ \phi^0\, H_m^{(1)}\, G_m^{(1)}} \equiv \delta_{nr}\, \xi_{(1)}^{(3)},$$

where we used an isotropy argument similar to (2.16), as well as the self-adjointness of the collision operator. Thus

$$\xi_{(1)}^{(3)} = \frac{\int \mathrm{d}\boldsymbol{c}\, \phi^0\, H_n^{(3)}\, G_n^{(1)}}{\int \mathrm{d}\boldsymbol{c}\, \phi^0\, H_m^{(1)}\, G_m^{(1)}}. \tag{3.11}$$

Hence, (3.10) reduces to

$$G_r^{(3)} = -\xi_{(1)}^{(3)} G_r^{(1)} + \left(\phi^0\right)^{-1} \hat{\mathbb{K}}\left\{\phi^0 H_r^{(3)}\right\}.$$

The last term is again identical to the function $\nu_r^{(3)}$ of (2.8). This calculation can be continued, and one easily obtains the following general form of the functions $G_r^{(n)}$:

$$G_r^{(1)} = \nu_r^{(1)},$$

$$G_r^{(2n+1)} = \nu_r^{(2n+1)} - \sum_{j=0}^{n-1} \xi_{(2j+1)}^{(2n+1)}\, G_r^{(2j+1)}, \tag{3.12}$$

with:

$$\xi_{(2j+1)}^{(2n+1)} = \frac{\int \mathrm{d}\boldsymbol{c}\ \phi^0 H_n^{(2n+1)} G_n^{(2j+1)}}{\int \mathrm{d}\boldsymbol{c}\ \phi^0 H_m^{(2j+1)} G_m^{(2j+1)}}. \tag{3.13}$$

Thus, the functions $G_r^{(2n+1)}$ are directly related to the functions $\nu_r^{(2n+1)}$ appearing in expansion (2.7) of the collision operator, in which the coefficients are the ordinary Hermitian moments $h_r^{(2n+1)}[\chi]$. It was stated at that point that a major disadvantage of the latter expansion was the lack of any orthogonality properties of the functions $\nu_r^{(2n+1)}$. The Hirshman–Sigmar pro-

Table 3 1

The Hirshman–Sigmar representation of the collision term.

Basis functions $G_r^{(2n+1)}(\boldsymbol{c})$

$$\phi^0 G_r^{(1)}(\boldsymbol{c}) = \hat{\mathbb{K}}\{\phi^0 H_r^{(1)}(\boldsymbol{c})\}$$

$$\phi^0 G_r^{(2n+1)}(\boldsymbol{c}) = \hat{\mathbb{K}}\{\phi^0 H_r^{(2n+1)}(\boldsymbol{c})\} - \sum_{j=0}^{n-1} \xi_{(2j+1)}^{(2n+1)} \phi^0 G_r^{(2j+1)}(\boldsymbol{c})$$

$$\xi_{(2j+1)}^{(2n+1)} = \frac{\int \mathrm{d}\boldsymbol{c}\ \phi^0 H_n^{(2n+1)} G_n^{(2j+1)}}{\int \mathrm{d}\boldsymbol{c}\ \phi^0 H_m^{(2j+1)} G_m^{(2j+1)}}$$

One-sided orthogonality

$$\int \mathrm{d}\boldsymbol{c}\ \phi^0(c)\ H_r^{(2j+1)}(\boldsymbol{c})\ G_r^{(2n+1)}(\boldsymbol{c}) = 0, \quad j \leqslant n$$

Collision term representation

$$\hat{\mathbb{K}}^{(2N+1)}\{\phi^0 \chi\} = \sum_{j=0}^{N} \psi_r^{(2j+1)}[\chi]\ \phi^0 G_r^{(2j+1)}(\boldsymbol{c})$$

$$\psi_r^{(2j+1)}[\chi] = \frac{\int \mathrm{d}\boldsymbol{c}\ \phi^0(c)\ \chi(\boldsymbol{c})\ G_r^{(2j+1)}(\boldsymbol{c})}{\frac{1}{3}\int \mathrm{d}\boldsymbol{c}\ \phi^0(c)\ H_m^{(2j+1)}(\boldsymbol{c})\ G_m^{(2j+1)}(\boldsymbol{c})}$$

cedure cures this disadvantage as much as possible. The form (3.12) of the basis functions $G_r^{(2n+1)}$ is reminiscent of the Hilbert–Schmidt orthogonalization procedure. The result is the one-sided orthogonality property (3.2) relating the functions $G_r^{(2n+1)}$ to the Hermite polynomials $H_r^{(2n+1)}$ and, as a consequence, the validity of the four corollaries derived in the present section.

The modelization procedure described in sections 11.2 and 11.3 was developed in detail for the case of like-particle collision operator. Its extension to the unlike-particle collisions is straightforward. Also, the inclusion of basis functions of different tensorial nature is elementary.

For the reader's convenience, the main results of this discussion are summed up in table 3.1.

11.4. The approximate collision operator

After the general digression of sections 11.2 and 11.3, we come back to our main goal, which is the derivation of a useful approximation of the drift kinetic equation collision term.

Summarizing the previous discussions, we write the linearized collision terms in a form displaying explicitly the simple and important pitch-angle scattering (PAS) operator,

$$\mathbb{K}^{\alpha}\{f\} = \mathbb{K}^{\alpha}_{\mathrm{PAS}}\{f\} + \hat{\mathbb{K}}^{\alpha}\{f\}. \tag{4.1}$$

Each of these terms contains two parts corresponding, respectively, to collisions of particles of species α with electrons and with ions,

$$\mathbb{K}^{\alpha}_{\mathrm{PAS}}\{f\} = \mathbb{K}^{\alpha e}_{\mathrm{PAS}}\{f\} + \mathbb{K}^{\alpha i}_{\mathrm{PAS}}\{f\}, \tag{4.2}$$

and a similar formula for $\hat{\mathbb{K}}^{\alpha}$.

We first concentrate on the first term in (4.1). It is customary to write it in terms of the dimensionless pitch-angle scattering operator $\mathscr{L}$, defined as

$$\mathscr{L} = \frac{1}{2}\frac{\partial}{\partial v_r}\left(v^2\delta_{rs} - v_r v_s\right)\frac{\partial}{\partial v_s}. \tag{4.3}$$

As a first, fundamental property of this operator, we note that, for any function $g(v)$ of the absolute value of the velocity,

$$\mathscr{L}g(v) = 0. \tag{4.4}$$

This follows immediately from property (2.7.8) of the Landau tensor. As a

result, any function of v can be freely commuted through the derivatives; we thus find, from (1.14),

$$\mathbb{K}^{\alpha\beta}_{\mathrm{PAS}}\{f\} = 2A_{\alpha\beta}\frac{\alpha_0^{\alpha\beta}(v)}{v^2}\mathscr{L}f^{\alpha}. \tag{4.5}$$

The factor in front of $\mathscr{L}$ is a function of the absolute value of the velocity, having the dimension of an inverse time: it will be called, quite appropriately, the *collision frequency of the α–β process*, $\nu^{\alpha\beta}(v)$. This function can be calculated explicitly. Considering first the unlike-particle collisions, we have seen in (1.16) that $v^{-2}\,\alpha_0^{\mathrm{ei}} \sim n_{\mathrm{i}}v^{-3}$, whereas α_0^{ie} can be neglected. Collecting now the constants (1.7) and recalling definitions (**4**.6.4) and (**4**.6.10) of the electron and ion relaxation times, we find

$$\nu^{\mathrm{ei}}(v) = 3\sqrt{\frac{\pi}{2}}\,\frac{1}{\tau_{\mathrm{e}}}\left(\frac{T_{\mathrm{e}}}{m_{\mathrm{e}}}\right)^{3/2}\frac{1}{v^3}, \tag{4.6}$$

and

$$\nu^{\mathrm{ie}}(v) = 0. \tag{4.7}$$

The calculation of the like-particle collision frequencies is deferred to Appendix 11A.1. They are both expressed in terms of the *Chandrasekhar function*, $\mathscr{H}(c)$, which, in turn, is related to the *error function* $\Phi(c)$:

$$\mathscr{H}(c) = \frac{1}{2c^2}\left[(2c^2-1)\Phi(c) + c\,\Phi'(c)\right], \tag{4.8}$$

with

$$\Phi(c) = 2\pi^{-1/2}\int_0^c \mathrm{d}t\ \mathrm{e}^{-t^2}. \tag{4.9}$$

The function $\mathscr{H}(c)$ behaves, for small and large c, respectively, as

$$\mathscr{H}(c) \cong 2\pi^{-1/2}\left(\tfrac{2}{3}c - \tfrac{2}{15}c^3 + \tfrac{1}{35}c^5 + \cdots\right), \tag{4.10}$$

$$\mathscr{H}(c) \cong 1 - \frac{1}{2c^2} + \pi^{-1/2}\,\mathrm{e}^{-c^2}\left(\frac{1}{c^3} - \frac{1}{c^5} + \cdots\right). \tag{4.11}$$

The function $\mathscr{H}(c)$ is plotted in fig. 4.1.

The result for the collision frequencies is

$$\nu^{ee}(v) = 3\sqrt{\frac{\pi}{2}}\,\frac{1}{Z\tau_e}\left(\frac{T_e}{m_e}\right)^{3/2}\frac{1}{v^3}\,\mathscr{H}\left(\sqrt{\frac{m_e}{2T_e}}\,v\right), \tag{4.12}$$

$$\nu^{ii}(v) = 3\sqrt{\frac{\pi}{2}}\,\frac{1}{\tau_i}\left(\frac{T_i}{m_i}\right)^{3/2}\frac{1}{v^3}\,\mathscr{H}\left(\sqrt{\frac{m_i}{2T_i}}\,v\right). \tag{4.13}$$

We note, in particular, the relation

$$\nu^{ee}(v) = \frac{1}{Z}\,\mathscr{H}\left(\sqrt{\frac{m_e}{2T_e}}\,v\right)\nu^{ei}(v). \tag{4.14}$$

This relation exhibits the fact, already familiar from (4.6.31), that for highly charged ions ($Z \gg 1$), the electron–ion collisions dominate over the electron–electron collisions. Moreover, because of the form of the function $\mathscr{H}(c)$, the electron–electron collision frequency is always smaller than the electron–ion collision frequency (for the same value of v), even for $Z = 1$: this can be seen in fig. 4.2.

We now discuss the remainders $\hat{\mathbb{K}}$ of the collision terms. These terms will be modelled by the Hirshman–Sigmar method, in the form (2.23), truncated at

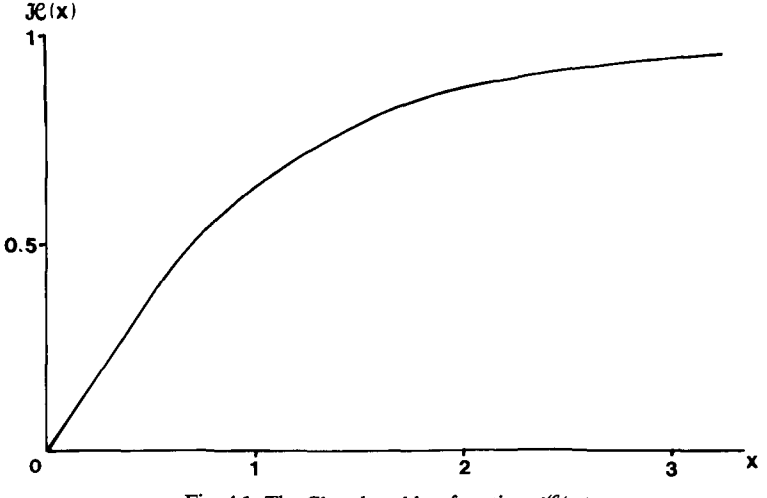

Fig. 4.1. The Chandrasekhar function $\mathscr{H}(x)$.

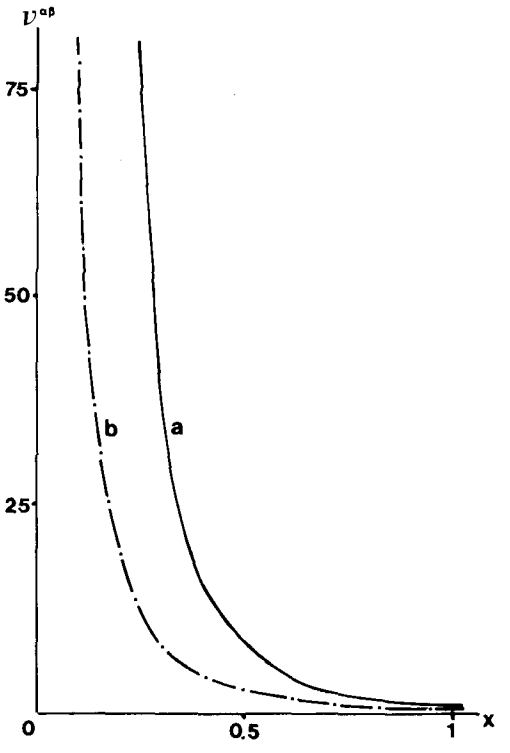

Fig. 4.2 The collision frequencies $\nu^{\alpha\beta}$. (a) electron–ion collision frequency $\nu^{ei} \sim x^{-3}$; (b) electron–electron collision frequency $\nu^{ee} \sim x^{-3}\mathscr{G}(x)$.

some appropriate value of N. It will turn out that their detailed form will be irrelevant for the specific problems treated in the neoclassical transport theory. Only a few general features will be of importance. In particular, we note that all the functions $G_r^{(n)}(\boldsymbol{c})$ are *vectors*. As they depend on the single vectorial variable $\boldsymbol{c}$, they are necessarily of the form

$$G_r^{(n)}(\boldsymbol{c}) = c_r \gamma^{(n)}(c), \tag{4.15}$$

where $\gamma^{(n)}(c)$ is a scalar function of the absolute value of the velocity. Hence, (2.23) can be written in the form

$$\hat{\mathbb{K}}^{(2N+1)}\{f\} = c_r n_r(c)\ \phi^0(c), \tag{4.16}$$

where the vector $n_r(c)$ depends only on the absolute value of $\boldsymbol{c}$. Going back to the dimensional variables $\boldsymbol{v}$, we collect all the results of our discussion in the following definitive form of the total collision terms *:

$$\mathbb{K}^{\mathrm{e}} f_1^{\mathrm{e}} = \left\{ \nu^{\mathrm{ee}}(v) + \nu^{\mathrm{ei}}(v) \right\} \mathscr{L} f_1^{\mathrm{e}}(\boldsymbol{v}) + v_r \mathscr{N}_r^{\mathrm{e}}(v)\, f_0^{\mathrm{e}}(v) \tag{4.17}$$

$$\mathbb{K}^{\mathrm{i}} f_1^{\mathrm{i}} = \nu^{\mathrm{ii}}(v)\, \mathscr{L} f_1^{\mathrm{i}}(\boldsymbol{v}) + v_r \mathscr{N}_r^{\mathrm{i}}(v)\, f_0^{\mathrm{i}}(v). \tag{4.18}$$

It must be kept in mind that the functions $\mathscr{N}_r^{\alpha}(v)$ are functionals of both $f_1^{\mathrm{e}}(\boldsymbol{v})$ and $f_1^{\mathrm{i}}(\boldsymbol{v})$ through the generalized moments $\psi_r^{(n)}[f]$ of (2.12).

11.5. Explicit form of the drift kinetic equation

We are now ready for the final transformations leading us to the drift kinetic equation (**10**.4.22). These imply two steps:

(1) Transformation of the collision terms (4.17), (4.18) to NGC variables;

(2) Averaging of the result over the gyrophase ϕ.

We choose as NGC variables the set ($\mathscr{E}$, M, ϕ) defined in table **1**.8.4 [and used in eq. (**10**.4.22)].

We note that the last term in (**10**.4.22) $\mathscr{K}_1^{\alpha}\{f_0, f_0\}$ can be omitted for the reasons explained in section 11.1 (point A). Hence, we are left with the problem of evaluating $\mathscr{K}_0^{\alpha}\{f_0, f_1\}$, i.e. the linearized collision operator, whose form is needed to zeroth order in ϵ.

We first transform the pitch-angle scattering (PAS) operator $\mathscr{L}$. From (**1**.8.3–**1**.8.5), together with (**1**.8.19), (**1**.8.20), we find after a somewhat lengthy calculation,

$$\frac{\partial}{\partial v_r} = \left[m_\alpha U_\alpha b_r + \sqrt{2m_\alpha}\sqrt{BM}\, n_{1r}(\phi) \right] \frac{\partial}{\partial \mathscr{E}}$$

$$+ \sqrt{\frac{2m_\alpha M}{B}}\, n_{1r}(\phi) \frac{\partial}{\partial M} - \sqrt{\frac{m_\alpha}{2MB}}\, n_{2r}(\phi) \frac{\partial}{\partial \phi} + \mathrm{O}(\epsilon). \tag{5.1}$$

* From here on, we abandon the functional notation $\mathbb{K}\{f\}$ of sections 11 2, 11.3 in favour of the lighter operator notation $\mathbb{K}f$.

(All the notations were defined in chapter 1.) On the other hand, the Landau tensor transforms as

$$G_{rs}(v) \equiv \frac{v^2\delta_{rs} - v_r v_s}{v^3}$$

$$= \sqrt{(m_\alpha/2)}\,(\mathscr{E} - e_\alpha\Phi)^{-3/2}$$

$$\times \Big\{(\mathscr{E} - e_\alpha\Phi)\delta_{rs} - (\mathscr{E} - MB - e_\alpha\Phi)b_r b_s - MB\,n_{1r}(\phi)\; n_{1s}(\phi)$$

$$- \sigma\sqrt{MB(\mathscr{E} - MB - e_\alpha\Phi)}\,\big[n_{1r}(\phi)b_s + n_{1s}(\phi)b_r\big]\Big\}. \tag{5.2}$$

We also recall the definition of the parallel velocity U_α of species α, expressed as a function of $\mathscr{E}$, M, $\boldsymbol{Y}$,

$$U_\alpha = \sigma\sqrt{(2/m_\alpha)\big[\mathscr{E} - MB(\boldsymbol{Y}) - e_\alpha\Phi(\boldsymbol{Y})\big]}\,. \tag{5.3}$$

We substitute (5.1), (5.2) into (4.3) and find, after a lengthy calculation,

$$\mathscr{L}^\alpha f_1^\alpha = \frac{m_\alpha}{B}\left\{U_\alpha\frac{\partial}{\partial M}\,U_\alpha M\,\frac{\partial}{\partial M} + \frac{1}{2m_\alpha}\,\frac{\mathscr{E} - e_\alpha\Phi}{M}\,\frac{\partial^2}{\partial\phi^2}\right\} f_1^\alpha(\mathscr{E},\, M,\, \phi). \tag{5.4}$$

Given the complication of (5.1), (5.2), this is a remarkably simple form.

The collision frequencies are simply expressed in terms of NGC variables. Thus, from (4.6) we get

$$\nu^{\mathrm{ei}}(\mathscr{E}) = \frac{3\sqrt{\pi}}{4}\,\frac{1}{\tau_{\mathrm{e}}}\,T_{\mathrm{e}}^{3/2}(\mathscr{E} + e\Phi)^{-3/2}, \tag{5.5}$$

and similar expressions for ν^{ee}, ν^{ii}.

Consider now the additional, modelled terms in the kinetic equations (4.17), (4.18). We recall relation (**1**.4.13),

$$v_r = v_\parallel b_r + v_\perp n_{1r}(\varphi), \tag{5.6}$$

which, to zeroth order in ϵ, becomes

$$v_r = U_\alpha b_r + \sqrt{(2/m_\alpha)MB}\,n_{1r}(\phi). \tag{5.7}$$

Hence, as the vectors $\mathcal{N}_r^\alpha$ only depend on the absolute value of v, we get

$$v_r \mathcal{N}_r^\alpha(v) = U_\alpha \mathcal{N}_\parallel^\alpha(\mathscr{E}) + \sqrt{(2/m_\alpha) MB}\, \boldsymbol{n}_1 \cdot \mathcal{N}_\perp^\alpha(\mathscr{E}). \tag{5.8}$$

Before leaving this point, we make an additional remark, which is important. The vectors $\mathcal{N}_r^\alpha$ are functionals of the distribution function through the generalized moments $\psi_r^{(n)}$ (table 3.1). We know that in deriving the drift kinetic equation an important part is played by the decomposition (**10**.4.4) of the distribution function into averaged and oscillating parts:

$$f_1(\mathscr{E}, M, \phi) = \bar{f}_1(\mathscr{E}, M) + \tilde{f}_1(\mathscr{E}, M, \phi). \tag{5.9}$$

Using this equation, together with (5.7) and (4.15), we get

$$\begin{aligned}
\psi_r^{(n)} &\sim \int \mathrm{d}\boldsymbol{v} f_1 G_r^{(n)} = \int \mathrm{d}\boldsymbol{v} f_1 v_r \gamma^{(n)}(v) \\
&= \int \mathrm{d}\boldsymbol{v} \left[\bar{f}_1 + \tilde{f}_1(\phi)\right]\left(v_\parallel b_r + v_\perp n_{1r}(\phi)\right) \gamma^{(n)}(v) \\
&= b_r \int \mathrm{d}\boldsymbol{v} \bar{f}_1 v_\parallel \gamma^{(n)}(v) + \int \mathrm{d}\boldsymbol{v} \tilde{f}_1(\phi)\, v_\perp n_{1r}(\phi)\, \gamma^{(n)}(v).
\end{aligned} \tag{5.10}$$

Thus, in going over to NGC variables, the parallel component of the vector $\psi^{(n)}$ becomes

$$\psi_\parallel^{(n)} = \int \mathrm{d}\boldsymbol{v} \bar{f}_1(\mathscr{E}, M)\, U_\alpha(\mathscr{E}, M)\, \gamma^{(n)}(\mathscr{E}). \tag{5.11}$$

We thus obtained the important fact that the *parallel* component $\psi_\parallel^{(n)}$ is (to leading order in ϵ) a functional of the *average distribution function* $\bar{f}_1$ *alone*, whereas the perpendicular component involves the oscillating part $\tilde{f}_1$.

We are now ready for the final gyro-averaging operation, which is trivially simple. Considering first the PAS terms, we have

$$\overline{\nu(\mathscr{E})\, \mathscr{L} f} = \nu \frac{m}{B} \left\{ U \frac{\partial}{\partial M} UM \frac{\partial}{\partial M} \right\} \bar{f} + 0. \tag{5.12}$$

This simple result comes from the fact that all coefficients in (5.4) are

independent of the gyrophase ϕ. The remaining terms (5.8) are just as simply averaged:

$$\begin{aligned}\overline{v_r \mathcal{N}_r(v)\, f_0(v)} &= \overline{U\mathcal{N}_{\parallel}(\mathscr{E})\, f_0} + \overline{\sqrt{(2/m)MB}\, \boldsymbol{n}_1(\phi)\cdot \boldsymbol{\mathcal{N}}_{\perp}(\mathscr{E})\, f_0} \\ &= U\mathcal{N}_{\parallel}(\mathscr{E})\, f_0 + \sqrt{(2/m)MB}\, \overline{\boldsymbol{n}_1(\phi)}\cdot \boldsymbol{\mathcal{N}}_{\perp}(\mathscr{E})\, f_0 \\ &= U\mathcal{N}_{\parallel}(\mathscr{E})\, f_0, \end{aligned} \tag{5.13}$$

because $\overline{\boldsymbol{n}_1(\phi)} = 0$.

These results can now be substituted into the drift kinetic equation (**10**.4.22). But before writing the result, we note that we shall actually be interested in the forthcoming chapters in a *stationary* solution of this equation. Indeed, using the macroscopic assumptions made in chapter 10, in particular, eq. (**10**.6.2), we assume that *the time derivatives, both of f_0 and of f_1 are of higher order in ϵ* and can be dropped from (**10**.4.22). This statement can be made more precise, by using the multiple times introduced in chapter 10. We have already assumed in eq. (**10**.6.3) that the reference distribution function f_0 is independent of the time scale t_1, and varies only on the slower time scale t_2. We now make the additional assumption that the deviation f_1 is independent of the faster time scale t_0, and thus changes, at most, on the time-scale t_1. With these assumptions, it is clear that the first two terms on the right-hand side of eq. (**10**.4.22) are identically zero. We are thus left with the *final form of the (stationary) drift kinetic equations* for the two species,

$$\begin{aligned} &U_\alpha \boldsymbol{b}\cdot \frac{\partial}{\partial \boldsymbol{Y}} \tilde{f}_1^\alpha + \boldsymbol{V}_{\mathrm{D}}^\alpha \cdot \frac{\partial}{\partial \boldsymbol{Y}} f_0^\alpha + e_\alpha U_\alpha E_{\parallel}^{(A)} \frac{\partial}{\partial \mathscr{E}} f_0^\alpha \\ &\quad = (\nu^{\alpha e} + \nu^{\alpha i}) \frac{m_\alpha}{B} U_\alpha \frac{\partial}{\partial M} U_\alpha M \frac{\partial}{\partial M} \tilde{f}_1^\alpha + U_\alpha \mathcal{N}_{\parallel}^\alpha f_0^\alpha. \end{aligned} \tag{5.14}$$

The solution of these closed equations for the average distribution functions $\tilde{f}_1^\alpha$ will be the object of chapter 14, treating the neoclassical transport theory.

Appendix 11A.1. Calculation of the like-particle collision frequencies

The evaluation of the collision frequencies boils down to the evaluation of the two functions $\alpha_0(v)$, $\alpha_1(v)$ defined in (1.13). Going over to dimensionless

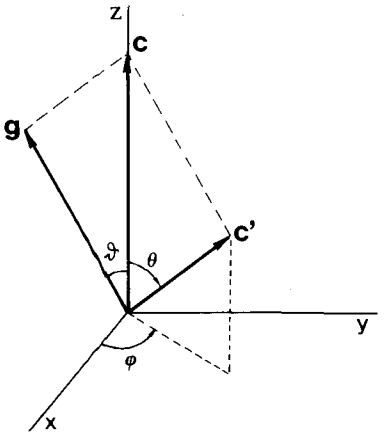

Fig A1 1 Reference frame used in the evaluation of the integrals (A1.1) and (A1 2).

variables, we must compute the following two integrals:

$$L^{(2p)}(c) = \pi^{-3/2} \int \mathrm{d}\boldsymbol{c}'\; c'^{2p} G_{rr}(\boldsymbol{c} - \boldsymbol{c}')\, \mathrm{e}^{-c'^2}, \tag{A1.1}$$

$$M^{(2p)}(c) = \pi^{-3/2} \int \mathrm{d}\boldsymbol{c}'\; c'^{2p} c_r c_s G_{rs}(\boldsymbol{c} - \boldsymbol{c}')\, \mathrm{e}^{-c'^2}. \tag{A1.2}$$

For our particular purpose, we need only $L^{(0)}$ and $M^{(0)}$. We use a method due originally to Chandrasekhar (1943) and developed by Balescu (1963, ch. 8). Using spherical coordinates in the reference frame depicted in fig. A1.1, we find

$$L^{(2p)}(c) = 2\pi^{-1/2} \int_0^\infty \mathrm{d}c'\; c'^{2p+2}\, \mathrm{e}^{-c'^2} \int_{-1}^{1} \mathrm{d}\cos\theta\; \frac{2}{g},$$

$$M^{(2p)}(c) = 2\pi^{-1/2} \int_0^\infty \mathrm{d}c'\; c'^{2p+2}\, \mathrm{e}^{-c'^2} \int_{-1}^{1} \mathrm{d}\cos\theta\; \frac{g^2 c'^2 - (\boldsymbol{g}\cdot\boldsymbol{c}')^2}{g^3}. \tag{A1.3}$$

From the definition $\boldsymbol{g} = \boldsymbol{c} - \boldsymbol{c}'$, we easily derive the following relations (using the notations shown in the figure):

$$\boldsymbol{g}\cdot\boldsymbol{c}' = cc'\cos\theta - c'^2,$$

$$\boldsymbol{g}\cdot\boldsymbol{c} = gc\cos\vartheta = c^2 - cc'\cos\theta. \tag{A1.4}$$

Through these relations, we may change the integration variable in (A1.3) from $\cos\theta$ to g,

$$\cos\theta = \frac{1}{2cc'}\left(c^2 + c'^2 - g^2\right),$$

$$\mathrm{d}\cos\theta = -\frac{1}{cc'}\, g\, \mathrm{d}g. \tag{A1.5}$$

We thus find

$$L^{(2p)}(c) = 2\pi^{-1/2}\frac{1}{c}\int_0^\infty \mathrm{d}c'\; c'^{2p+1}\, \mathrm{e}^{-c'^2}\, l(c, c'),$$

$$M^{(2p)}(c) = 2\pi^{-1/2}\frac{1}{4c}\int_0^\infty \mathrm{d}c'\; c'^{2p+1}\, \mathrm{e}^{-c'^2}\, m(c, c'), \tag{A1.6}$$

where

$$l(c, c') = \int_{|c-c'|}^{c+c'} \mathrm{d}g = \begin{cases} 2c', & c' < c, \\ 2c, & c' > c, \end{cases}$$

$$m(c, c') = \int_{|c-c'|}^{c+c'} \mathrm{d}g \left\{ -\frac{(c^2 - c'^2)^2}{g^2} + 2(c^2 + c'^2) - g^2 \right\}$$

$$= \begin{cases} \frac{16}{3}c'^3, & c' < c, \\ \frac{16}{3}c^3, & c' > c. \end{cases} \tag{A1.7}$$

We then immediately obtain

$$L^{(2p)}(c) = \frac{4}{c}\, 2\pi^{-1/2}\left(\int_0^c \mathrm{d}c'\; c'^{2p+2}\, \mathrm{e}^{-c'^2} + c\int_c^\infty \mathrm{d}c'\; c'^{2p+1}\, \mathrm{e}^{-c'^2}\right),$$

$$M^{(2p)}(c) = \frac{4}{3c}\, 2\pi^{-1/2}\left(\int_0^c \mathrm{d}c'\; c'^{2p+4}\, \mathrm{e}^{-c'^2} + c^3\int_c^\infty \mathrm{d}c'\; c'^{2p+1}\, \mathrm{e}^{-c'^2}\right). \tag{A1.8}$$

These integrals are simply related to the error function $\Phi(c)$, defined in (4.9),

$$L^{(0)}(c) = \frac{2}{c}\Phi(c),$$

$$L^{(2)}(c) = \frac{1}{c}\left[3\Phi(c) - c\Phi'(c)\right],$$

$$L^{(4)}(c) = \frac{1}{2c}\left[15\Phi(c) - (7c + 2c^3)\Phi'(c)\right],$$

$$M^{(0)}(c) = \frac{1}{c}\left[\Phi(c) - c\Phi'(c)\right],$$

$$M^{(2)}(c) = \frac{1}{2c}\left[5\Phi(c) - (5c + 2c^3)\Phi'(c)\right],$$

$$M^{(4)}(c) = \frac{1}{4c}\left[35\Phi(c) - (35c + 18c^3 + 4c^5)\Phi'(c)\right]. \tag{A1.9}$$

Using these relations in definition (1.13) of function $\alpha_0^\beta(v)$, we find

$$\alpha_0^\beta = n_\alpha\left(\frac{m_\alpha}{T_\alpha}\right)^{1/2}\frac{1}{2c^2}(c^2L^{(0)} - M^{(0)})$$

$$= n_\alpha\left(\frac{m_\alpha}{T_\alpha}\right)^{1/2}\frac{1}{2c^3}\left[(2c^2 - 1)\Phi(c) + c\Phi'(c)\right] = n_\alpha v^{-1}\mathscr{H}(c),$$

where $\mathscr{H}(c)$ is the Chandrasekhar function (4.8). From here on, eqs. (4.12) and (4.13) follow immediately.

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12

Particle and energy fluxes in magnetically confined plasmas

12.1. Short and long mean free path regimes

We now start our investigation of the *transport processes in toroidally confined plasmas*. For simplicity, we consider only *axisymmetric configurations* (some indications about non-axisymmetric configurations will be given in chapter 15). We recall again, if necessary, that we only deal here with *quiescent plasmas*. In such systems, we assume that *the toroidal magnetic surfaces defining the configuration are given and stationary*.

Our main object, which is of primary importance in fusion physics, is the study of the *transport of matter and energy across these surfaces*. These processes determine the (unavoidable) "leakage" through the magnetic structure and, eventually, the very feasibility of the magnetic confinement of the plasma.

Let us say from the beginning that the theory which will be presented in the forthcoming chapters (the *neoclassical transport theory*) is far from giving an exhaustive picture of the transport in confined plasmas: it can only be considered as a first step towards the solution of the problem. In a more complete and consistent picture, one should allow for the reaction of the particle and energy fluxes on the confining field. This effect may lead to instabilities, to the deformation of the surfaces, to the appearance of islands and possibly to the complete (three-dimensional) ergodization of the magnetic field lines, hence to the destruction of the magnetic surfaces. These extremely complex processes give rise to the so-called *anomalous transport*, which falls beyond the scope of this volume.

But even within the framework of the neoclassical theory, a new feature introduces essential differences with the phenomena studied in the first part of this book. In a fusion device, the *hydrodynamic length scale* L_H is determined essentially by the minor radius of the torus, i.e. a quantity of the order of 1 meter. On the other hand, if temperatures of the order of 1–10 keV are realized (as is necessary for fusion), the *mean free path* λ_{mfp} can easily be

made to exceed L_H. Hence, the main assumption (**5**.1.2) of the classical theory breaks down. As a result (as discussed in section **10**.1), a useful transport theory must cover *both* the *short mean free path regime* defined by the ordering (**10**.1.11) and the *long mean free path regime* defined by (**10**.1.12).

It turns out that, even in the short mean free path regime there appear important differences with the classical predictions of chapter 5. The reason is a strong influence of the magnetic field geometry on the transport processes. But, of course, it is mainly in the long mean free path regime that we shall find the most essential new features.

Before starting the quantitative discussion, let us make a useful comparison with the "analogous" problem for a *very dilute neutral gas*. In this case, if the effect of gravitation can be neglected, whenever the mean free path becomes comparable to the size of the system, the collisions can be omitted in studying the dynamics. The molecules move in straight trajectories with constant speeds and their motion is limited in space only by the solid walls enclosing the gas. Because of the extreme dilution, each particle just bounces back and forth between the walls without having any appreciable chance of colliding with another particle. In this, so-called *Knudsen regime*, all dissipation is absent: there is no internal mechanism capable of changing the particle velocities, let alone drive the system towards equilibrium. * As a result, the macroscopic fluid motion bears no resemblance to the kind of *diffusive* motion existing in denser, collision-dominated gases. It is customary to characterize the flow in the Knudsen regime as *effusive*. Such a flow is determined by the geometry of the box and the nature of the walls, but also very crucially by the initial preparation of the gas.

This point can be clearly understood if we go back to the calculations in section **5**.2. We had obtained there a solution of the Hermitian moment equations in the form (**5**.2.6). Under condition (**5**.2.10), we were able to reduce this equation to the Markovian form (**5**.2.16). But in the Knudsen regime, (**5**.2.10) is invalid, hence (**5**.2.6) cannot be further reduced. As a result, each moment $h^{(p)}(t)$ depends on the initial value of *all* the moments $h^{(q)}(0)$, i.e. on the detailed form of the initial velocity distribution, as stated above. It also depends on the value of the source terms evaluated at *earlier times*.

* The only dissipative feature in this case is provided by the collisions with the walls. In certain situations, a reasonable scenario is the following: whenever a particle arrives at a wall, it is temporally absorbed, it comes to equilibrium with the wall through collisions with the particles of the solid, and is then re-emitted into the gas. The distribution function of the particles with velocities directed away from the wall, and in a layer close to the latter, can then be assumed to be nearly Maxwellian, with a temperature equal to the wall temperature.

In conclusion, *the effusive flow of a neutral gas has a definitely non-hydrodynamical character*.

One may think, at a first glance, that the situation in a long mean free path plasma is analogous to the one pertaining to a Knudsen gas. A second thought will lead us, however, to the conclusion that the situation is basically different, the reason being the presence of a *non-uniform toroidal magnetic field*, as described in chapter 8.

As a result, even in the absence of collisions, the motion of the charged particles is not uniform (as in the Knudsen gas), but is determined by the magnetic field. Whenever the ratio ϵ of the Larmor radius to the length of the gradient of the magnetic field is small, i.e. whenever the *drift approximation* is applicable, we can construct a good picture of the particle trajectories, as was done in great detail in chapter 9. There is a fundamental qualitative difference between the *topology of the trajectories* in a collisionless neutral gas and in a collisionless plasma in a toroidal magnetic field: in the latter case, the orbits are *bounded in space*. The charged particles either spiral along the torus (if they are passing), or else bounce back and forth along a banana trajectory (if they are trapped). There is no need for a material wall to keep the particles within a bounded region (this is the very idea of a "magnetic confinement" or, more casually, of a "magnetic bottle").

A second important difference between neutral gas and plasma lies in the very nature of the collision process. A long mean free path regime in a neutral gas necessarily requires a very low density. If the interactions can be described by a potential falling off at an effective distance a (or by a corresponding hard sphere model), the mean free path is estimated by the well-known relation

$$\lambda_{\text{mfp}} \sim n^{-1} a^{-2}.$$

It depends on the density, but not on the temperature.

On the contrary, in a plasma, a long mean free path can exist even for a relatively high density (typically, $n \simeq 10^{12}$ to 10^{14} cm^{-3} in a tokamak), provided the temperature is high enough. Indeed, from (4.6.4) and the relation $\lambda_{\text{mfp}} = V_{\text{T}} \tau_{\alpha}$, we find

$$\lambda_{\text{mfp}} \sim n^{-1} T^{5/2}.$$

The reason for this difference is the fact that the interactions between charged particles, as described by the Landau equation, can by no means be approximated by a billiard ball collision process. The curvature of the trajectory during the "collision" process results from the cumulative effect of a large number of small deflections reminiscent of the Brownian motion [hence the

name of the *Fokker–Planck equation* given – rather improperly * – to the Landau equation by many authors; see Spitzer (1967)].

Combining the two previous remarks, we sum up the situation for a tokamak plasma as one in which the charged particles at rather high density are forced to move in a limited region of space. Under such conditions, close encounters between particles are bound to occur in the plasma. The difference with the collision-dominated plasma is that an average particle may circle the torus – or bounce back and forth along a banana orbit – many times before undergoing a collision.

The main conclusion is that the collisions can never be neglected in the description of a long mean free path plasma. Thus, the presence of the confining magnetic field requires – in a very subtle way – the balancing effect of a *dissipative mechanism* which drives the system towards a local equilibrium. This is the main difference between a neutral Knudsen gas and a plasma in the banana regime. As will be seen in the forthcoming sections, the latter system *behaves, in many ways, as a hydrodynamic system, in spite of the long mean free path.* For this reason, the extremely misleading name of "collisionless regime", used by some authors to denote the long mean free path regime, should be rigorously banned from the literature.

The balancing effects of the magnetic field and of the collisions appears very clearly in the analysis in chapter 10. It was shown there that, under the *sole assumption* that $\epsilon \ll 1$ (drift approximation), *the gyro-averaged distribution function is locally Maxwellian to order* ϵ^0. The fact is independent of the ratio between mean free path and hydrodynamic length. This result is fundamental to transport theory **, as it leads straightforwardly to a set of relations between fluxes and thermodynamic forces, hence to a closure of the hydrodynamic equations.

Let us now quote the four review papers which are justly considered as classics of the neoclassical transport theory. The first one is by Galeev and Sagdeev (1975, 1979) (these two papers are identical). These authors must be credited with the initial idea of neoclassical transport theory in the banana

* The Fokker-Planck equation of the theory of the Brownian motion is a *linear* equation, see Chandrasekhar (1943), van Kampen (1981). Note that even the so-called "non-linear Fokker–Planck equation" is linear with respect to the unknown probability distribution, but has coefficients which are non-linear functions of the independent variables (van Kampen 1981, ch 8). This point was also discussed in section **11.1**.

** In their classic review, Hinton and Hazeltine (1976), p 247, say: "This circumstance [that the distribution function is Maxwellian to dominant order] is so crucial to the linear transport processes under consideration that, *if it could not be deduced* from the orderings, *it would be assumed to hold anyway*" (!) [our italics]. Rather than this, somewhat shocking statement, we prefer to say that, in the absence of a Maxwellian reference state, one could not obtain hydrodynamical relations of the usual form and, in particular, a set of transport coefficients (see the previous discussion of the Knudsen gas)

regime. Unfortunately, their review paper is not always very clear. Hinton and Hazeltine (1976) wrote a quite systematic account of the theory, based on their own work (in collaboration with Rosenbluth). There were, however, a few gaps left over in the theory at that time. The work of Hirshman and Sigmar (1981), although not introducing basically new ideas, succeeded in redistributing the accents on the various features in such a way as to simplify the theory quite considerably. As a result, they were able to tackle analytically even the problem (of great practical importance) of transport in a multispecies plasma (impurity transport). Finally, the recent paper by Kovrizhnykh (1984) is mainly devoted to non-axisymmetric configurations.

Let us say, once for all, that our forthcoming treatment is largely based on these papers (especially on the one by Hirshman and Sigmar) and we shall not always refer to them specifically at each step.

12.2. The Hermitian moment expansion

We consider a plasma consisting of *electrons and a single species of ions*, in the presence of an *axisymmetric confining magnetic field* (whose precise form need not be specified at the present stage). Our main assumption is that the *drift approximation* is applicable, i.e.

$$\epsilon \ll 1, \tag{2.1}$$

where the basic parameter ϵ was defined in (**10**.1.9) (see also chapter 1). Under this assumption, the analysis of chapter 10 leads to the fundamental result that the distribution functions are of the form

$$f^\alpha = f_0^\alpha(\mathscr{E};\ \rho,\ t) + \epsilon f'^\alpha(\mathscr{E},\ M,\ \phi;\ \rho,\ \theta,\ t;\ \epsilon), \tag{2.2}$$

where $f_0^\alpha(\mathscr{E};\ \rho,\ t)$ is the leading term, and $\epsilon f'^\alpha$ is a deviation, of order ϵ at most. The functions are expressed in terms of the following NGC variables: the total energy $\mathscr{E}$, the magnetic moment M, the gyrophase ϕ and the spatial variables $\boldsymbol{Y}$ for which we use toroidal coordinates; ρ is the "radial" coordinate, θ the poloidal angle and ζ the toroidal angle. These variables were discussed in detail in chapters 1 and 8. The function f_0^α will be called the distribution function of the *reference state*: it is of the form (**10**.5.14)

$$f_0^\alpha(\mathscr{E};\ \rho,\ t) = n_{\alpha 0}\left(\frac{m_\alpha}{2\pi T_{\alpha 0}}\right)^{3/2} \exp\left(\frac{e_\alpha \Phi_0}{T_{\alpha 0}}\right) \exp\left(-\frac{\mathscr{E}}{T_{\alpha 0}}\right). \tag{2.3}$$

The dependence on ρ and t comes through the (zeroth order) temperature $T_{\alpha 0}$, density $n_{\alpha 0}$ and electric potential Φ_0. When expressed in terms of particle

velocity variables, the reference state distribution function has the form (**10**.5.15):

$$f_0^\alpha(v;\ \rho,\ t) = n_{\alpha 0}\left(\frac{m_\alpha}{2\pi T_{\alpha 0}}\right)^{3/2} \exp\left(-\frac{m_\alpha}{2T_{\alpha 0}} v^2\right). \tag{2.4}$$

The reference state in a confined plasma, in the drift approximation, is thus *locally Maxwellian*. We recall, however, that the presence of the magnetic field introduces some important differences as compared to the reference state (**4**.2.6) relating to a collision-dominated plasma. These additional constraints are discussed in sections **10**.5 and **10**.6.

We now adopt the following point of view. The reference state will be written in the "classical form" (**4**.2.6),

$$f_0^\alpha(v;\ Y,\ t;\ \epsilon) = n_\alpha(Y,\ t;\ \epsilon)\left(\frac{m_\alpha}{2\pi T_\alpha(Y,\ t;\ \epsilon)}\right)^{3/2}$$

$$\times \exp\left(-\frac{m_\alpha}{2T_\alpha(Y,\ t;\ \epsilon)} |\, v - u^\alpha(Y,\ t;\ \epsilon)\,|^2\right). \tag{2.5}$$

The plasmadynamical quantities n_α, T_α, u^α can then be expanded in powers of the drift parameter ϵ. The constraints of section **10**.5 allow us to say something definite about the zeroth order terms in these expansions:

$$n_\alpha(Y,\ t;\ \epsilon) = n_{\alpha 0}(\rho,\ t) + \epsilon\, n_{\alpha 1}(\rho,\ \theta,\ t) + O(\epsilon^2),$$

$$T_\alpha(Y,\ t;\ \epsilon) = T_{\alpha 0}(\rho,\ t) + \epsilon\, T_{\alpha 1}(\rho,\ \theta,\ t) + O(\epsilon^2),$$

$$u^\alpha(Y,\ t;\ \epsilon) = 0 + \epsilon\, u^\alpha(\rho,\ \theta,\ t) + O(\epsilon^2). \tag{2.6}$$

Note that none of these functions depends on the toroidal angle ζ, because of the assumed axisymmetry.

If the zeroth-order contributions of these expansions are substituted into (2.5), the reference distribution function f_0^α of eq. (2.4) is recovered.

We now represent an arbitrary non-equilibrium function $f^\alpha(v;\ Y,\ t)$ in the form (**4**.3.1) or, after the change of variables (**4**.3.4),

$$f^\alpha(v;\ Y,\ t;\ \epsilon) = n_\alpha(Y,\ t;\ \epsilon)\left(\frac{m_\alpha}{T_\alpha(Y,\ t;\ \epsilon)}\right)^{3/2} \phi^0(c)$$

$$\times [1 + \chi^\alpha(c;\ \rho,\ \theta,\ t;\ \epsilon)]. \tag{2.7}$$

This form is *not* equivalent to (2.2), because the reference state still depends on ϵ through the plasmadynamical quantities. This is, however, not an obstacle to our further developments.

Indeed, we are now in the same situation as in eq. (4.3.8). As the plasmadynamical quantities n_α, T_α, $\boldsymbol{u}^\alpha$ entering (2.7) are the *exact* densities, temperatures and average velocities, we are free to impose the constraints (4.3.2) or (4.3.9). [This would not be allowed if we took eq. (2.4) as a reference state].

The deviation χ^α can now be expanded in a series of irreducible Hermite polynomials, as in (4.3.11),

$$\chi^\alpha(\boldsymbol{c};\, \rho,\, \theta,\, t;\, \epsilon) = \sum_{m=0}^{\infty} h^{\alpha(2m)}(\rho,\, \theta,\, t;\, \epsilon)\; H^{(2m)}(\boldsymbol{c}) + \sum_{m=0}^{\infty} h_r^{\alpha(2m+1)}(\rho,\, \theta,\, t;\, \epsilon)\; H_r^{(2m+1)}(\boldsymbol{c}) + \sum_{m=1}^{\infty} h_{rs}^{\alpha(2m)}(\rho,\, \theta,\, t;\, \epsilon)\; H_{rs}^{(2m)}(\boldsymbol{c}) + \cdots . \tag{2.8}$$

The moments, defined in this manner, obey the *same* equations of evolution as derived in chapter 4, viz. (4.5.22)–(4.5.27), with the generalized frictions given in (4.6.34), (4.6.35).

These equations will again be taken as the starting point in our work. This feature will provide the unifying link between classical and neoclassical theories of transport. The possibility of this unification was first demonstrated by Hirshman and Sigmar (1981). The analysis of the moment equations will, of course, be different from the one of chapter 5, in order to take into account the influence of the magnetic field inhomogeneity.

One of the first operations will be a careful analysis of the order of magnitude with respect to the *small parameter* ϵ. Both the Hermitian (non-equilibrium) moments *and* the plasmadynamical quantities n_α, T_α, $\boldsymbol{u}^\alpha$ will be expanded in powers of ϵ and the moment equations will be properly scaled.

12.3. The Chew–Goldberger–Low (CGL) pressure tensor

The Hermitian moments $h_{r_1 \ldots r_n}^{\alpha(p)}$ are determined by a hierarchical set of equations induced by the kinetic equation. As discussed at the beginning of

section 4.5, the *complete* set of those equations is merely a representation of the kinetic equation, and is exactly equivalent to it.

On the other hand, these equations are only useful if they can be simplified, and thus truncated, under special circumstances. In the case of collision-dominated plasmas, the simplification came from the smallness of the ratio λ_H of the relaxation time to the hydrodynamical time, eq. (5.1.3). This assumption led us to the simplified equations (5.1.7)–(5.1.10) which were the basis of the transport theory in the collision-dominated regime.

In the present analysis we wish to cover both the short mean free path regime ($\lambda_H < 1$) and the long mean free path regime ($\lambda_H > 1$); hence the parameter λ_H is of no use in ordering the terms of the moment equations. There does exist, however, another small parameter in the theory: the drift parameter ϵ. We shall therefore analyze the moment equations and determine the size of the various terms, as measured by ϵ. Retaining the dominant terms in this process will again lead to a serious simplification of the moment equation hierarchy, as in section 5.1, but now according to a new criterion.

For convenience, we begin our analysis with the equations for the *tensorial moments* $h_{rs}^{\alpha(2p)}$, i.e. (4.5.25)–(4.5.27). The first of these, written more explicitly by using (4.5.29), (4.5.32), is

$$\partial_t h_{rs}^{\alpha(2)} = -\sqrt{2}\,\mathscr{T}_{rs|pq}\ \nabla_p u_q + 2\Omega_\alpha\,\mathscr{T}_{rs|pq}\ \varepsilon_{pmn}\,h_{qm}^{\alpha(2)}\,b_n$$

$$+ Q_{rs}^{\alpha(2)} - \frac{2}{\sqrt{5}}\,\mathscr{T}_{rs|pq}\left(\frac{T_\alpha}{m_\alpha}\right)^{1/2}\frac{1}{n_\alpha T_\alpha^{3/2}}\,\nabla_p\left(n_\alpha T_\alpha^{3/2}\,h_q^{\alpha(3)}\right)$$

$$+\sqrt{2}\,\delta^{\alpha e}\,\mathscr{T}_{rs|pq}\left(\frac{T_e}{m_e}\right)^{1/2}\nabla_p h_q^{(1)} + C_{rs}^{\alpha(2)} + N_{rs}^{\alpha(2)}. \tag{3.1}$$

We now do a scaling analysis of this equation. We know that in the drift approximation, the Larmor frequency is a large quantity, of order ϵ^{-1}. On the other hand, as results from (2.6), the scalar plasmadynamical quantities n_α, T_α are predominantly of order ϵ^0; the vector quantities u_r^α, hence also their linear combinations u_r and $h_r^{(1)}$, are at most of order ϵ. The Hermitian vector and tensor moments $h_r^{\alpha(3)}$, $h_{rs}^{\alpha(2)}$, etc., are also at most of order ϵ. We moreover assume that the gradient operator does not necessarily introduce an additional order of smallness; thus, formally, $\nabla_r = O(\epsilon^0)$. As a result, $\nabla_r u_s$ is of order ϵ, just like u_s itself. Next we note that the generalized friction term $Q_{rs}^{\alpha(2)}$, which contains linear and non-linear combinations of such moments, is also at most of order ϵ. Finally, from eq. (10.6.2) we deduce that $\partial_t h_{rs}^{\alpha(2)}$ is of order ϵ^2 in the transport regime (i.e. neglecting the fast transients). Therefore, writing the

dominant magnetic term of (3.1) on the left-hand side, dividing all terms by Ω_α and recalling the definition (**4**.5.28) of the symmetrizer $\mathscr{T}_{rs\,|\,pq}$, we find

$$2\mathscr{T}_{rs\,|\,pq}\ \varepsilon_{pmn}\, h^{\alpha(2)}_{qm}\, b_n \equiv \varepsilon_{rmn}\, h^{\alpha(2)}_{sm}\, b_n + \varepsilon_{smn}\, h^{\alpha(2)}_{rm}\, b_n$$

$$= \frac{1}{\Omega_\alpha}\{\text{terms of order } \epsilon \text{ at most}\}.$$

As Ω_α^{-1} is of order ϵ, we reach the conclusion

$$\varepsilon_{rmn}\, h^{\alpha(2)}_{sm}\, b_n + \varepsilon_{smn}\, h^{\alpha(2)}_{rm}\, b_n = 0 + \mathrm{O}(\epsilon^2). \tag{3.2}$$

(From here on, up to the end of this section, we shall write relations valid through order ϵ and omit writing terms of order ϵ^2). We note that these equations can be *formally* obtained from those of chap. 5, i.e. (**5**.1.8), (**5**.1.10) by setting $\partial_t h^{\alpha(2p)}_{rs} = 0$ and $\tau_\alpha = \infty$.

In order to see the implication of these very simple relations, we choose a local reference frame whose z-axis is parallel to the magnetic field. In this frame, $b_n = \delta_{nz}$. Considering that (3.2) is a set of 6 independent equations, we write these out explicitly by successively giving to r, s the values x, y, z. This was done explicitly in section **5**.1b. Setting again $\partial_t h^{\alpha(2p)}_{rs} = 0$ and $\tau_\alpha = \infty$, eq. (**5**.1.23) reduces to $0 = 0$; the remaining ones, (**5**.1.24), (**5**.1.25) lead to

$$h^{\alpha(2)}_{xy} = h^{\alpha(2)}_{xz} = h^{\alpha(2)}_{yz} = 0, \qquad h^{\alpha(2)}_{xx} = h^{\alpha(2)}_{yy}. \tag{3.3}$$

Thus, through order ϵ, *in any local reference frame, one of whose axes is parallel to the magnetic field, the tensor* $h^{\alpha(2)}_{rs}$ *is diagonal.* The xx and yy components are equal to each other, and their common value will be denoted by $h^{\alpha(2)}_{\perp}$; the zz component is not constrained by (3.2): its value will be denoted by $h^{\alpha(2)}_{\parallel}$.

The tensor $h^{\alpha(2)}_{rs}$ obeying conditions (3.3) can be expressed in an arbitrary reference frame in the form

$$h^{\alpha(2)}_{rs} = h^{\alpha(2)}_{\parallel}\, b_r b_s + h^{\alpha(2)}_{\perp}\,(\delta_{rs} - b_r b_s). \tag{3.4}$$

Moreover, the tensor $h^{\alpha(2)}_{rs}$ must be *traceless*; thus

$$h^{\alpha(2)}_{\parallel} + 2h^{\alpha(2)}_{\perp} = 0,$$

or

$$h^{\alpha(2)}_{\perp} = -\tfrac{1}{2}\, h^{\alpha(2)}_{\parallel}. \tag{3.5}$$

Hence, (3.4) becomes

$$h_{rs}^{\alpha(2)} = \tfrac{3}{2}\, h_{\|}^{\alpha(2)} \left(b_r b_s - \tfrac{1}{3}\delta_{rs} \right). \tag{3.6}$$

The quantity $h_{\|}^{\alpha(2)}$ is of order ϵ.

Going over to the corresponding dimensional quantity by using (**4**.3.17), we find the following form for the *dissipative pressure tensor*:

$$\pi_{rs}^{\alpha} = \tfrac{3}{2}\, \pi_{\|}^{\alpha} \left(b_r b_s - \tfrac{1}{3}\delta_{rs} \right). \tag{3.7}$$

Consider now the *complete pressure tensor* P_{rs}^{α} defined by

$$P_{rs}^{\alpha} = P_{\alpha}\, \delta_{rs} + \pi_{rs}^{\alpha}. \tag{3.8}$$

Because of the form (3.7), we see that the pressure tensor is completely defined by two independent scalar quantities: the pressure P_{α} and the quantity $\pi_{\|}^{\alpha}$. Alternatively, the pressure tensor can be written in the form

$$P_{rs}^{\alpha} = P_{\|}^{\alpha}\, b_r b_s + P_{\perp}^{\alpha} \left(\delta_{rs} - b_r b_s \right). \tag{3.9}$$

We find the following relations between the two sets of coefficients. The scalar pressure (of order ϵ^0) is given, as usual, by

$$P_{\alpha} = \tfrac{1}{3} \mathrm{Tr}\, \boldsymbol{P}^{\alpha} = \tfrac{1}{3} \left(P_{\|}^{\alpha} + 2 P_{\perp}^{\alpha} \right). \tag{3.10}$$

On the other hand, comparing (3.8) and (3.9) we have

$$P_{\|}^{\alpha}\, b_r b_s + P_{\perp}^{\alpha} (\delta_{rs} - b_r b_s) = \tfrac{1}{3} \left(P_{\|}^{\alpha} + 2 P_{\perp}^{\alpha} \right) \delta_{rs} + \tfrac{3}{2}\, \pi_{\|}^{\alpha} \left(b_r b_s - \tfrac{1}{3}\delta_{rs} \right),$$

or

$$\tfrac{1}{3} \left(P_{\perp}^{\alpha} - P_{\|}^{\alpha} + \tfrac{3}{2} \pi_{\|}^{\alpha} \right) (\delta_{rs} - 3 b_r b_s) = 0,$$

from which follows

$$\pi_{\|}^{\alpha} = \tfrac{2}{3} \left(P_{\|}^{\alpha} - P_{\perp}^{\alpha} \right). \tag{3.11}$$

Equivalently, the dissipative pressure tensor (3.7) can be written in the form

$$\pi_{rs}^{\alpha} = \left(P_{\|}^{\alpha} - P_{\perp}^{\alpha} \right) \left(b_r b_s - \tfrac{1}{3}\delta_{rs} \right). \tag{3.12}$$

Let us now summarize this discussion. We found that in the presence of a magnetic field, when the drift approximation is valid, and through first order in ϵ, the *pressure tensor is anisotropic*. The anisotropy has, however, a very special and simple form (3.9), characterized by *two scalar quantities*. This form of the pressure tensor was first found by *Chew, Goldberger and Low* (1956). $P_{\|}^{\alpha}$ and $P_{\perp}^{\alpha}$ will be called, respectively, the *parallel pressure* and the *perpendicular pressure*. The dissipative part of the pressure tensor (3.12) is entirely determined by the *pressure anisotropy* $(P_{\|}^{\alpha} - P_{\perp}^{\alpha})$, which is of order ϵ. More generally, we will say that any second-rank tensor which can be written in the form (3.9) is a tensor of *CGL form*.

If we think in terms of particle motions, the appearance of an anisotropy in the pressure should not be surprising. Indeed, the peculiar motion of the charged particles (the parallel velocity $v_{\|}$ is decoupled from the Larmor gyration and the drift motion) is the source of the difference in behaviour in the directions parallel and perpendicular to the magnetic field.

In a collision-dominated regime, the collisions tend to randomize the motions of the particles, therefore the pressure anisotropy is expected to be small. More precisely, any initial pressure anisotropy will be smoothed out in a time of the order of a collisional relaxation time. In a long mean free path regime, however, the anisotropy should play a major role. This qualitative hint will be amply confirmed in the forthcoming study.

The analysis given here for the pressure tensor can be repeated for all the tensorial Hermitian moments. The important feature in this discussion is the *dominance of the magnetic terms* in the corresponding moment equations. But, as it appears in eqs. (4.5.25)–(4.5.27), these magnetic terms have the same form for all the tensorial moments. We thus immediately find that eq. (3.2) is quite general,

$$\varepsilon_{rmn}\, h_{sm}^{\alpha(2p)}\, b_n + \varepsilon_{smn}\, h_{rm}^{\alpha(2p)}\, b_n = 0, \quad p = 1, 2, 3, \ldots. \tag{3.13}$$

From this equation follows immediately

$$h_{rs}^{\alpha(2p)} = \tfrac{3}{2}\, h_{\|}^{\alpha(2p)} \left(b_r b_s - \tfrac{1}{3}\delta_{rs}\right), \quad p = 1, 2, 3 \ldots. \tag{3.14}$$

Thus, *through order* ϵ, *all the tensorial Hermitian moments have the CGL form* (3.14). In particular, they are diagonal in a local reference frame with one axis directed along the magnetic field and all their components are multiples of a single scalar quantity $h_{\|}^{\alpha(2p)}$. We will soon see the importance of this conclusion for the theory of transport in the long mean free path regime.

12.4. The vector moment equations

We now apply the same kind of analysis to the vector moment equations, which are of fundamental importance, as they determine the fluxes of matter and energy. We first make a choice of the relevant moments. In the classical theory, we described the flow of particles by the centre-of-mass velocity u_r and by the non-hydrodynamic moment $h_r^{(1)}$, the electric current. In the present case of a confined plasma, we know that $(m_e/T_e)^{1/2}u_r$ is of the same order of smallness as $h_r^{(1)}$ ($\sim\epsilon$). Therefore, a more symmetrical treatment of the electron and ion fluxes proves to be more convenient (at least in some problems). We thus choose to represent the flow of matter by the two particle fluxes Γ_r^α or, in dimensionless form, by $h_r^{\alpha(1)}$, defined in table 3.2.1. To these fluxes we adjoin the two heat fluxes $h_r^{\alpha(3)}$ and the (non-privileged) moments $h_r^{\alpha(5)}$. All the other moments $h_r^{\alpha(p)}$ for $p=7, 9,\ldots$ will be neglected. This means that we decide to work in the *21M approximation*. *

The relevant moment equations are therefore (4.5.16), (4.5.22) and (4.5.23). For convenience, we multiply all the terms of these equations by the relaxation time τ_α to make them dimensionless (as was done in section 5.3).

We now neglect all the terms of order ϵ^2 or higher in those equations. This means that we neglect the time-derivatives $\tau_\alpha\,\partial_t h_r^{\alpha(p)}$ which are, as usual, assumed to be of order ϵ^2. We keep the source terms and the large magnetic terms. The collisional friction terms $\tau_\alpha Q_r^{\alpha(p)}$ are kept in the linear approximation: the non-linear terms are of order ϵ^2 at most. For the same reason, the convective and the non-linear terms $\tau_\alpha C_r^{\alpha(p)}$ and $\tau_\alpha N_r^{\alpha(p)}$ are neglected.

We now come to the most peculiar point: the "up" and "down" terms $\tau_\alpha U_r^{\alpha(p)}$ and $\tau_\alpha D_r^{\alpha(p)}$, defined by (4.5.19) and (4.5.30)–(4.5.34). In the short mean free path regime, these contributions were discarded because they contain the combination

$$\tau_\alpha\left(\frac{T_\alpha}{m_\alpha}\right)^{1/2}\nabla_s h_{rs}^{\alpha(p\pm1)}\sim\lambda_{\mathrm{H}}h_{rs}^{\alpha(p\pm1)}.$$

As $h_{rs}^{\alpha(p\pm1)}$ is already of order λ_{H}, these terms could be neglected in the linear transport theory of collision-dominated plasmas. But in the long mean free path regime, the order in λ_{H} is irrelevant: only the order in ϵ is a classification criterion. From that point of view, these terms are of order ϵ, hence compara-

* All the analytical results published up to the present were done in the 13M approximation (or its equivalent).

ble to the linear friction terms $Q_r^{\alpha(p)}$. Thus, the vector moment equations, through order ϵ, are

$$\tau_\alpha\left(\frac{m_\alpha}{T_\alpha}\right)^{1/2}\left[-\frac{1}{m_\alpha n_\alpha}\nabla_r P_\alpha+\frac{e_\alpha}{m_\alpha}E_r-\sqrt{2}\,\frac{1}{m_\alpha n_\alpha}\nabla_s\left(n_\alpha T_\alpha h_{rs}^{\alpha(2)}\right)\right]$$

$$+\Omega_\alpha\tau_\alpha\varepsilon_{rmn}h_m^{\alpha(1)}b_n+\tau_\alpha Q_r^{\alpha(1)}=0, \tag{4.1}$$

$$\tau_\alpha\left(\frac{T_\alpha}{m_\alpha}\right)^{1/2}\left\{-\sqrt{\frac{5}{2}}\,\frac{1}{T_\alpha}\nabla_r T_\alpha\right.$$

$$\left.-\sqrt{\frac{2}{5}}\left[\sqrt{7}\,\frac{1}{n_\alpha T_\alpha^2}\nabla_s\left(n_\alpha T_\alpha^2 h_{rs}^{\alpha(4)}\right)+\sqrt{2}\,T_\alpha^{-7/2}\nabla_s\left(T_\alpha^{7/2}h_{rs}^{\alpha(2)}\right)\right]\right\}$$

$$+\Omega_\alpha\tau_\alpha\varepsilon_{rmn}h_m^{\alpha(3)}b_n+\tau_\alpha Q_r^{\alpha(3)}=0, \tag{4.2}$$

$$\tau_\alpha\left(\frac{T_\alpha}{m_\alpha}\right)^{1/2}\sqrt{\frac{2}{5}}\left[-\frac{3}{n_\alpha T_\alpha^3}\nabla_s\left(n_\alpha T_\alpha^3 h_{rs}^{\alpha(6)}\right)\right.$$

$$\left.-2T_\alpha^{-11/2}\,\nabla_s\left(T_\alpha^{11/2}h_{rs}^{\alpha(4)}\right)-\sqrt{14}\,h_{rs}^{\alpha(2)}T_\alpha^{-1}\,\nabla_s T_\alpha\right]$$

$$+\Omega_\alpha\tau_\alpha\varepsilon_{rmn}h_m^{\alpha(5)}b_n+\tau_\alpha Q_r^{\alpha(5)}=0. \tag{4.3}$$

The explicit expressions of the generalized frictions $Q_r^{\alpha(p)}$ were given in eqs. (4.6.34), (4.6.35). We just keep in mind here that they are linear combinations of the vector moments, and that momentum conservation requires the following relation

$$\sum_\alpha n_\alpha m_\alpha\left(\frac{m_\alpha}{T_\alpha}\right)^{1/2}Q_r^{\alpha(1)}=0. \tag{4.4}$$

We further note that, through order ϵ, the factors n_α, T_α appearing in the terms involving $h_{rs}^{\alpha(2p)}$, as well as those multiplying $\nabla_r P_\alpha$, $\nabla_r T_\alpha$, must be understood as $n_{\alpha 0}$, $T_{\alpha 0}$ (hence, they depend only on the radial coordinate ρ). For simplicity, in most of the forthcoming formulae *we shall drop the subscripts zero in* $n_{\alpha 0}$, $T_{\alpha 0}$, $P_{\alpha 0}$.

The essential difference between eqs. (4.1)–(4.3) and the corresponding equations (5.1.7), (5.1.9), (5.1.18) of the short mean free path problem is the

presence of the terms involving the tensor moments. As discussed in the previous section, these terms reflect quantitatively the expected *importance of the anisotropy in the long mean free path regime*.

These terms introduce a novel feature: the *vector moment equations are now coupled to the tensor moment equations*. This could lead to serious complications. However, we now appreciate the usefulness of our discussion in section 12.3. The tensor moment equations were solved there (through order ϵ), with the result that the tensor moments have the simple *CGL form* (3.14). Although this fact simplifies the problem a great deal, *it does not solve the problem completely*. Indeed, the CGL expression of the tensor moments requires the knowledge of the scalar *anisotropy factors* $h_{\|}^{\alpha(2p)}$. These quantities conceal the whole complexity of the problem. They can only be calculated by *solving directly the kinetic equation in the long mean free path regime*. This feature is to be contrasted with the collision-dominated regime, in which the vector moments could be calculated by a "pure moment method", i.e. by solving the set of *algebraic* equations (5.1.7), (5.1.9) (with $\partial_t h_r^{\alpha(p)} = 0$).

Before continuing our treatment, we introduce a set of more convenient notations. We recall definitions (5.1.13), (5.1.15), (5.1.20) of the *source terms*, related to the thermodynamic forces,

$$g_r^{\alpha(1)} = \left(\frac{m_\alpha}{T_\alpha}\right)^{1/2} \tau_\alpha \left\{ -\frac{1}{m_\alpha n_\alpha} \nabla_r P_\alpha + \frac{e_\alpha}{m_\alpha} E_r \right\}, \tag{4.5}$$

$$g_r^{\alpha(3)} = -\sqrt{\frac{5}{2}} \left(\frac{T_\alpha}{m_\alpha}\right)^{1/2} \tau_\alpha \frac{1}{T_\alpha} \nabla_r T_\alpha, \tag{4.6}$$

$$g_r^{\alpha(5)} = 0. \tag{4.7}$$

The new terms, involving the tensor moments, are to be regarded as *additional source terms* in the vector moment equations,

$$\bar{g}_r^{\alpha(1)} = -\sqrt{2} \left(\frac{m_\alpha}{T_\alpha}\right)^{1/2} \tau_\alpha \frac{1}{m_\alpha n_\alpha} \nabla_s \left(n_\alpha T_\alpha h_{rs}^{\alpha(2)} \right), \tag{4.8}$$

$$\bar{g}_r^{\alpha(3)} = -\sqrt{\frac{2}{5}} \left(\frac{T_\alpha}{m_\alpha}\right)^{1/2} \tau_\alpha \times \left[\sqrt{7} \frac{1}{n_\alpha T_\alpha^2} \nabla_s \left(n_\alpha T_\alpha^2 h_{rs}^{\alpha(4)} \right) + \sqrt{2}\, T_\alpha^{-7/2} \nabla_s \left(T_\alpha^{7/2} h_{rs}^{\alpha(2)} \right) \right], \tag{4.9}$$

$$\bar{g}_r^{\alpha(5)} = -\sqrt{\frac{2}{5}}\left(\frac{T_\alpha}{m_\alpha}\right)^{1/2}\tau_\alpha\left\{\frac{3}{n_\alpha T_\alpha^3}\nabla_s\left(n_\alpha T_\alpha^3\, h_{rs}^{\alpha(6)}\right)\right.$$
$$+ 2T_\alpha^{-11/2}\,\nabla_s\left(T_\alpha^{11/2}\, h_{rs}^{\alpha(4)}\right)$$
$$\left. + \sqrt{14}\, h_{rs}^{\alpha(2)}\, T_\alpha^{-1}\,\nabla_s T_\alpha\right\}. \tag{4.10}$$

It will also be useful to proceed to a fuller decomposition of the source term $g_r^{\alpha(1)}$. It consists of the gradient of the partial pressure and of the electric field which , in turn, is of the form

$$E_r = -\nabla_r\Phi - c^{-1}\,\partial_t A_r \equiv -\nabla_r\Phi + E_r^{(A)}. \tag{4.11}$$

The first term represents the self-consistent field built up inside the plasma, whereas the second term, $E_r^{(A)}$ represents the field induced by purely external means (such as transformer coils) in the confined plasma. *The latter term is the only one in* (4.5) *which is not of the form of a gradient*. We now write

$$g_r^{\alpha(1)P} = -\left(\frac{m_\alpha}{T_\alpha}\right)^{1/2}\tau_\alpha\,\frac{1}{m_\alpha n_\alpha}\,\nabla_r P_\alpha, \tag{4.12}$$

$$g_r^{\alpha(1)\Phi} = -\left(\frac{m_\alpha}{T_\alpha}\right)^{1/2}\tau_\alpha\,\frac{e_\alpha}{m_\alpha}\,\nabla_r\Phi, \tag{4.13}$$

$$g_r^{\alpha(1)A} = \left(\frac{m_\alpha}{T_\alpha}\right)^{1/2}\tau_\alpha\,\frac{e_\alpha}{m_\alpha}\,E_r^{(A)}, \tag{4.14}$$

and thus

$$g_r^{\alpha(1)} = g_r^{\alpha(1)P} + g_r^{\alpha(1)\Phi} + g_r^{\alpha(1)A}. \tag{4.15}$$

The vector moment equations are now written in the compact form

$$\Omega_\alpha\tau_\alpha\,\varepsilon_{rmn}h_m^{\alpha(1)}b_n + \tau_\alpha Q_r^{\alpha(1)} + g_r^{\alpha(1)} + \bar{g}_r^{\alpha(1)} = 0, \tag{4.16}$$

$$\Omega_\alpha\tau_\alpha\,\varepsilon_{rmn}h_m^{\alpha(3)}b_n + \tau_\alpha Q_r^{\alpha(3)} + g_r^{\alpha(3)} + \bar{g}_r^{\alpha(3)} = 0, \tag{4.17}$$

$$\Omega_\alpha\tau_\alpha\,\varepsilon_{rmn}h_m^{\alpha(5)}b_n + \tau_\alpha Q_r^{\alpha(5)} + \quad + \bar{g}_r^{\alpha(5)} = 0. \tag{4.18}$$

Some comments are useful at this point in order to grasp the deeper meaning of the transport theory we are constructing here. Indeed, there is a

significant conceptual difference between the view adopted here and the procedure used in the first part of this book, especially in chapters 4–7.

It was stressed at the beginning of this section that the two particle fluxes Γ_r^e and Γ_r^i are treated *symmetrically*: they are both determined by the *asymptotic* * equations (4.16)–(4.18) and will ultimately be expressed as functions of the thermodynamic forces $g_r^{\alpha(p)}$, i.e. in terms of the gradients of pressure and temperature and of the electric field. In other words, Γ_r^e and Γ_r^i are determined by two *transport equations*.

The procedure of chapters 4–7 is altogether different: the two fluxes Γ_r^e and Γ_r^i are combined into the momentum density ρu_r and the electric current j_r. Only the latter quantity is treated asymptotically and leads to an algebraic transport equation [(7.1.9a) combined with (7.1.14)]: the generalized Ohm law. On the contrary, the velocity $\boldsymbol{u}$ is determined as a function of space and time by the momentum balance equation, a true partial differential equation (7.1.2). The latter, however, requires two additional closure relations for the *pressure tensors* π^e, π^i: eqs. (7.1.13).

In the present procedure, there is no need for such additional relations. It was first established that the pressure tensors have the CGL form; then the tensor moments are included in the equations for the two particle fluxes (and the two heat fluxes). Thus, the viscosity effects are automatically included in the vector fluxes (and will actually play a major role in the neoclassical transport theory).

Another way of describing the situation is as follows. The final macroscopic description of the plasma in section 7.1 is a *one-fluid description*, based on a set of four partial differential equations for the scalar moments ρ, σ, T_e, T_i, and a partial differential equation for the *vector* moment $\rho \boldsymbol{u}$. All the additional quantities are determined by algebraic transport equations. In the theory we are now constructing we shall ultimately obtain a closed *two-fluid picture* of the plasma described by a set of four partial differential equations for the *scalar* moments n_e, n_i, T_e, T_i alone. All the other quantities, in particular, *all the vector moments*, will be described by algebraic transport equations.

As a result, the distinction between *convective fluxes* proportional to $\boldsymbol{u}$ and conductive, or *diffusive* fluxes determined by a transport equation is no longer relevant: *all the vector fluxes are diffusive*.

These equations will be further discussed at the end of our present work, when the final closed equations describing the plasmadynamics of a confined plasma will be established, in chapter 18.

* The asymptotic character has been introduced by annulling the time-derivative term $\partial_t h_r^{\alpha(1)}$: the fluxes determined by (4.16)–(4.18) correspond to a steady state that sets in after a transient adjustment period· see section 5.2

12.5. The average parallel fluxes

We now derive a set of very important relations, involving the various components of the vector moments (or "fluxes") in *toroidal geometry*. These relations are obtained by *projecting the equations on various directions, and taking surface averages of the resulting expressions.*

The *axisymmetric magnetic field* is now specified as in section **8**.8. We thus introduce an orthogonal coordinate system, as described in chapter 8, consisting of a *radial coordinate* ρ, a *poloidal angle* θ and a *toroidal angle* ζ. The magnetic field is then represented as in (**8**.8.31),

$$\boldsymbol{B} = -R_0 \mathscr{B}_P(\rho)\, \nabla\rho \wedge \nabla\zeta + \mathscr{I}(\rho)\, \nabla\zeta, \tag{5.1}$$

or, more explicitly (using physical components),

$$\boldsymbol{B} = B_\theta\, \boldsymbol{e}_\theta + B_\zeta\, \boldsymbol{e}_\zeta, \tag{5.2}$$

with:

$$B_\theta = \frac{R_0 \mathscr{B}_P(\rho)}{l_\rho l_\zeta}, \qquad B_\zeta = \mathscr{I}(\rho)\frac{1}{l_\zeta}, \tag{5.3}$$

where l_ρ, l_θ, l_ζ are the scale factors of the coordinate system [see General Appendix G2, in particular (**G2**.1.36), (**G2**.2.2), (**G2**.2.15)]. In each point of the confinement region we introduce a *local triorthogonal triad* of (physical) unit vectors $\boldsymbol{e}_\rho$, $\boldsymbol{e}_\theta$, $\boldsymbol{e}_\zeta$, such that $\boldsymbol{e}_\rho$ is perpendicular, and $\boldsymbol{e}_\theta$, $\boldsymbol{e}_\zeta$ are tangent to the magnetic surface passing through that point. The vectors are oriented in such a way that

$$\boldsymbol{e}_\rho \wedge \boldsymbol{e}_\theta = \boldsymbol{e}_\zeta. \tag{5.4}$$

For convenience, the set $(\boldsymbol{e}_\rho, \boldsymbol{e}_\theta, \boldsymbol{e}_\zeta)$ will be called the *local geometrical triad.*

Together with this local triad, which is directly associated with the *toroidal geometry*, another local triad plays an important role, because it is directly associated with the *dynamics*. In this triad, one of the basis vectors, $\boldsymbol{e}_\parallel$, is identified with the unit vector $\boldsymbol{b}$ along the magnetic field line passing through the point; a second basis vector is the radial vector $\boldsymbol{e}_\rho$ (the same as before), and the third basis vector, $\boldsymbol{e}_\wedge$, is perpendicular to both. The vectors are oriented in such a way that:

$$\boldsymbol{e}_\rho \wedge \boldsymbol{e}_\parallel = \boldsymbol{e}_\wedge. \tag{5.5}$$

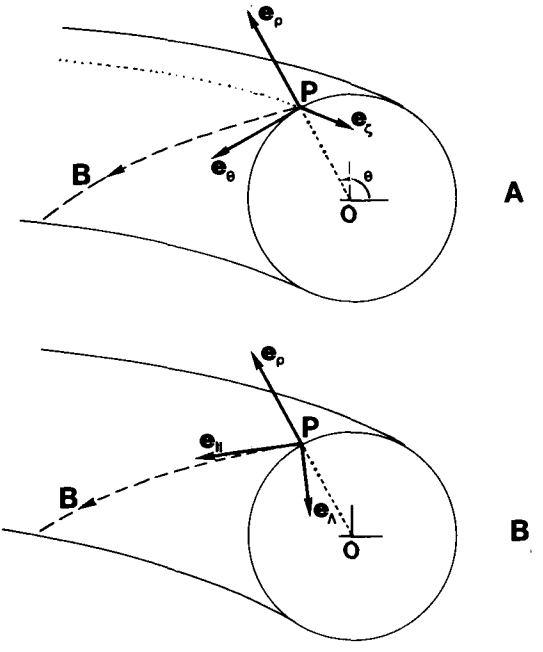

Fig. 5 1. *The local geometrical triad (A) and the local dynamical triad (B).* The triads are localized at a point P on a toroidal magnetic surface The dashed line is the magnetic field line passing through P.

Thus, the *geometrical triad* and the *dynamical triad* have the radial unit vector $\boldsymbol{e}_\rho$ in common; they differ only by the orientation of the two basis vectors which are tangent to the magnetic surface [see fig. 5.1]. The two sets of unit vectors are interrelated by the following relations, derived from (5.2) and (5.3),

$$\boldsymbol{e}_\parallel \equiv \boldsymbol{b} = \frac{1}{Bl_\zeta}\left(\frac{R_0\mathscr{B}_P}{l_\rho}\,\boldsymbol{e}_\theta + \mathscr{I}\,\boldsymbol{e}_\zeta\right),$$

$$\boldsymbol{e}_\wedge \equiv \boldsymbol{e}_\rho \wedge \boldsymbol{b} = \frac{1}{Bl_\zeta}\left(-\mathscr{I}\,\boldsymbol{e}_\theta + \frac{R_0\mathscr{B}_P}{l_\rho}\,\boldsymbol{e}_\zeta\right), \tag{5.6}$$

and, conversely,

$$\boldsymbol{e}_\theta = \frac{1}{Bl_\zeta}\left(\frac{R_0\mathscr{B}_P}{l_\rho}\,\boldsymbol{e}_\parallel - \mathscr{I}\,\boldsymbol{e}_\wedge\right), \qquad \boldsymbol{e}_\zeta = \frac{1}{Bl_\zeta}\left(\mathscr{I}\,\boldsymbol{e}_\parallel + \frac{R_0\mathscr{B}_P}{l_\rho}\,\boldsymbol{e}_\wedge\right), \tag{5.7}$$

where the magnetic field intensity is given by

$$B = \frac{1}{l_\zeta}\left[\mathscr{I}^2 + \left(\frac{R_0 \mathscr{B}_P}{l_\rho}\right)^2\right]^{1/2}. \tag{5.8}$$

In order to fix a convenient terminology, we shall call $\boldsymbol{e}_\rho$ the *radial direction*, $\boldsymbol{e}_\parallel$ ($\equiv \boldsymbol{b}$) the *parallel direction* and $\boldsymbol{e}_\wedge$ the *perp–tangential direction*. The latter, unusual name is motivated by the fact that the direction $\boldsymbol{e}_\wedge$ is both *perpendicular* to the magnetic field and *tangent* to the magnetic surface at each point P.

It is in the dynamical basis that the moment equations assume their simplest form; on the other hand, in the geometrical frame, the simplifying feature of axisymmetry can be easily and fully exploited. We also note the additional fact that, in the dynamical basis, all the tensors having the CGL property are diagonal (see section 12.3).

We first consider the *dynamical basis* and project eqs. (4.16)–(4.18) on the $\boldsymbol{e}_\parallel$ direction; equivalently, we multiply all the terms scalarly by $\boldsymbol{B}$. As usual, the magnetic term drops out and we are left with

$$\begin{aligned}
&\tau_\alpha \boldsymbol{B}\cdot\boldsymbol{Q}^{\alpha(1)} + \boldsymbol{B}\cdot\boldsymbol{g}^{\alpha(1)} + \boldsymbol{B}\cdot\bar{\boldsymbol{g}}^{\alpha(1)} = 0,\\
&\tau_\alpha \boldsymbol{B}\cdot\boldsymbol{Q}^{\alpha(3)} + \boldsymbol{B}\cdot\boldsymbol{g}^{\alpha(3)} + \boldsymbol{B}\cdot\bar{\boldsymbol{g}}^{\alpha(3)} = 0,\\
&\tau_\alpha \boldsymbol{B}\cdot\boldsymbol{Q}^{\alpha(5)} + \qquad\quad + \boldsymbol{B}\cdot\bar{\boldsymbol{g}}^{\alpha(5)} = 0.
\end{aligned} \tag{5.9}$$

An additional simplification sets in upon taking the surface average of these equations. Indeed, we must recall property (8.6.3) of the surface average operation. As a result,

$$\langle \boldsymbol{B}\cdot\boldsymbol{g}^{\alpha(3)}\rangle = -\sqrt{\tfrac{5}{2}}\,\tau_\alpha\left(\frac{T_{\alpha 0}}{m_\alpha}\right)^{1/2}\frac{1}{T_{\alpha 0}}\langle \boldsymbol{B}\cdot\nabla T_\alpha\rangle = 0.$$

For the same reason, the terms involving $\boldsymbol{g}^{\alpha(1)P}$ and $\boldsymbol{g}^{\alpha(1)\Phi}$ of $\boldsymbol{g}^{\alpha(1)}$ give a vanishing contribution to the surface average, because they are gradients; *only* $\boldsymbol{g}^{\alpha(1)A}$ [eq. 4.14) *survives* and we obtain

$$\tau_\alpha\langle \boldsymbol{B}\cdot\boldsymbol{Q}^{\alpha(1)}\rangle + \langle \boldsymbol{B}\cdot\bar{\boldsymbol{g}}^{\alpha(1)}\rangle + \langle \boldsymbol{B}\cdot\boldsymbol{g}^{\alpha(1)A}\rangle = 0, \tag{5.10}$$

$$\tau_\alpha\langle \boldsymbol{B}\cdot\boldsymbol{Q}^{\alpha(3)}\rangle + \langle \boldsymbol{B}\cdot\bar{\boldsymbol{g}}^{\alpha(3)}\rangle = 0, \tag{5.11}$$

$$\tau_\alpha\langle \boldsymbol{B}\cdot\boldsymbol{Q}^{\alpha(5)}\rangle + \langle \boldsymbol{B}\cdot\bar{\boldsymbol{g}}^{\alpha(5)}\rangle = 0. \tag{5.12}$$

The terms involving the generalized sources can be written more explicitly. Thus

$$\langle \boldsymbol{B}\cdot\bar{\boldsymbol{g}}^{\alpha(1)}\rangle = -\sqrt{2}\left(\frac{m_\alpha}{T_\alpha}\right)^{1/2}\tau_\alpha\frac{1}{m_\alpha n_{\alpha 0}}\left\langle B_r\,\nabla_s\left(n_{\alpha 0}T_{\alpha 0}h_{rs}^{\alpha(2)}\right)\right\rangle.$$

We note that

$$J\equiv B_r\,\nabla_s\left(n_{\alpha 0}T_{\alpha 0}h_{rs}^{\alpha(2)}\right)=B\,\nabla_s\left(n_{\alpha 0}T_{\alpha 0}h_{\|s}^{\alpha(2)}\right),$$

But, in the dynamical basis, $\mathbf{h}^{\alpha(2)}$ is diagonal; hence only $s=\|$ contributes to this expression:

$$J=B\,\nabla_\|\left(n_{\alpha 0}T_{\alpha 0}h_{\|\|}^{\alpha(2)}\right).$$

As the zeroth order function $n_{\alpha 0}T_{\alpha 0}$ only depends on the radial coordinate ρ, it is not affected by the parallel gradient operator, thus

$$J=Bn_{\alpha 0}T_{\alpha 0}\,\nabla_\|h_{\|\|}^{\alpha(2)}=n_{\alpha 0}T_{\alpha 0}\,B\,\nabla_\|h_{\|\|}^{\alpha(2)}$$

$$=n_{\alpha 0}\,T_{\alpha 0}\boldsymbol{B}\cdot(\nabla\cdot\mathbf{h}^{\alpha(2)}).$$

A similar calculation is performed on $\boldsymbol{B}\cdot\bar{\boldsymbol{g}}^{\alpha(3)}$ and $\boldsymbol{B}\cdot\bar{\boldsymbol{g}}^{\alpha(5)}$, using definitions (4.9), (4.10). As a result, the following "composite" Hermitian moments $\bar{h}_{rs}^{\alpha(4)}$, $\bar{h}_{rs}^{\alpha(6)}$ appear naturally in the relevant expressions:

$$\bar{h}_{rs}^{\alpha(4)}=\sqrt{7}\,h_{rs}^{\alpha(4)}+\sqrt{2}\,h_{rs}^{\alpha(2)},\qquad \bar{h}_{rs}^{\alpha(6)}=3h_{rs}^{\alpha(6)}+2h_{rs}^{\alpha(4)}. \tag{5.13}$$

For conciseness, the three quantities $h_{rs}^{\alpha(2)}$, $\bar{h}_{rs}^{\alpha(4)}$ and $\bar{h}_{rs}^{\alpha(6)}$ will be referred to collectively by the name: (dimensionless) *generalized stress tensors.*

Equations (5.10)–(5.12) are now rewritten explicitly as

$$\sqrt{2}\,\langle\boldsymbol{B}\cdot(\nabla\cdot\mathbf{h}^{\alpha(2)})\rangle=\left(\frac{m_\alpha}{T_\alpha}\right)^{1/2}\langle\boldsymbol{B}\cdot\boldsymbol{Q}^{\alpha(1)}\rangle+\frac{e_\alpha}{T_\alpha}\langle\boldsymbol{B}\cdot\boldsymbol{E}^{(A)}\rangle,$$

$$\langle\boldsymbol{B}\cdot(\nabla\cdot\bar{\mathbf{h}}^{\alpha(4)})\rangle=\sqrt{\frac{5}{2}}\left(\frac{m_\alpha}{T_\alpha}\right)^{1/2}\langle\boldsymbol{B}\cdot\boldsymbol{Q}^{\alpha(3)}\rangle,$$

$$\langle\boldsymbol{B}\cdot(\nabla\cdot\bar{\mathbf{h}}^{\alpha(6)})\rangle=\sqrt{\frac{5}{2}}\left(\frac{m_\alpha}{T_\alpha}\right)^{1/2}\langle\boldsymbol{B}\cdot\boldsymbol{Q}^{\alpha(5)}\rangle. \tag{5.14}$$

12.6. The quasi-transport equations

We now write eqs. (5.9) explicitly, using (4.6.34)–(4.6.38):
Parallel electronic fluxes: *

$$
\begin{aligned}
&c^{e}_{11}h^{e(1)}_{\|} - ac^{e}_{11}h^{i(1)}_{\|} - c^{e}_{13}h^{e(3)}_{\|} - c^{e}_{15}h^{e(5)}_{\|} = g^{e(1)}_{\|} + \bar{g}^{e(1)}_{\|},\\
&-c^{e}_{11}h^{e(1)}_{\|} + ac^{e}_{11}h^{i(1)}_{\|} + c^{e}_{13}h^{e(3)}_{\|} + c^{e}_{15}h^{e(5)}_{\|} = aA\left(g^{i(1)}_{\|} + \bar{g}^{i(1)}_{\|}\right),\\
&-c^{e}_{13}h^{e(1)}_{\|} + ac^{e}_{13}h^{i(1)}_{\|} + c^{e}_{33}h^{e(3)}_{\|} + c^{e}_{35}h^{e(5)}_{\|} = g^{e(3)}_{\|} + \bar{g}^{e(3)}_{\|},\\
&-c^{e}_{15}h^{e(1)}_{\|} + ac^{e}_{15}h^{i(1)}_{\|} + c^{e}_{35}h^{e(3)}_{\|} + c^{e}_{55}h^{e(5)}_{\|} = \bar{g}^{e(5)}_{\|}.
\end{aligned}
\tag{6.1}
$$

Parallel ionic fluxes:

$$
c^{i}_{33}h^{i(3)}_{\|} + c^{i}_{35}h^{i(5)}_{\|} = g^{i(3)}_{\|} + \bar{g}^{i(3)}_{\|}, \qquad c^{i}_{35}h^{i(3)}_{\|} + c^{i}_{55}h^{i(5)}_{\|} = \bar{g}^{i(5)}_{\|}. \tag{6.2}
$$

We recall the notations

$$
a = \left(\frac{T_i}{m_i}\frac{m_e}{T_e}\right)^{1/2}, \qquad A = \frac{1}{Z\mu}\frac{\tau_e}{\tau_i} = \frac{|\Omega_e|\tau_e}{\Omega_i\tau_i}. \tag{6.3}
$$

We note that the *parallel components* of the additional source terms $\bar{g}^{\alpha(P)}_{\|}$, derived from eqs. (4.8)–(4.10) can be written in a considerably simpler form by using the results of the end of section 12.5:

$$
\begin{aligned}
\bar{g}^{\alpha(1)}_{\|} &= -\sqrt{2}\left(\frac{T_\alpha}{m_\alpha}\right)^{1/2}\tau_\alpha \boldsymbol{b}\cdot(\nabla\cdot\boldsymbol{h}^{\alpha(2)}),\\
\bar{g}^{\alpha(3)}_{\|} &= -\sqrt{\frac{2}{5}}\left(\frac{T_\alpha}{m_\alpha}\right)^{1/2}\tau_\alpha \boldsymbol{b}\cdot(\nabla\cdot\bar{\boldsymbol{h}}^{\alpha(4)}),\\
\bar{g}^{\alpha(5)}_{\|} &= -\sqrt{\frac{2}{5}}\left(\frac{T_\alpha}{m_\alpha}\right)^{1/2}\tau_\alpha \boldsymbol{b}\cdot(\nabla\cdot\bar{\boldsymbol{h}}^{\alpha(6)}).
\end{aligned}
\tag{6.4}
$$

The generalized stresses $\bar{\boldsymbol{h}}^{\alpha(4)}$, $\bar{\boldsymbol{h}}^{\alpha(6)}$ are defined by (5.13).

We note that the ion moments $h^{i(3)}_{\|}$, $h^{i(5)}_{\|}$ are decoupled from the other moments. Next, we note that the determinant of eqs. (6.1) is identically zero,

* As will be seen below, it is convenient to include the ion flux $h^{i(1)}_{\|}$ in the group of "electronic fluxes"

because the left-hand sides of the first two equations are equal (but of opposite sign). This is a consequence of the momentum conservation (4.4), and expresses the mutual dependence of the two moments $h_{\|}^{e(1)}$ and $h_{\|}^{i(1)}$. As a result, eqs. (6.1) are solvable only if the following *solubility condition* is satisfied:

$$g_{\|}^{e(1)} + aA\, g_{\|}^{i(1)} + \bar{g}_{\|}^{e(1)} + aA\, \bar{g}_{\|}^{i(1)} = 0. \tag{6.5}$$

If this relation is written out explicitly in dimensional form, using definitions (4.5), (4.8), we find that it is equivalent to

$$\boldsymbol{b}\cdot\nabla(P_e + P_i) + \boldsymbol{b}\cdot\left[\nabla\cdot(\boldsymbol{\pi}^e + \boldsymbol{\pi}^i)\right] = 0, \tag{6.6}$$

which expresses the vanishing of the divergence of the total pressure tensor. This is simply the condition of *mechanical equilibrium*, which must be satisfied because $\mathrm{d}\boldsymbol{u}/\mathrm{d}t = 0$.

We note, moreover that the moments $h_{\|}^{e(1)}$, $h_{\|}^{i(1)}$ appear in eqs. (6.1) only in the combination

$$h_{\|}^{(1)} = ah_{\|}^{i(1)} - h_{\|}^{e(1)}, \tag{6.7}$$

which is simply the dimensionless *electric current*.

In conclusion of this discussion, the set of equations (6.1) can be replaced by an equivalent one by
- suppressing the second equation in the set;
- taking the single moment $h_{\|}^{(1)}$ as an independent variable;
- introducing the source term

$$g_{\|}^{(1)} = -g_{\|}^{e(1)}, \tag{6.8}$$

which is identical to the parallel component of (5.1.12). We thus obtain

$$\begin{aligned}
c_{11}^e h_{\|}^{(1)} + c_{13}^e h_{\|}^{e(3)} + c_{15}^e h_{\|}^{e(5)} &= g_{\|}^{(1)} - \bar{g}_{\|}^{e(1)},\\
c_{13}^e h_{\|}^{(1)} + c_{33}^e h_{\|}^{e(3)} + c_{35}^e h_{\|}^{e(5)} &= g_{\|}^{e(3)} + \bar{g}_{\|}^{e(3)},\\
c_{15}^e h_{\|}^{(1)} + c_{35}^e h_{\|}^{e(3)} + c_{55}^e h_{\|}^{e(5)} &= \qquad\quad \bar{g}_{\|}^{e(5)}.
\end{aligned} \tag{6.9}$$

This set must, of course, be supplemented by constraint (6.5).

The remarkable point about eqs. (6.9) and (6.2) is their similarity with eqs. (5.1.7) and (5.1.9), respectively. The only difference is the appearance of the

additional source terms $\bar{g}_{\|}^{\alpha(P)}$. These involve the generalized stresses and, therefore, represent the essential new feature appearing in the long mean free path regime.

Given this similarity, the solution of these equations is immediate. It leads to a set of *parallel quasi-transport equations* relating the (privileged *and* the non-privileged) fluxes $h_{\|}^{\alpha(P)}$ not only to the thermodynamical forces $g_{\|}^{\alpha(P)}$, but also to the generalized stresses $\bar{g}_{\|}^{\alpha(P)}$:

Parallel electronic fluxes:

$$h_{\|}^{(1)} = \tilde{\sigma}_{\|}\left(g_{\|}^{(1)} - \bar{g}_{\|}^{\mathrm{e}(1)}\right) + \tilde{\alpha}_{\|}\left(g_{\|}^{\mathrm{e}(3)} + \bar{g}_{\|}^{\mathrm{e}(3)}\right) + \tilde{\gamma}_{\|}\,\bar{g}_{\|}^{\mathrm{e}(5)},$$

$$h_{\|}^{\mathrm{e}(3)} = \tilde{\alpha}_{\|}\left(g_{\|}^{(1)} - \bar{g}_{\|}^{\mathrm{e}(1)}\right) + \tilde{\kappa}_{\|}^{\mathrm{e}}\left(g_{\|}^{\mathrm{e}(3)} + \bar{g}_{\|}^{\mathrm{e}(3)}\right) + \tilde{\delta}_{\|}^{\mathrm{e}}\,\bar{g}_{\|}^{\mathrm{e}(5)},$$

$$h_{\|}^{\mathrm{e}(5)} = \tilde{\gamma}_{\|}\left(g_{\|}^{(1)} - \bar{g}_{\|}^{\mathrm{e}(1)}\right) + \tilde{\delta}_{\|}^{\mathrm{e}}\left(g_{\|}^{\mathrm{e}(3)} + \bar{g}_{\|}^{\mathrm{e}(3)}\right) + \tilde{\epsilon}_{\|}^{\mathrm{e}}\,\bar{g}_{\|}^{\mathrm{e}(5)}. \tag{6.10}$$

Parallel ionic fluxes:

$$h_{\|}^{\mathrm{i}(3)} = \tilde{\kappa}_{\|}^{\mathrm{i}}\left(g_{\|}^{\mathrm{i}(3)} + \bar{g}_{\|}^{\mathrm{i}(3)}\right) + \tilde{\delta}_{\|}^{\mathrm{i}}\,\bar{g}_{\|}^{\mathrm{i}(5)},$$

$$h_{\|}^{\mathrm{i}(5)} = \tilde{\delta}_{\|}^{\mathrm{i}}\left(g_{\|}^{\mathrm{i}(3)} + \bar{g}_{\|}^{\mathrm{i}(3)}\right) + \tilde{\epsilon}_{\|}^{\mathrm{i}}\,\bar{g}_{\|}^{\mathrm{i}(5)}. \tag{6.11}$$

The coefficients $\tilde{\sigma}_{\|}$, $\tilde{\alpha}_{\|}$, $\tilde{\kappa}_{\|}^{\mathrm{e}}$, $\tilde{\kappa}_{\|}^{\mathrm{i}}$ are precisely the *parallel transport coefficients* (in the 21M approximation), as given in table **5**.3.2. If the generalized stresses are set equal to zero, the ordinary parallel transport equations (**5**.5.1)–(**5**.5.3) are recovered. The presence of these stresses introduces, however, a set of new *quasi-transport coefficients* into the equations

$$\tilde{\gamma}_{\|} = -\frac{H_{135}^{\mathrm{e}}}{F_{135}^{\mathrm{e}}},$$

$$\tilde{\delta}_{\|}^{\mathrm{e}} = -\frac{H_{315}^{\mathrm{e}}}{F_{135}^{\mathrm{e}}}, \qquad \tilde{\delta}_{\|}^{\mathrm{i}} = -\frac{c_{35}^{\mathrm{i}}}{D_{35}^{\mathrm{i}}},$$

$$\tilde{\epsilon}_{\|}^{\mathrm{e}} = \frac{D_{13}^{\mathrm{e}}}{F_{135}^{\mathrm{e}}}, \qquad \tilde{\epsilon}_{\|}^{\mathrm{i}} = \frac{c_{33}^{\mathrm{i}}}{D_{35}^{\mathrm{i}}}. \tag{6.12}$$

The cross-coefficients $\tilde{\gamma}_{\|}$, $\tilde{\delta}_{\|}^{\mathrm{e}}$, $\tilde{\delta}_{\|}^{\mathrm{i}}$ also exist in the classical theory, where they express the response of the *non-privileged* moments $h_{\|}^{\alpha(5)}$ to the thermodynamic forces $g_{\|}^{(1)}$, $g_{\|}^{\mathrm{e}(3)}$; they were therefore not very relevant in that framework (see section **5**.4). In the present situation, the latter moments formally

loose their non-privileged status, because of the presence of the additional "source terms" $\bar{g}_{\|}^{\alpha(5)}$. As a result, we obtain the quasi-transport equations (6.10) and (6.11). It is important to note that the *extended quasi-transport matrix* [including the coefficients (6.12)] possesses the *Onsager symmetry*.

The parallel quasi-transport equations (multiplied by B) will now be *surface-averaged*. Noting the simplifications discussed after eq. (5.9), it is found that the averages of all the terms involving *gradients* of the temperatures or of the potential vanish identically. We are then left with the following averaged quasitransport equations:

Average parallel electronic fluxes:

$$\left\langle Bh_{\|}^{(1)}\right\rangle = \tilde{\sigma}_{\|}\left\langle Bg_{\|}^{(1)A}\right\rangle - \tilde{\sigma}_{\|}\left\langle B\bar{g}_{\|}^{e(1)}\right\rangle + \tilde{\alpha}_{\|}\left\langle B\bar{g}_{\|}^{e(3)}\right\rangle + \tilde{\gamma}_{\|}\left\langle B\bar{g}_{\|}^{e(5)}\right\rangle,$$

$$\left\langle Bh_{\|}^{e(3)}\right\rangle = \tilde{\alpha}_{\|}\left\langle Bg_{\|}^{(1)A}\right\rangle - \tilde{\alpha}_{\|}\left\langle B\bar{g}_{\|}^{e(1)}\right\rangle + \tilde{\kappa}_{\|}^{e}\left\langle B\bar{g}_{\|}^{e(3)}\right\rangle + \tilde{\delta}_{\|}^{e}\left\langle B\bar{g}_{\|}^{e(5)}\right\rangle,$$

$$\left\langle Bh_{\|}^{e(5)}\right\rangle = \tilde{\gamma}_{\|}\left\langle Bg_{\|}^{(1)A}\right\rangle - \tilde{\gamma}_{\|}\left\langle B\bar{g}_{\|}^{e(1)}\right\rangle + \tilde{\delta}_{\|}^{e}\left\langle B\bar{g}_{\|}^{e(3)}\right\rangle + \tilde{\epsilon}_{\|}^{e}\left\langle B\bar{g}_{\|}^{e(5)}\right\rangle. \quad (6.13)$$

Average parallel ionic fluxes:

$$\left\langle Bh_{\|}^{i(3)}\right\rangle = \tilde{\kappa}_{\|}^{i}\left\langle B\bar{g}_{\|}^{i(3)}\right\rangle + \tilde{\delta}_{\|}^{i}\left\langle B\bar{g}_{\|}^{i(5)}\right\rangle,$$

$$\left\langle Bh_{\|}^{i(5)}\right\rangle = \tilde{\delta}_{\|}^{i}\left\langle B\bar{g}_{\|}^{i(3)}\right\rangle + \tilde{\epsilon}_{\|}^{i}\left\langle B\bar{g}_{\|}^{i(5)}\right\rangle. \quad (6.14)$$

For completeness, we must not forget to add to these equations the surface-averaged solubility constraint (6.5):

$$\left\langle B\bar{g}_{\|}^{e(1)}\right\rangle + aA\left\langle B\bar{g}_{\|}^{i(1)}\right\rangle = 0. \quad (6.15)$$

Note that the external electric field source terms $g_{\|}^{(1)A}$ drop out from this relation, because of the electroneutrality condition.

12.7. The perpendicular fluxes

We now repeat this type of calculation for the perpendicular components of the moments $\boldsymbol{h}^{\alpha(1)}$, $\boldsymbol{h}^{\alpha(3)}$, $\boldsymbol{h}^{\alpha(5)}$, We project eqs. (4.16)–(4.18) on the two remaining directions of the *dynamical reference frame*, i.e. on the perp–tangential direction, $\boldsymbol{e}_{\wedge}$ and on the radial direction, $\boldsymbol{e}_{\rho}$. It turns out again that these equations require *solubility conditions*, which are nothing other than the perpendicular components of the *mechanical equilibrium condition*:

$$c^{-1}\boldsymbol{j} \wedge \boldsymbol{B} = \nabla P + \nabla \cdot \pi. \quad (7.1)$$

The resulting equations are then solved as in the classical case. We thus find, for the electric current,

$$h_{\wedge}^{(1)} = \tilde{\sigma}_{\perp}\left(g_{\wedge}^{(1)} - \bar{g}_{\wedge}^{e(1)}\right) + \tilde{\alpha}_{\perp}\left(g_{\wedge}^{e(3)} + \bar{g}_{\wedge}^{e(3)}\right) + \tilde{\gamma}_{\perp}\bar{g}_{\wedge}^{e(5)}$$

$$-\tilde{\sigma}_{\wedge}\left(g_{\rho}^{(1)} - \bar{g}_{\rho}^{e(1)}\right) - \tilde{\alpha}_{\wedge}\left(g_{\rho}^{e(3)} + \bar{g}_{\rho}^{e(3)}\right) - \tilde{\gamma}_{\wedge}\bar{g}_{\rho}^{e(5)}, \tag{7.2}$$

$$h_{\rho}^{(1)} = \tilde{\sigma}_{\wedge}\left(g_{\wedge}^{(1)} - \bar{g}_{\wedge}^{e(1)}\right) + \tilde{\alpha}_{\wedge}\left(g_{\wedge}^{e(3)} + \bar{g}_{\wedge}^{e(3)}\right) + \tilde{\gamma}_{\wedge}\bar{g}_{\wedge}^{e(5)}$$

$$+\tilde{\sigma}_{\perp}\left(g_{\rho}^{(1)} - \bar{g}_{\rho}^{e(1)}\right) + \tilde{\alpha}_{\perp}\left(g_{\rho}^{e(3)} + \bar{g}_{\rho}^{e(3)}\right) + \tilde{\gamma}_{\perp}\bar{g}_{\rho}^{e(5)}. \tag{7.3}$$

Similar expressions, which we do not write down explicitly, hold for the other moments.

We now introduce a major simplification. In order to be consistent with the treatment of this chapter, we need to analyze the orders of magnitude of the various terms with respect to the parameter ϵ. The analysis in section **5**.6 gives us a first set of orders of magnitude. It was shown there that

$$\tilde{L}_{\wedge} \sim \Omega_{\alpha}^{-1} = O(\epsilon), \qquad \tilde{L}_{\perp} \sim \Omega_{\alpha}^{-2} = O(\epsilon^2), \tag{7.4}$$

where $\tilde{L}$ is any dimensionless transport coefficient.

An additional set of estimates follows from eqs. (**10**.6.5)–(**10**.6.7): it was found that all the gradients are predominantly in the radial direction, hence

$$g_{\rho}^{\alpha(P)} = O(\epsilon^0), \qquad g_{\wedge}^{\alpha(P)} = O(\epsilon). \tag{7.5}$$

Finally, from Sec. 12.3 we know that:

$$\bar{g}_{\rho}^{\alpha(P)} = O(\epsilon), \qquad \bar{g}_{\wedge}^{\alpha(P)} = O(\epsilon). \tag{7.6}$$

When we apply these estimates to eq. (7.2), we see that only two terms are of dominant order ϵ, all the others being of order ϵ^2, or higher,

$$h_{\wedge}^{(1)} = -\tilde{\sigma}_{\wedge}\, g_{\rho}^{(1)} - \tilde{\alpha}_{\wedge}\, g_{\rho}^{e(3)} + O(\epsilon^2). \tag{7.7}$$

We now take the values of the limiting transport coefficients from eq. (**5**.6.2) and find a very simple leading order expression, with $\tilde{\alpha}_{\wedge} = 0$ and $\tilde{\sigma}_{\wedge} = -(\Omega_e \tau_e)^{-1}$.

The same drift scaling analysis, applied to the *radial component* shows that there is no contribution of order ϵ in that expression: *the radial electric current*

is at most of order ϵ^2. This qualitative conclusion is sufficient for our present purpose: the detailed analysis of the radial fluxes will be done in section 12.9.

Before continuing, we note an additional simplification. The source term $g_\rho^{(1)}$, defined in (5.1.12) is, apparently, a rather complicated quantity. On the other hand, the perp–tangential component of the electric current [which is proportional to this source term because of (7.7)] can also be obtained straightforwardly from the *solubility constraint* (7.1),

$$j_\wedge = -\frac{c}{B}\nabla_\rho P + O(\epsilon^2). \tag{7.8}$$

This expression is transformed into dimensionless form by using table 3.2.1,

$$h_\wedge^{(1)} = -\frac{1}{\Omega_e \tau_e} g_\rho^{(1)P}, \tag{7.9}$$

where we introduce the new source term, proportional to the gradient of the *total pressure*,

$$g_\rho^{(1)P} = -\tau_e \left(\frac{m_e}{T_e}\right)^{1/2} \frac{1}{m_e n_e} \nabla_\rho P. \tag{7.10}$$

Comparing (7.9) and (7.7), we find *:

$$g_\rho^{(1)} = -g_\rho^{(1)P}. \tag{7.11}$$

Thus, *the perp–tangential component of the electric current is driven (to leading order) by the radial component of the total pressure gradient.*

We now analyze all the other moments in a similar manner and obtain the following list of the perp–tangential components of the vector moments, through order ϵ:

$$h_\wedge^{(1)} = -\frac{1}{\Omega_e \tau_e} g_\rho^{(1)P},$$

$$h_\wedge^{e(3)} = \frac{1}{\Omega_e \tau_e} g_\rho^{e(3)}, \qquad h_\wedge^{e(5)} = 0,$$

$$h_\wedge^{i(3)} = \frac{1}{\Omega_i \tau_i} g_\rho^{i(3)}, \qquad h_\wedge^{i(5)} = 0, \tag{7.12}$$

* This relation can also be obtained directly from definition (5.1.12) by a rather tedious calculation, in which the centre-of-mass velocity is eliminated.

This result is exceedingly simple, and will play an extremely important role in the later developments. It shows that, to leading order, *the perp–tangential components of the fluxes are driven solely by the radial components of the thermodynamical sources*. The result holds for both the short mean free path regime and the long mean free path regime, because the generalized stresses do not enter these expressions. We also know from the analysis in chapters 5 and 6 that these fluxes are non-dissipative.

The second conclusion of the analysis is: *All the radial components of the fluxes are at most of order* ϵ^2. Another way of saying this is to note that, through order ϵ, the fluxes lie completely on the magnetic surfaces. In particular, the electric current and the particle and heat fluxes are of the form

$$\boldsymbol{j} = j_{\parallel}\boldsymbol{b} + \boldsymbol{j}_{\wedge} + \mathrm{O}(\epsilon^2), \qquad \Gamma^{\alpha} = \Gamma^{\alpha}_{\parallel}\boldsymbol{b} + \Gamma^{\alpha}_{\wedge} + \mathrm{O}(\epsilon^2),$$

$$\boldsymbol{q}^{\alpha} = q^{\alpha}_{\parallel}\boldsymbol{b} + \boldsymbol{q}^{\alpha}_{\wedge} + \mathrm{O}(\epsilon^2). \tag{7.13}$$

This is, in principle, a quite favourable conclusion for the feasibility of the magnetic confinement: the radial components of the particle and heat fluxes, i.e. the "leaks" through the confining configuration, are small (of order ϵ^2). It will be seen later that this optimistic conclusion has to be seriously weakened.

12.8. The zero-divergence constraint. The "poloidal fluxes"

We now arrive at a very important point. We have seen in section **10**.6 that the vector moments in a confined plasma have a peculiar property, valid through order ϵ: *their divergence is zero*, as a result of the geometrical constraints imposed by the confining magnetic field. The divergence of a vector field is an operation involving all the components of the vector. As a result, *the geometrical constraints impose a relation between the components of the vector moments*. We now show how this new type of relation can be exploited.

The zero-divergence constraints relevant for our purpose are

$$\nabla \cdot \boldsymbol{j} = 0 + \mathrm{O}(\epsilon^2), \qquad \nabla \cdot \Gamma^{\alpha} = 0 + \mathrm{O}(\epsilon^2),$$

$$\nabla \cdot \boldsymbol{q}^{\alpha} = 0 + \mathrm{O}(\epsilon^2), \qquad \nabla \cdot \boldsymbol{r}^{\alpha} = 0 + \mathrm{O}(\epsilon^2), \tag{8.1}$$

where $\boldsymbol{r}^{\alpha}$ is the dimensional fifth-order Hermitian moment corresponding to $\boldsymbol{h}^{\alpha(5)}$,

$$r_s^{\alpha} = n_{\alpha}\left(\frac{T_{\alpha}}{m_{\alpha}}\right)^{1/2} h_s^{\alpha(5)}. \tag{8.2}$$

In subsequent formulae we no longer write the term $O(\epsilon^2)$, it being understood that all the results of the present section are valid through order ϵ.

We present here the detailed analysis of the first of eqs. (8.1); the remaining ones are treated in the same way. We thus consider the equation

$$\nabla \cdot \boldsymbol{j} = 0. \tag{8.3}$$

This equation holds through order ϵ; but (7.13) tells us that the radial component of the current is at most of order ϵ^2, hence (8.3) reduces to

$$\nabla \cdot \boldsymbol{j}_{\|} + \nabla \cdot \boldsymbol{j}_{\wedge} = 0. \tag{8.4}$$

This is the equation we were looking for; it expresses the mutual dependence of the parallel and perp–tangential components of the current. We now transform this equation by writing it in the form

$$\nabla \cdot (\boldsymbol{b} j_{\|}) + \nabla \cdot \boldsymbol{j}_{\wedge} = 0.$$

Noting that $\boldsymbol{b} = \boldsymbol{B}/B$ and that $\nabla \cdot \boldsymbol{B} = 0$, this can also be written as

$$\boldsymbol{B} \cdot \nabla \left(\frac{j_{\|}}{B} \right) = -\nabla \cdot \boldsymbol{j}_{\wedge}. \tag{8.5}$$

We now recall that the perp–tangential current component has been determined explicitly in terms of the radial component of the pressure gradient in eq. (7.8). Hence, eq. (8.5) can be regarded as an equation for the "unknown" parallel current, the right-hand side being a known source term.

We now note that eq. (8.5) is of the standard form of a *magnetic differential equation*, defined in (**8**.5.2). From (7.8) we write the source term in the following form (using eq. **G2**.2.15):

$$\sigma \equiv -\nabla \cdot \boldsymbol{j}_{\wedge} = \nabla \cdot \left(\frac{c}{B l_{\rho}} \frac{\mathrm{d}P}{\mathrm{d}\rho} \boldsymbol{e}_{\wedge} \right). \tag{8.6}$$

We recall that the (zeroth-order) total pressure is a surface quantity, depending only on the radial coordinate ρ. We shall often use the abbreviation

$$A'(\rho) \equiv \frac{\mathrm{d}A(\rho)}{\mathrm{d}\rho} \tag{8.7}$$

for any surface function $A(\rho)$.

We first check the solubility condition (**8**.5.9) which, because of (**8**.6.8), can be simply written in terms of the surface average of the source term,

$$\langle \sigma \rangle = V' \left\langle \nabla \cdot \left(\frac{c}{Bl_\rho} \frac{\mathrm{d}P}{\mathrm{d}\rho} \, \boldsymbol{e}_\wedge \right) \right\rangle$$

$$= \frac{\mathrm{d}}{\mathrm{d}\rho} V' \left\langle \frac{c}{Bl_\rho} \frac{\mathrm{d}P}{\mathrm{d}\rho} \boldsymbol{e}_\wedge \cdot \nabla\rho \right\rangle = 0. \tag{8.8}$$

We used here identity (**8**.6.9) and the obvious fact that

$$\boldsymbol{e}_\wedge \cdot \nabla\rho = \frac{1}{l_\rho} \boldsymbol{e}_\wedge \cdot \boldsymbol{e}_\rho = 0.$$

The equation being soluble, its general solution is given by (**8**.5.3) in the form

$$\frac{j_\parallel}{B} = \frac{\hat{j}_\parallel}{B} + J^{(1)}(\rho), \tag{8.9}$$

where $\hat{j}_\parallel$ is a particular solution of (8.5) and $J^{(1)}(\rho)$ is an *arbitrary surface quantity*, depending only on the radial coordinate ρ.

We now recall our assumption that the confining magnetic field is *axisymmetric*, a property which will greatly simplify the solution of (8.5). In order to exploit this property, we have a definite advantage to turn over to the *geometrical triad* $(\boldsymbol{e}_\rho, \boldsymbol{e}_\theta, \boldsymbol{e}_\zeta)$ defined in section 12.5 and use representation (5.2) of the magnetic field. We evaluate the source term by using (5.6) and calculate the divergence by eq. (**G2**.2.16),

$$\sigma = \nabla \cdot \left[\frac{c}{Bl_\rho} P' \frac{1}{Bl_\zeta} \left(-\mathscr{I} \boldsymbol{e}_\theta + \frac{R_0 \mathscr{B}_\mathrm{P}}{l_\rho} \boldsymbol{e}_\zeta \right) \right]$$

$$= \frac{1}{l_\rho l_\theta l_\zeta} \left[\frac{\partial}{\partial\theta} l_\zeta l_\rho \left(-\frac{\mathscr{I} c P'}{l_\rho l_\zeta B^2} \right) + \frac{\partial}{\partial\zeta} l_\rho l_\theta \left(\frac{c P' R_0 \mathscr{B}_\mathrm{P}}{l_\rho^2 l_\zeta B^2} \right) \right].$$

But, because of the assumed axisymmetry, neither the magnetic field, nor the scale factors depend on the toroidal angle ζ, hence the second term

vanishes. As the quantities $\mathscr{I}$, P are surface functions, hence independent of θ, the source term reduces to

$$\sigma = -\frac{\mathscr{I}c}{l_\rho l_\theta l_\zeta} P' \frac{\partial}{\partial\theta}\frac{1}{B^2}. \tag{8.10}$$

We now transform the left-hand side of (8.5) by using the axisymmetry property, as well as representation (5.2) of the magnetic field which, be definition, has no radial component ($B_\rho = 0$),

$$\boldsymbol{B}\cdot\nabla\left(\frac{j_\parallel}{B}\right) = \left[B_\rho\frac{1}{l_\rho}\frac{\partial}{\partial\rho} + B_\theta\frac{1}{l_\theta}\frac{\partial}{\partial\theta} + B_\zeta\frac{1}{l_\zeta}\frac{\partial}{\partial\zeta}\right]\frac{j_\parallel}{B}$$

$$= B_\theta\frac{1}{l_\theta}\frac{\partial}{\partial\theta}\frac{j_\parallel}{B} = \frac{R_0\mathscr{B}_\mathrm{P}}{l_\rho l_\theta l_\zeta}\frac{\partial}{\partial\theta}\frac{j_\parallel}{B}. \tag{8.11}$$

Combining this result with (8.10), we reduce the magnetic differential equation (8.5) to

$$\frac{\partial}{\partial\theta}\frac{j_\parallel}{B} = -\frac{\mathscr{I}c}{R_0\mathscr{B}_P} P' \frac{\partial}{\partial\theta}\frac{1}{B^2}. \tag{8.12}$$

This equation can be immediately integrated, and we find the particular solution

$$\frac{\hat{j}_\parallel}{B} = -\frac{\mathscr{I}c}{R_0\mathscr{B}_\mathrm{P}} P' \frac{1}{B^2}. \tag{8.13}$$

Substituting this into (8.9) we find the *general solution of the magnetic differential equation* (8.5),

$$j_\parallel(\rho,\theta) = -\frac{\mathscr{I}(\rho)\,c}{R_0\mathscr{B}_\mathrm{P}(\rho)} P'(\rho)\frac{1}{B(\rho,\theta)} + J^{(1)}(\rho)\,B(\rho,\theta). \tag{8.14}$$

It can also be expressed in dimensionless form, using table 3.2.1 and definition (7.10),

$$h_\parallel^{(1)} = -\frac{\mathscr{I}}{\Omega_e\tau_e}\frac{l_\rho}{R_0\mathscr{B}_\mathrm{P}} g_\rho^{(1)P} + \frac{1}{en_e}\left(\frac{m_e}{T_e}\right)^{1/2} BJ^{(1)}(\rho). \tag{8.15}$$

We now introduce a set of notations that will be widely used in the forthcoming text, because they lead to compact and elegant formulae. We first

define the quantity $\mathscr{B}_0$ in terms of the surface average of the square of the magnetic field $B(\rho, \theta)$,

$$\mathscr{B}_0 = \langle B^2 \rangle^{1/2}. \tag{8.16}$$

Clearly, $\mathscr{B}_0$ is a surface quantity, whose value depends on the geometrical characteristics of the confining magnetic field. This quantity will be very often used as a natural scaling factor for the magnetic field. A related surface quantity is the Larmor frequency associated with the average magnetic field,

$$\Omega_{\alpha 0} = \frac{e_\alpha \mathscr{B}_0}{m_\alpha c}. \tag{8.17}$$

Another useful quantity which will very frequently appear in forthcoming formulae is

$$\mathscr{K}_\alpha = \frac{\mathscr{I}(\rho)\, l_\rho}{R_0 \mathscr{B}_P(\rho)\, \Omega_{\alpha 0}(\rho)\, \tau_\alpha}. \tag{8.18}$$

At this point, we make the following remark. The quantity $\mathscr{K}_\alpha$ is, in general *not* a surface quantity, because the scale factor l_ρ may depend on the poloidal angle θ. The forthcoming expressions are, however, nicely simplified if it is *assumed* that *the scale factor l_ρ is a surface quantity*,

$$l_\rho = l_\rho(\rho). \tag{8.19}$$

This assumption does not appear very restrictive in usual configurations. [In the standard model, $l_\rho = 1$, as we know from (**8**.9.17).] For simplicity, we thus decide to make this assumption in all cases treated in this book. However, when it is not satisfied, the reader will very easily make the necessary, rather trivial changes. Clearly, whenever (8.19) is valid, the quantity $\mathscr{K}_\alpha$ *is also a surface quantity*.

We now introduce the dimensionless surface quantity ω_1, defined as

$$\omega_1(\rho) = \frac{1}{e n_e} \left(\frac{m_e}{T_e} \right)^{1/2} \mathscr{B}_0\, J^{(1)}(\rho). \tag{8.20}$$

The parallel dimensionless current (8.15) is now written in the compact form

$$h_\parallel^{(1)} = -\frac{\mathscr{B}_0}{B} \mathscr{K}_e\, g_\rho^{(1)P} + \frac{B}{\mathscr{B}_0}\, \omega_1 \tag{8.21}$$

This is the final dimensionless form of the solution of the magnetic differential equation (8.5). Note that the poloidal angle dependence is quite explicit here: θ only enters through the magnetic field $B(\rho, \theta)$.

The remaining equations of (8.1) are treated in exactly the same way. The source terms are taken from eqs. (**10**.6.10) for $\Gamma^{\alpha}_{\wedge}$ and from (**10**.6.11) for $\boldsymbol{q}^{\alpha}_{\wedge}$; moreover, it was shown in (7.12) that $\boldsymbol{r}^{\alpha}_{\wedge} = 0$. The results, in dimensional form [corresponding to eq. (8.14)] are

$$\Gamma^{\alpha}_{\parallel} = -\frac{\mathscr{I}}{\Omega_\alpha m_\alpha}\frac{1}{R_0\mathscr{B}_{\mathrm{P}}}\left(P'_\alpha + e_\alpha n_\alpha \Phi'\right) + J^{\alpha(1)}(\rho)\ B,$$

$$q^{\alpha}_{\parallel} = -\frac{\mathscr{I}}{\Omega_\alpha m_\alpha}\frac{1}{R_0\mathscr{B}_{\mathrm{P}}}\ P_\alpha T'_\alpha + J^{\alpha(3)}(\rho)\ B,$$

$$r^{\alpha}_{\parallel} = J^{\alpha(5)}(\rho)\ B. \tag{8.22}$$

In dimensionless form, the parallel fluxes are

$$h^{\alpha(1)}_{\parallel} = \frac{\mathscr{B}_0}{B}\mathscr{K}_\alpha\left(g^{\alpha(1)}_{\rho} - g^{\alpha(1)A}_{\rho}\right) + \frac{B}{\mathscr{B}_0}\,\omega^{\alpha}_1,$$

$$h^{\alpha(3)}_{\parallel} = \frac{\mathscr{B}_0}{B}\mathscr{K}_\alpha\ g^{\alpha(3)}_{\rho} + \frac{B}{\mathscr{B}_0}\,\omega^{\alpha}_3,$$

$$h^{\alpha(5)}_{\parallel} = \frac{B}{\mathscr{B}_0}\,\omega^{\alpha}_5, \tag{8.23}$$

with the following definitions for the *surface quantities* ω^{α}_n:

$$\omega^{\alpha}_1(\rho) = \frac{1}{n_\alpha}\left(\frac{m_\alpha}{T_\alpha}\right)^{1/2}\mathscr{B}_0 J^{\alpha(1)}(\rho),$$

$$\omega^{\alpha}_3(\rho) = \sqrt{\frac{2}{5}}\,\frac{1}{P_\alpha}\left(\frac{m_\alpha}{T_\alpha}\right)^{1/2}\mathscr{B}_0 J^{\alpha(3)}(\rho),$$

$$\omega^{\alpha}_5(\rho) = \frac{1}{P_\alpha}\left(\frac{m_\alpha}{T_\alpha}\right)^{3/2}\mathscr{B}_0 J^{\alpha(5)}(\rho). \tag{8.24}$$

We also note that the quantity ω_1 is actually a combination of the ω^{α}_1, because of relation (6.7),

$$\omega_1 = a\omega^{\mathrm{i}}_1 - \omega^{\mathrm{e}}_1. \tag{8.25}$$

In conclusion, we have seen that *the zero-divergence constraint provides a partial determination of the parallel fluxes of matter and energy in terms of the thermodynamic forces.* Two remarks are important.

(1) The *parallel fluxes* appear to be driven by the *radial forces*. This seemingly surprisingly result is in contrast with the classical theory, in which the parallel fluxes were completely decoupled from everything that happened in the perpendicular directions. It is the zero-divergence constraint – introduced by the confining geometry – which couples the flux components in a new way and introduces this unexpected effect of the radial gradients on the parallel fluxes.

(2) The zero-divergence constraint provides only an *incomplete determination* of the parallel fluxes. These are fixed up to an arbitrary integration constant, i.e. an arbitrary surface function ω_n^α. These quantities will play a major role in the neoclassical transport theory.

In order to obtain a further interpretation of these surface quantities, the fluxes Γ^α and q^α may be alternatively projected on the geometrical basis (e_θ, e_ζ) instead of on the dynamical basis $(e_\parallel, e_\wedge)$. Using relations (5.6), a short calculation yields the values for the *poloidal components* of the fluxes,

$$h_\theta^{\alpha(1)} = \frac{R_0 \mathscr{B}_P}{\mathscr{B}_0 l_\rho l_\zeta} \omega_1^\alpha, \qquad h_\theta^{\alpha(3)} = \frac{R_0 \mathscr{B}_P}{\mathscr{B}_0 l_\rho l_\zeta} \omega_3^\alpha. \tag{8.26}$$

Hence, the surface quantities ω_j^α are directly proportional (but not equal!) to the poloidal components of the fluxes $h_\theta^{\alpha(j)}$. We shall therefore call them briefly (but improperly!) the "*poloidal fluxes*" (between quotes!). The following important point should be kept in mind: *the true poloidal fluxes* $h_\theta^{\alpha(j)}$ *depend on both* θ *and* ρ [through the factor $(l_\rho l_\zeta)^{-1} \sim B_\theta$], whereas *the quantities* ω_j^α *are surface quantities, independent of the poloidal angle* θ.

The "poloidal fluxes" can be finally determined by noting that we now possess two independent expressions for the parallel fluxes, viz. (8.23) and (6.10), (6.11). By comparing the right-hand sides of these equations, we obtain a set of equations for the ω_n^α. The process is even simpler if we compare the *surface-averaged* equations (6.13) and (6.14) to the surface average of eqs. (8.23). Multiplying the latter by $B/\mathscr{B}_0$, [where $\mathscr{B}_0$ is defined by eq. (8.16)], we find

$$\left\langle \frac{B}{\mathscr{B}_0} h_\parallel^{\alpha(1)} \right\rangle = \mathscr{K}_\alpha \left\langle \left(g_\rho^{\alpha(1)} - g_\rho^{\alpha(1)A} \right) \right\rangle + \omega_1^\alpha,$$

$$\left\langle \frac{B}{\mathscr{B}_0} h_\parallel^{\alpha(3)} \right\rangle = \mathscr{K}_\alpha \left\langle g_\rho^{\alpha(3)} \right\rangle + \omega_3^\alpha,$$

$$\left\langle \frac{B}{\mathscr{B}_0} h_\parallel^{\alpha(5)} \right\rangle = \omega_5^\alpha. \tag{8.27}$$

Let us perform in some detail the identification for the electric current equation (which is a little bit special). Comparing the first equations (8.27) with (6.13) and using (6.7), we obtain

$$\left\langle \frac{B}{\mathscr{B}_0} h_{\|}^{(1)} \right\rangle = \left\langle \frac{B}{\mathscr{B}_0} \left(a h_{\|}^{\mathrm{i}(1)} - h_{\|}^{\mathrm{e}(1)} \right) \right\rangle$$

$$= a\mathscr{K}_{\mathrm{i}} \left\langle \left(g_{\rho}^{\mathrm{i}(1)} - g_{\rho}^{\mathrm{i}(1)A} \right) \right\rangle - \mathscr{K}_{\mathrm{e}} \left\langle \left(g_{\rho}^{\mathrm{e}(1)} - g_{\rho}^{\mathrm{e}(1)A} \right) \right\rangle + a\omega_1^{\mathrm{i}} - \omega_1^{\mathrm{e}}$$

$$= \tilde{\sigma}_{\|} \left\langle \frac{B}{\mathscr{B}_0} g_{\|}^{(1)A} \right\rangle - \tilde{\sigma}_{\|} \left\langle \frac{B}{\mathscr{B}_0} \bar{g}_{\|}^{\mathrm{e}(1)} \right\rangle$$

$$+ \tilde{\alpha}_{\|} \left\langle \frac{B}{\mathscr{B}_0} \bar{g}_{\|}^{\mathrm{e}(3)} \right\rangle + \tilde{\gamma}_{\|} \left\langle \frac{B}{\mathscr{B}_0} \bar{g}_{\|}^{\mathrm{e}(5)} \right\rangle . \tag{8.28}$$

A short calculation, using definitions (4.12)–(4.15) and (7.10) leads to a considerable simplification,

$$a\mathscr{K}_{\mathrm{i}} \left(g_{\rho}^{\mathrm{i}(1)} - g_{\rho}^{\mathrm{i}(1)A} \right) - \mathscr{K}_{\mathrm{e}} \left(g_{\rho}^{\mathrm{e}(1)} - g_{\rho}^{\mathrm{e}(1)A} \right) = -\mathscr{K}_{\mathrm{e}} \, g_{\rho}^{(1)P} . \tag{8.29}$$

We now write the equation resulting from (8.28) together with those obtained from the other moment equations:

$$a\omega_1^{\mathrm{i}} - \omega_1^{\mathrm{e}} + \tilde{\sigma}_{\|} \left\langle \frac{B}{\mathscr{B}_0} \bar{g}_{\|}^{\mathrm{e}(1)} \right\rangle - \tilde{\alpha}_{\|} \left\langle \frac{B}{\mathscr{B}_0} \bar{g}_{\|}^{\mathrm{e}(3)} \right\rangle - \tilde{\gamma}_{\|} \left\langle \frac{B}{\mathscr{B}_0} \bar{g}_{\|}^{\mathrm{e}(5)} \right\rangle$$

$$= \mathscr{K}_{\mathrm{e}} \left\langle g_{\rho}^{(1)P} \right\rangle + \tilde{\sigma}_{\|} \left\langle \frac{B}{\mathscr{B}_0} g_{\|}^{(1)A} \right\rangle ,$$

$$\omega_3^{\mathrm{e}} + \tilde{\alpha}_{\|} \left\langle \frac{B}{\mathscr{B}_0} \bar{g}_{\|}^{\mathrm{e}(1)} \right\rangle - \tilde{\kappa}_{\|}^{\mathrm{e}} \left\langle \frac{B}{\mathscr{B}_0} \bar{g}_{\|}^{\mathrm{e}(3)} \right\rangle - \tilde{\delta}_{\|}^{\mathrm{e}} \left\langle \frac{B}{\mathscr{B}_0} \bar{g}_{\|}^{\mathrm{e}(5)} \right\rangle$$

$$= -\mathscr{K}_{\mathrm{e}} \left\langle g_{\rho}^{\mathrm{e}(3)} \right\rangle + \tilde{\alpha}_{\|} \left\langle \frac{B}{\mathscr{B}_0} g_{\|}^{(1)A} \right\rangle ,$$

$$\omega_5^e + \tilde{\gamma}_{\|}\left\langle \frac{B}{\mathscr{B}_0}\bar{g}_{\|}^{e(1)}\right\rangle - \tilde{\delta}_{\|}^{e}\left\langle \frac{B}{\mathscr{B}_0}\bar{g}_{\|}^{e(3)}\right\rangle - \tilde{\epsilon}_{\|}^{e}\left\langle \frac{B}{\mathscr{B}_0}\bar{g}_{\|}^{e(5)}\right\rangle = \tilde{\gamma}_{\|}\left\langle \frac{B}{\mathscr{B}_0}g_{\|}^{(1)A}\right\rangle$$

$$\omega_3^i - \tilde{\kappa}_{\|}^{i}\left\langle \frac{B}{\mathscr{B}_0}\bar{g}_{\|}^{i(3)}\right\rangle - \tilde{\delta}_{\|}^{i}\left\langle \frac{B}{\mathscr{B}_0}\bar{g}_{\|}^{i(5)}\right\rangle = -\mathscr{K}_i\left\langle g_\rho^{i(3)}\right\rangle,$$

$$\omega_5^i - \tilde{\delta}_{\|}^{i}\left\langle \frac{B}{\mathscr{B}_0}\bar{g}_{\|}^{i(3)}\right\rangle - \tilde{\epsilon}_{\|}^{i}\left\langle \frac{B}{\mathscr{B}_0}\bar{g}_{\|}^{i(5)}\right\rangle = 0. \tag{8.30}$$

These equations must be completed with the solubility constraint (6.15),

$$\left\langle \frac{B}{\mathscr{B}_0}\bar{g}_{\|}^{e(1)}\right\rangle + aA\left\langle \frac{B}{\mathscr{B}_0}\bar{g}_{\|}^{i(1)}\right\rangle = 0. \tag{8.31}$$

These equations determine the "poloidal fluxes" ω_n^α in terms of:
– *four thermodynamic forces*: in *radial pressure gradient* $g_\rho^{(1)P}$, the *two radial temperature gradients* $g_\rho^{\alpha(3)}$ and the *parallel external electric field* $g_{\|}^{(1)A}$;
– *six generalized stresses*: $\bar{g}_{\|}^{\alpha(1)}$, $\bar{g}_{\|}^{\alpha(3)}$, $\bar{g}_{\|}^{\alpha(5)}$ (for $\alpha = e, i$).

The latter quantities cannot be determined by a purely macroscopic moment method: their evaluation from kinetic theory will be the object of chapter 14. It appears that, in the long mean free path regime, *the generalized stresses appear to be linear combinations of the "poloidal fluxes"*. As a result, eqs. (8.30), (8.31) express the "poloidal fluxes" entirely in terms of the thermodynamic forces. This turns out to be a crucial step in the neoclassical transport theory.

12.9. *Decomposition of the average radial fluxes*

The quantities of major interest for a fusion device are the radial components of the particle and heat fluxes, averaged over a magnetic surface, i.e. $\langle h_r^{\alpha(p)}\rangle$, $p = 1, 3$. Indeed, these are the quantities measuring the *leakage* of matter and energy through the confinement region. It turns out that, especially in the long mean free path regime, the operation of surface-averaging plays a quite important role. Actually, a set of transport equations of the hydrodynamic type can only be obtained for these surface-averaged quantities. In this sense, one can speak about a *delocalized hydrodynamics*, typical of the long mean free path regime.

In order to calculate these quantities, we go over to the *local geometrical frame* in order to exploit the simplifications introduced by the *axisymmetry* of

the confined plasma. We consider eq. (4.1) (using also eq. 4.11) and multiply it scalarly by $l_\zeta \boldsymbol{e}_\zeta$ (it turns out that the inclusion of the scale factor l_ζ is important for the forthcoming treatment):

$$\left(\frac{m_\alpha}{T_\alpha}\right)^{1/2}\left[-\frac{1}{m_\alpha n_\alpha} l_\zeta \boldsymbol{e}_\zeta \cdot \nabla P_\alpha + \frac{e_\alpha}{m_\alpha} l_\zeta \boldsymbol{e}_\zeta \cdot (-\nabla \Phi + \boldsymbol{E}^{(A)}) - \sqrt{2}\,\frac{T_\alpha}{m_\alpha} l_\zeta \boldsymbol{e}_\zeta \cdot (\nabla \cdot \mathbf{h}^{\alpha(2)})\right]$$

$$+ \Omega_\alpha l_\zeta \boldsymbol{e}_\zeta \cdot (\boldsymbol{h}^{\alpha(1)} \wedge \boldsymbol{b}) + l_\zeta \boldsymbol{e}_\zeta \cdot \boldsymbol{Q}^{\alpha(1)} = 0. \tag{9.1}$$

Because of the axisymmetry, neither P_α nor Φ depend on ζ, hence the gradient terms vanish identically,

$$\boldsymbol{e}_\zeta \cdot \nabla P_\alpha = \boldsymbol{e}_\zeta \cdot \nabla \Phi = 0.$$

We now transform the magnetic term, using eqs. (5.2)–(5.4),

$$\Omega_\alpha l_\zeta \boldsymbol{e}_\zeta \cdot (\boldsymbol{h}^{\alpha(1)} \wedge \boldsymbol{b}) = \Omega_\alpha l_\zeta B^{-1} \boldsymbol{e}_\zeta \cdot \left[\boldsymbol{h}^{\alpha(1)} \wedge (\boldsymbol{e}_\zeta B_\zeta + \boldsymbol{e}_\theta B_\theta)\right]$$

$$= \Omega_\alpha l_\zeta \frac{B_\theta}{B} \boldsymbol{e}_\zeta \cdot (\boldsymbol{h}^{\alpha(1)} \wedge \boldsymbol{e}_\theta) = \Omega_\alpha l_\zeta \frac{B_\theta}{B} \boldsymbol{e}_\rho \cdot \boldsymbol{h}^{\alpha(1)}$$

$$= \frac{e_\alpha B}{m_\alpha c} l_\zeta \frac{R_0 \mathscr{B}_P}{B} \frac{1}{l_\rho l_\zeta} \boldsymbol{e}_\rho \cdot \boldsymbol{h}^{\alpha(1)} = \frac{e_\alpha}{m_\alpha c} \frac{R_0 \mathscr{B}_P}{l_\rho} \boldsymbol{e}_\rho \cdot \boldsymbol{h}^{\alpha(1)}.$$

We note that this result involves the contravariant radial component of the vector $\boldsymbol{h}^{\alpha(1)}$, as defined in (**G2**.2.7). We might therefore use the alternative notations

$$l_\rho^{-1} \boldsymbol{e}_\rho \cdot \boldsymbol{h}^{\alpha(1)} \equiv \hat{h}^{\alpha(1)\rho} \equiv (\nabla \rho) \cdot \boldsymbol{h}^{\alpha(1)}. \tag{9.2}$$

These relations are useful for making contact with the notations of other authors. However, we prefer using physical coordinates, for the reasons discussed in General Appendix **G2**.2.

The term involving the pressure tensor can be transformed by using the following exact identity, valid for any symmetric tensor $\boldsymbol{A}$ in an axisymmetric configuration (Hinton and Hazeltine 1976),

$$l_\zeta \boldsymbol{e}_\zeta \cdot (\nabla \cdot \boldsymbol{A}) = \nabla \cdot \left[(\boldsymbol{A} \cdot \boldsymbol{e}_\zeta) l_\zeta\right]. \tag{9.3}$$

The proof of this identity is given in Appendix 12A.1. The merit of this property is to show that the pressure tensor term in (9.1) has the form of a *divergence*, a fact that will be further exploited below. At this stage, (9.1) is rewritten as

$$\frac{R_0\mathscr{B}_P}{l_\rho}\, e_\rho \cdot \boldsymbol{h}^{\alpha(1)} = -\left(\frac{m_\alpha}{T_\alpha}\right)^{1/2} c l_\zeta \left(e_\zeta \cdot \boldsymbol{E}^{(A)}\right) - \frac{m_\alpha}{e_\alpha c}\, l_\zeta \left(e_\zeta \cdot \boldsymbol{Q}^{\alpha(1)}\right)$$

$$+\sqrt{2}\left(\frac{m_\alpha}{T_\alpha}\right)^{1/2} \frac{cT_\alpha}{e_\alpha}\, \nabla \cdot \left[\left(\mathbf{h}^{\alpha(2)} \cdot e_\zeta\right) l_\zeta\right]. \tag{9.4}$$

We now take the *surface average* of eq. (9.4). The interesting things happen to the divergence term. Indeed, using the general property (**8**.6.9) of the averaging operation, we find

$$K \equiv \left\langle \nabla \cdot \left(\mathbf{h}^{\alpha(2)} \cdot e_\zeta l_\zeta\right)\right\rangle$$

$$= \frac{1}{V'} \frac{\mathrm{d}}{\mathrm{d}\rho} V' \left\langle (\nabla\rho) \cdot \mathbf{h}^{\alpha(2)} \cdot e_\zeta l_\zeta \right\rangle$$

$$= \frac{1}{V'} \frac{\mathrm{d}}{\mathrm{d}\rho} V' \left\langle l_\rho^{-1} e_\rho \cdot \mathbf{h}^{\alpha(2)} \cdot e_\zeta l_\zeta \right\rangle.$$

We now use the fact that $\mathbf{h}^{\alpha(2)}$ has the CGL form (3.6),

$$K = \frac{1}{V'} \frac{\mathrm{d}}{\mathrm{d}\rho} V' \tfrac{3}{2} \left\langle l_\rho^{-1} h_\parallel^{\alpha(2)} e_\rho \cdot \left(e_\parallel e_\parallel - \tfrac{1}{3}\boldsymbol{I}\right) \cdot e_\zeta \right\rangle = 0, \tag{9.5}$$

the vanishing being due to the obvious properties

$$e_\rho \cdot e_\parallel = e_\rho \cdot e_\zeta = 0.$$

The averaged eq. (9.4) thus reduces to

$$\frac{R_0\mathscr{B}_P}{l_\rho} \left\langle h_\rho^{\alpha(1)} \right\rangle = -c\left(\frac{m_\alpha}{T_\alpha}\right)^{1/2} \left\langle l_\zeta E_\zeta^{(A)} \right\rangle - \frac{m_\alpha c}{e_\alpha} \left\langle l_\zeta Q_\zeta^{\alpha(1)} \right\rangle, \tag{9.6a}$$

where we used our assumption (8.19). Applying exactly the same treatment to the heat flux equation (4.2), we find

$$\frac{R_0\mathscr{B}_P}{l_\rho} \left\langle h_\rho^{\alpha(3)} \right\rangle = -\frac{m_\alpha c}{e_\alpha} \left\langle l_\zeta Q_\zeta^{\alpha(3)} \right\rangle. \tag{9.6b}$$

An important property, which follows immediately from (9.6a) is the *ambipolar character of the average radial particle fluxes*, i.e. the vanishing of the average electric current in the radial direction,

$$\frac{R_0\mathscr{B}_P}{l_\rho}\sum_\alpha e_\alpha\langle\Gamma_\rho^\alpha\rangle$$

$$= -\frac{R_0\mathscr{B}_P}{l_\rho}\sum_\alpha e_\alpha n_\alpha\left(\frac{T_\alpha}{m_\alpha}\right)^{1/2}\langle h_\rho^{\alpha(1)}\rangle$$

$$= -c\left[\left(\sum_\alpha e_\alpha n_\alpha\right)\langle l_\zeta E_\zeta^{(A)}\rangle + \sum_\alpha m_\alpha n_\alpha\left(\frac{T_\alpha}{m_\alpha}\right)^{1/2}\langle l_\zeta Q_\zeta^{\alpha(1)}\rangle\right] = 0,$$

the vanishing being due to the electroneutrality condition and to (4.4). Thus

$$\sum_\alpha e_\alpha\langle\Gamma_\rho^\alpha\rangle = 0 + O(\epsilon^3). \tag{9.7}$$

Thus, *the electric current circulates only on the magnetic surfaces*; it has no radial component. This statement does not imply, however, that there is no radial flow of particles. It simply means that the average fluxes of electrons and of ions are synchronized in the radial direction,

$$\langle\Gamma_\rho^e\rangle = Z\langle\Gamma_\rho^i\rangle.$$

This special type of motion of the charged particles is called an *ambipolar diffusion*. Whenever such a situation arises, the electric current provides no information about the transport of matter. In order to obtain a complete description of the transport processes, we must consider the *radial electron flux*, rather than the radial electric current, as an independent variable.

At this stage, we have exploited all the symmetries of the problem, and we revert to the dynamical reference frame. Using (5.7), we transform (9.6a) into

$$\frac{R_0\mathscr{B}_P}{l_\rho}\langle h_\rho^{\alpha(1)}\rangle = -c\left(\frac{m_\alpha}{T_\alpha}\right)^{1/2}\left\langle l_\zeta\frac{1}{l_\zeta B}\left(\mathscr{J}E_\parallel^{(A)} + \frac{R_0\mathscr{B}_P}{l_\rho}E_\wedge^{(A)}\right)\right\rangle$$

$$-\frac{m_\alpha c}{e_\alpha}\left\langle l_\zeta\frac{1}{l_\zeta B}\left(\mathscr{J}Q_\parallel^{\alpha(1)} + \frac{R_0\mathscr{B}_P}{l_\rho}Q_\wedge^{\alpha(1)}\right)\right\rangle. \tag{9.8}$$

A similar transformation is applied to eq. (9.6b). As a result, the average radial fluxes can be written as a sum of several terms,

$$\left\langle h_\rho^{\alpha(n)} \right\rangle = \delta_{n,1} \left\langle h_\rho^{\alpha(1)} \right\rangle_{\mathrm{DR}} + \left\langle h_\rho^{\alpha(n)} \right\rangle_{\mathrm{CL}} + \left\langle h_\rho^{\alpha(n)} \right\rangle_{\mathrm{NCL}}, \tag{9.9}$$

where we *define* the various terms as follows (Hinton and Hazeltine 1976, Hirshman and Sigmar 1981):
The electric drift fluxes:

$$\left\langle h_\rho^{\alpha(1)} \right\rangle_{\mathrm{DR}} = -c \left(\frac{m_\alpha}{T_\alpha} \right)^{1/2} \left\langle \frac{1}{B} E_\wedge^{(A)} \right\rangle, \tag{9.10}$$

The classical fluxes:

$$\left\langle h_\rho^{\alpha(n)} \right\rangle_{\mathrm{CL}} = -\left\langle \frac{1}{\Omega_\alpha} Q_\wedge^{\alpha(n)} \right\rangle, \quad n = 1, 3, \tag{9.11}$$

The neoclassical fluxes:

$$\left\langle h_\rho^{\alpha(n)} \right\rangle_{\mathrm{NCL}} = -\frac{\mathscr{I} l_\rho}{R_0 \mathscr{B}_{\mathrm{P}}} \left\langle \frac{1}{\Omega_\alpha} \left[Q_\parallel^{\alpha(n)} + \delta_{n,1} \frac{e_\alpha}{m_\alpha} \left(\frac{m_\alpha}{T_\alpha} \right)^{1/2} E_\parallel^{(A)} \right] \right\rangle,$$

$$n = 1, 3. \tag{9.12}$$

The reason for the names given to the various terms will appear clearly in the next chapter's discussion. At present, we continue our business of classification by further decomposing the neoclassical fluxes as

$$\left\langle h_\rho^{\alpha(1)} \right\rangle_{\mathrm{NCL}} = -\frac{\mathscr{I} l_\rho}{R_0 \mathscr{B}_{\mathrm{P}}} \frac{m_\alpha c}{e_\alpha} \left\langle \frac{1}{B^2} B \left[Q_\parallel^{\alpha(n)} + \delta_{n,1} \frac{e_\alpha}{m_\alpha} \left(\frac{m_\alpha}{T_\alpha} \right)^{1/2} E_\parallel^{(A)} \right] \right\rangle$$

$$= -\frac{\mathscr{I} l_\rho}{R_0 \mathscr{B}_{\mathrm{P}}} \frac{m_\alpha c}{e_\alpha}$$

$$\times \left\langle \left(\frac{1}{B^2} - \frac{1}{\mathscr{B}_0^2} + \frac{1}{\mathscr{B}_0^2} \right) B \left[Q_\parallel^{\alpha(n)} + \delta_{n,1} \frac{e_\alpha}{m_\alpha} \left(\frac{m_\alpha}{T_\alpha} \right)^{1/2} E_\parallel^{(A)} \right] \right\rangle$$

$$= -\frac{\mathscr{I} l_\rho}{R_0 \mathscr{B}_P} \frac{m_\alpha c}{e_\alpha} \left\langle \left(\frac{1}{B^2} - \frac{1}{\mathscr{B}_0^2} \right) B Q_\parallel^{\alpha(n)} \right\rangle$$

$$- \frac{\mathscr{I} l_\rho}{R_0 \mathscr{B}_P} \frac{m_\alpha c}{e_\alpha} \frac{1}{\mathscr{B}_0^2} \left\langle B \left[Q_\parallel^{\alpha(n)} + \delta_{n,1} \frac{e_\alpha}{m_\alpha} \left(\frac{m_\alpha}{T_\alpha} \right)^{1/2} E_\parallel^{(A)} \right] \right\rangle$$

$$- \delta_{n,1} \frac{\mathscr{I} l_\rho}{R_0 \mathscr{B}_P} \left(\frac{m_\alpha}{T_\alpha} \right)^{1/2} c \left\langle \left(\frac{1}{B^2} - \frac{1}{\mathscr{B}_0^2} \right) B E_\parallel^{(A)} \right\rangle. \tag{9.13}$$

We recall that $\mathscr{B}_0$ was defined in (8.16).

In the second term on the right-hand side we recognize precisely the combination appearing in the right-hand side of the parallel relations (5.14). Hence, this contribution involves the *generalized stress tensors*, i.e. the part which is most characteristic of the long mean free path regime. We now *define* the various terms as

$$\left\langle h_\rho^{\alpha(n)} \right\rangle_{NCL} = \left\langle h_\rho^{\alpha(n)} \right\rangle_{PS} + \left\langle h_\rho^{\alpha(n)} \right\rangle_B \left[+ \delta_{n,1} \left\langle h_\rho^{\alpha(1)} \right\rangle_{MDR} \right]. \tag{9.14}$$

The Pfirsch–Schlüter fluxes:

$$\left\langle h_\rho^{\alpha(n)} \right\rangle_{PS} = -\frac{\mathscr{I} l_\rho}{R_0 \mathscr{B}_P} \left\langle \frac{1}{\Omega_\alpha} \left(1 - \frac{B^2}{\mathscr{B}_0^2} \right) Q_\parallel^{\alpha(n)} \right\rangle$$

$$= -\mathscr{K}_\alpha T_\alpha \left\langle \frac{B_0}{B} \left(1 - \frac{B^2}{B_0^2} \right) Q_\parallel^{\alpha(n)} \right\rangle. \tag{9.15}$$

The banana fluxes:

$$\left\langle h_\rho^{\alpha(1)} \right\rangle_B = -\frac{\mathscr{I} l_\rho}{R_0 \mathscr{B}_P} \frac{m_\alpha c}{e_\alpha} \left(\frac{T_\alpha}{m_\alpha} \right)^{1/2} \frac{1}{\mathscr{B}_0^2} \sqrt{2} \langle \boldsymbol{B} \cdot (\nabla \cdot \boldsymbol{h}^{\alpha(2)}) \rangle, \tag{9.16}$$

$$\left\langle h_\rho^{\alpha(3)} \right\rangle_B = -\frac{\mathscr{I} l_\rho}{R_0 \mathscr{B}_P} \frac{m_\alpha c}{e_\alpha} \left(\frac{T_\alpha}{m_\alpha} \right)^{1/2} \frac{1}{\mathscr{B}_0^2} \sqrt{\tfrac{2}{5}} \langle \boldsymbol{B} \cdot (\nabla \cdot \bar{\mathbf{h}}^{\alpha(4)}) \rangle. \tag{9.17}$$

Alternative, more compact expressions are obtained in terms of the *additional parallel source terms* defined in (6.4),

$$\left\langle h_\rho^{\alpha(1)} \right\rangle_{\mathrm{B}} = \mathscr{K}_\alpha \left\langle \frac{B}{\mathscr{B}_0} \bar{g}_\parallel^{\alpha(1)} \right\rangle, \tag{9.18}$$

$$\left\langle h_\rho^{\alpha(3)} \right\rangle_{\mathrm{B}} = \mathscr{K}_\alpha \left\langle \frac{B}{\mathscr{B}_0} \bar{g}_\parallel^{\alpha(3)} \right\rangle. \tag{9.19}$$

The quantity $\mathscr{K}_\alpha$ was defined in (8.18).
The modified electric drift fluxes:

$$\left\langle h_\rho^{\alpha(1)} \right\rangle_{\mathrm{MDR}} = -\frac{\mathscr{I} l_\rho c}{R_0 \mathscr{B}_{\mathrm{P}}} \left(\frac{m_\alpha}{T_\alpha} \right)^{1/2} \left\langle \left(\frac{1}{B^2} - \frac{1}{\mathscr{B}_0^2} \right) B E_\parallel^{(A)} \right\rangle. \tag{9.20}$$

It will be shown in sections **13**.1 and **13**.4 that the electric drift fluxes and the modified drift fluxes are very small contributions, compared to the other fluxes. They will most often be neglected in the forthcoming chapters. As a result, *the neoclassical fluxes are simply the sums of the Pfirsch–Schlüter fluxes and of the banana fluxes*. (More generally speaking, this statement can be taken as a *definition of the neoclassical fluxes*. Indeed, it will be shown in chapter 18 that, even when the drift and modified drift fluxes are retained, their status is quite different from the neoclassical ones.)

The meaning of these formal decompositions will be discussed more fully in the next chapter. At this stage, we note the following very important point. The argument leading to eq. (9.7) can be applied to each separate term in the decompositions (9.9) and (9.14). Hence, *not only the total average radial particle fluxes, but also the separate components of these fluxes are ambipolar*. In particular,

$$\sum_\alpha e_\alpha \left\langle \Gamma_\rho^\alpha \right\rangle_{\mathrm{CL}} = 0, \tag{9.21}$$

$$\sum_\alpha e_\alpha \left\langle \Gamma_\rho^\alpha \right\rangle_{\mathrm{PS}} = 0, \tag{9.22}$$

$$\sum_\alpha e_\alpha \left\langle \Gamma_\rho^\alpha \right\rangle_{\mathrm{B}} = 0. \tag{9.23}$$

Table 9.1
Decomposition of the dimensionless radial particle and heat fluxes
$[V_\alpha \equiv \sqrt{T_\alpha/m_\alpha},\ \mathscr{B}_0 \equiv \langle B^2\rangle^{1/2},\ l_\rho = l_\rho(\rho)]$.

Fluxes	Particle fluxes	Heat fluxes
Electric drift		
$\left\langle h_\rho^{\alpha(n)}\right\rangle_{\mathrm{DR}}$	$-\frac{c}{V_\alpha}\left\langle \frac{1}{B}E_\wedge^{(A)}\right\rangle$	0
Modified electric drift		
$\left\langle h_\rho^{\alpha(n)}\right\rangle_{\mathrm{MDR}}$	$-\frac{\mathscr{I}l_\rho c}{R_0\mathscr{B}_\mathrm{P}V_\alpha}\left\langle B\left(\frac{1}{B^2}-\frac{1}{\mathscr{B}_0^2}\right)E_\parallel^{(A)}\right\rangle$	0
Classical		
$\left\langle h_\rho^{\alpha(n)}\right\rangle_{\mathrm{CL}}$	$-\left\langle \frac{1}{\Omega_\alpha}Q_\wedge^{\alpha(1)}\right\rangle$	$-\left\langle \frac{1}{\Omega_\alpha}Q_\wedge^{\alpha(3)}\right\rangle$
Pfirsch–Schlüter		
$\left\langle h_\rho^{\alpha(n)}\right\rangle_{\mathrm{PS}}$	$-\mathscr{K}_\alpha\tau_\alpha\left\langle \frac{\mathscr{B}_0}{B}\left(1-\frac{B^2}{\mathscr{B}_0^2}\right)Q_\parallel^{\alpha(1)}\right\rangle$	$-\mathscr{K}_\alpha\tau_\alpha\left\langle \frac{\mathscr{B}_0}{B}\left(1-\frac{B^2}{\mathscr{B}_0^2}\right)Q_\parallel^{\alpha(3)}\right\rangle$
Banana		
$\left\langle h_\rho^{\alpha(n)}\right\rangle_{\mathrm{B}}$	$\mathscr{K}_\alpha\left\langle \frac{B}{\mathscr{B}_0}\bar{g}_\parallel^{\alpha(1)}\right\rangle$	$\mathscr{K}_\alpha\left\langle \frac{B}{\mathscr{B}_0}\bar{g}_\parallel^{\alpha(3)}\right\rangle$

For the reader's convenience, the decomposition of the average fluxes is summarized in table 9.1.

12.10 The average parallel electric current

In the calculations of the previous section we saw that the *average parallel electric field* influences the *radial* average particle and heat fluxes. This appears explicitly in the modified drift flux; but it also enters the problem in a subtler and more important way.

It was announced at the end of section 12.8 that the generalized stresses will be expressed as linear combinations of the "poloidal fluxes". But these are determined, in turn, by eqs. (8.30) as linear combinations of the radial forces $g_\rho^{(1)P}$, $g_\rho^{\alpha(3)}$ *and* of the *parallel electric field*, $g_\parallel^{(1)\mathrm{A}}$. As a result of this coupling between different directions, *the average radial fluxes (in particular, the banana fluxes) are driven by the radial gradients of pressure and temperature as well as*

by the parallel electric field. The latter plays the role of a *bona fide* thermodynamic force for the radial fluxes.

This effect of the parallel electric field is, however, a *cross-effect* in the thermodynamic sense discussed in section **5**.5. The direct effect of this thermodynamic force consists of driving a *parallel electric current*. Hence, if we want a complete set of transport equations, we need to add an equation for the parallel electric current to the set of equations for the three independent radial fluxes $\langle h_\rho^{\alpha(1)} \rangle$, $\langle h_\rho^{\alpha(3)} \rangle$.

We already derived such an equation: it is the first equation of (6.13). The interesting point about it is that a decomposition similar to the one of the radial fluxes appears quite naturally,

$$\left\langle \frac{B}{\mathscr{B}_0} h_\parallel^{(1)} \right\rangle = \left\langle \frac{B}{\mathscr{B}_0} h_\parallel^{(1)} \right\rangle_{\mathrm{CL}} + \left\langle \frac{B}{\mathscr{B}_0} h_\parallel^{(1)} \right\rangle_{\mathrm{B}}. \tag{10.1}$$

where the two terms are defined as follows.
The classical parallel electric current:

$$\left\langle \frac{B}{\mathscr{B}_0} h_\parallel^{(1)} \right\rangle_{\mathrm{CL}} = \tilde{\sigma}_\parallel \left\langle \frac{B}{\mathscr{B}_0} g_\parallel^{(1)A} \right\rangle. \tag{10.2}$$

For simplicity, we introduce the abbreviation

$$\hat{g}_\parallel^{(1)A} \equiv \left\langle \frac{B}{\mathscr{B}_0} g_\parallel^{(1)A} \right\rangle = \left(\frac{m_\mathrm{e}}{T_\mathrm{e}} \right)^{1/2} \tau_\mathrm{e} \frac{e}{m_\mathrm{e} \mathscr{B}_0} \left\langle BE_\parallel^{(A)} \right\rangle. \tag{10.3}$$

Hence, the classical electric current is written as

$$\left\langle \frac{B}{\mathscr{B}_0} h_\parallel^{(1)} \right\rangle_{\mathrm{CL}} = \tilde{\sigma}_\parallel \hat{g}_\parallel^{(1)A}. \tag{10.4}$$

The banana parallel electric current:

$$\left\langle \frac{B}{\mathscr{B}_0} h_\parallel^{(1)} \right\rangle_{\mathrm{B}} = -\tilde{\sigma}_\parallel \left\langle \frac{B}{\mathscr{B}_0} \bar{g}_\parallel^{\mathrm{e}(1)} \right\rangle + \tilde{\alpha}_\parallel \left\langle \frac{B}{\mathscr{B}_0} \bar{g}_\parallel^{\mathrm{e}(3)} \right\rangle + \tilde{\gamma}_\parallel \left\langle \frac{B}{\mathscr{B}_0} \bar{g}_\parallel^{\mathrm{e}(5)} \right\rangle. \tag{10.5}$$

There is no contribution of the Pfirsch–Schlüter type to the parallel electric current.

Equation (10.5) is particularly interesting. It shows that the coupling between radial and parallel directions goes both ways. Indeed, by using again

the previous argument concerning the generalized stresses, we see that
– the radial electron flux and the radial heat fluxes are influenced by the parallel electric field;
– the parallel electric current is, in turn, influenced by the radial pressure and temperature gradients.

These very peculiar cross-effects are characteristic of the long mean free path regime, in which the generalized stresses play an essential role. They will be analyzed in detail in chapter 15.

12.11. Microscopic expression of the fluxes

The evaluation of the banana fluxes cannot be done by using a "pure" moment method. It requires the calculation of the generalized stresses $h_{rs}^{\alpha(2p)}$. It was shown in section 12.3 that, through order ϵ, these moments have the CGL form (3.14), but the equations do not determine the value of the scalar quantities $h_{\|}^{\alpha(2p)}$. We therefore need a direct solution of the kinetic equation. As will be seen in the forthcoming chapters, it is, in general, very hard to find an explicit solution in the long mean free path regime. It turns out, however, that the information needed for the determination of the banana fluxes can be obtained exactly and explicitly from the kinetic equation. For this purpose, we must specify the microscopic definition of the quantities determining the banana fluxes.

Consider first the particle flux, which is related to the dimensionless pressure tensor $h_{rs}^{\alpha(2)}$. From (**4**.3.17) and (**4**.3.4) we find

$$h_{rs}^{\alpha(2)} = \frac{1}{\sqrt{2}} \int \mathrm{d}\boldsymbol{c} \left(c_r c_s - \tfrac{1}{3} c^2 \delta_{rs} \right) \phi^0(c)\ \chi^\alpha(\boldsymbol{c};\ \boldsymbol{x},\ t). \tag{11.1}$$

We now note that the quantity entering (9.16) is

$$\begin{aligned} &\langle \boldsymbol{B} \cdot (\nabla \cdot \boldsymbol{h}^{\alpha(2)}) \rangle \\ &= \frac{1}{\sqrt{2}} \left\langle B_r \left[\nabla_s \int \mathrm{d}\boldsymbol{c} \left(c_r c_s - \tfrac{1}{3} c^2 \delta_{rs} \right) \phi^0(c)\ \chi^\alpha(\boldsymbol{c};\ \boldsymbol{x},\ t) \right] \right\rangle \\ &= \frac{1}{\sqrt{2}} \left\langle B_r \left[\nabla_s \int \mathrm{d}\boldsymbol{c}\ c_r c_s\ \phi^0(c)\ \chi^\alpha(\boldsymbol{c};\ \boldsymbol{x},\ t) \right] \right\rangle. \end{aligned} \tag{11.2}$$

Indeed, the subtracted scalar part yields zero under the average, because of

property (**8**.6.3). Finally, as the spatial variation comes in only through the distribution function, we finally get

$$\langle \boldsymbol{B}\cdot(\nabla\cdot\boldsymbol{h}^{\alpha(2)})\rangle = \frac{1}{\sqrt{2}}\left\langle B\int \mathrm{d}\boldsymbol{c}\, c_{\parallel}^2\, \boldsymbol{b}\cdot\nabla\phi^0(c)\,\chi^\alpha(\boldsymbol{c};\,\boldsymbol{x},\,t)\right\rangle. \tag{11.3}$$

For further convenience, this quantity can also be expressed in terms of the full velocity distribution function $f^\alpha(\boldsymbol{v};\,\boldsymbol{x},\,t)$ (depending on the *dimensional* variable $\boldsymbol{v}$). We then note that, to leading order in ϵ,

$$c_{\parallel}^2 = \frac{m_\alpha}{T_\alpha}\left(v_{\parallel}-u_{\parallel}^\alpha\right)^2 = \frac{m_\alpha}{T_\alpha}\,v_{\parallel}^2\,[1+\mathrm{O}(\epsilon)],$$

hence

$$\langle \boldsymbol{B}\cdot(\nabla\cdot\boldsymbol{h}^{\alpha(2)})\rangle = \frac{m_\alpha}{\sqrt{2}\,n_\alpha T_\alpha}\left\langle B\int \mathrm{d}\boldsymbol{v}\, v_{\parallel}^2\,\boldsymbol{b}\cdot\nabla f^\alpha(\boldsymbol{v};\,\boldsymbol{x},\,t)\right\rangle. \tag{11.4}$$

Similarly, the "composite" fourth-order stress tensor entering the heat flux is defined by

$$\begin{aligned}\bar{h}_{rs}^{\alpha(4)} &= \sqrt{7}\,h_{rs}^{\alpha(4)} + \sqrt{2}\,h_{rs}^{\alpha(2)}\\ &= \int \mathrm{d}\boldsymbol{c}\left(c_r c_s - \tfrac{1}{3}c^2\delta_{rs}\right)\left\{\sqrt{7}\cdot\frac{1}{2\sqrt{7}}(c^2-7)+\sqrt{2}\cdot\frac{1}{\sqrt{2}}\right\}\phi^0\chi^\alpha\\ &= \int \mathrm{d}\boldsymbol{c}\left(c_r c_s - \tfrac{1}{3}c^2\delta_{rs}\right)\left(\tfrac{1}{2}c^2-\tfrac{5}{2}\right)\phi^0\chi^\alpha.\end{aligned}$$

By a similar calculation we obtain

$$\langle \boldsymbol{B}\cdot(\nabla\cdot\bar{\boldsymbol{h}}^{\alpha(4)})\rangle = \frac{m_\alpha}{n_\alpha T_\alpha}\left\langle B\int \mathrm{d}\boldsymbol{v}\, v_{\parallel}^2\left(\frac{m_\alpha v^2}{2T_\alpha}-\frac{5}{2}\right)\boldsymbol{b}\cdot\nabla f^\alpha\right\rangle. \tag{11.5}$$

Finally, the divergence of the composite sixth-order stress tensor is evaluated as

$$\langle \boldsymbol{B}\cdot(\nabla\cdot\bar{\boldsymbol{h}}^{\alpha(6)})\rangle = \frac{m_\alpha}{\sqrt{7}\,n_\alpha T_\alpha}\left\langle B\int \mathrm{d}\boldsymbol{v}\, v_{\parallel}^2\left(\frac{m_\alpha^2 v^4}{4T_\alpha^2}-\frac{7m_\alpha v_\alpha^2}{2T_\alpha}+\frac{35}{4}\right)\boldsymbol{b}\cdot\nabla f^\alpha\right\rangle. \tag{11.6}$$

We now come to an important point. Just as the macroscopic study in the previous sections of this chapter appealed repeatedly to the drift approximation and to the related ordering in ϵ, the microscopic study must correspondingly be done in a set of variables in which this ordering is manifest. In other words, we must use a set of *natural guiding centre (NGC)* variables, as described in chapters 1 and 9, for describing the motion of the particles. From the form of eqs. (11.4)–(11.6), a convenient choice of such variables is the set $(\boldsymbol{Y}, K, M, \phi)$ described in table **1**.8.4. (the final choice used in chapter 14 will be slightly different.) As we are interested only in the leading terms in ϵ, it is sufficient to do the transformation as

$$\boldsymbol{x} \to \boldsymbol{Y} + \mathrm{O}(\epsilon), \qquad \tfrac{1}{2} m_\alpha v^2 \to K + \mathrm{O}(\epsilon),$$

$$v_\parallel \to U_\alpha = \sigma (2/m_\alpha)^{1/2} (K - MB(\boldsymbol{Y}))^{1/2}.$$

Hence, the relevant quantities take the form

$$\langle \boldsymbol{B} \cdot (\nabla \cdot \boldsymbol{h}^{\alpha(2)}) \rangle = \frac{m_\alpha}{\sqrt{2}\, n_\alpha T_\alpha} \left\langle B \int \mathrm{d}\boldsymbol{v}\, U_\alpha^2\, \boldsymbol{b} \cdot \nabla f^\alpha(K, M, \phi; \boldsymbol{Y}) \right\rangle,$$

$$\langle \boldsymbol{B} \cdot (\nabla \cdot \bar{\boldsymbol{h}}^{\alpha(4)}) \rangle = \frac{m_\alpha}{n_\alpha T_\alpha} \left\langle B \int \mathrm{d}\boldsymbol{v}\, U_\alpha^2 \left(\frac{K}{T_\alpha} - \frac{5}{2} \right) \boldsymbol{b} \cdot \nabla f^\alpha(K, M, \phi; \boldsymbol{Y}) \right\rangle$$

$$\langle \boldsymbol{B} \cdot (\nabla \cdot \bar{\boldsymbol{h}}^{\alpha(6)}) \rangle$$

$$= \frac{m_\alpha}{\sqrt{7}\, n_\alpha T_\alpha} \left\langle B \int \mathrm{d}\boldsymbol{v}\, U_\alpha^2 \left(\frac{K^2}{T_\alpha^2} - 7 \frac{K}{T_\alpha} + \frac{35}{4} \right) \boldsymbol{b} \cdot \nabla f^\alpha(K, M, \phi; \boldsymbol{Y}) \right\rangle. \tag{11.7}$$

We recall that the volume element in velocity space is

$$\mathrm{d}\boldsymbol{v} = \mathrm{d}K\, \mathrm{d}M\, \mathrm{d}\phi\, |J_\alpha|,$$

where J_α is the Jacobian (given in table **1**.8.4). Note that the only factor depending on the gyrophase ϕ in the integrands is the distribution function f^α. Hence, the effect of the integration over ϕ is simply to change the distribution function f^α into the *gyro-averaged distribution function* $\bar{f}^\alpha$ [see section **10**.4],

$$\int \mathrm{d}\phi\, f^\alpha(K, M, \phi; \boldsymbol{Y}) = 2\pi \bar{f}^\alpha(K, M; \boldsymbol{Y}). \tag{11.8}$$

Hence, the calculation of the banana fluxes does not require the complete distribution functions, but only their gyro-averages. The basic equation (**10**.4.22) obeyed by the gyro-averaged distribution functions was derived in section **10**.4 through order ϵ: it was called the *drift kinetic equation.*

We are now able to pinpoint precisely our problem:

For the calculation of the banana fluxes, we need a solution of the drift-kinetic equation, in the long mean free path regime. This solution is used for the evaluation of the surface-averaged divergence of the generalized stresses (11.7).

Appendix 12A.1. Proof of relation (9.3)

Let $A_{rs}(\rho, \theta, \zeta)$ be the components of a symmetric tensor ($A_{rs} = A_{sr}$), which may even depend on the toroidal angle ζ. It is assumed that the toroidal coordinate system is *axisymmetric*, i.e. that the scale factors $l_r = l_r(\rho, \theta)$ are independent of ζ. By eqs. (5.3), this condition is realized when the confining magnetic field is axisymmetric (i.e. the components B_r are independent of ζ).

Using eq. (**G2**.2.23), the left-hand side of (9.3) is evaluated as

$$
\begin{aligned}
& l_\zeta \boldsymbol{e}_\zeta \cdot (\nabla \cdot \boldsymbol{A}) \\
& = l_\zeta \cdot \frac{1}{l_\zeta} \left[\frac{\partial}{\partial \zeta} A_{\zeta\zeta} + \frac{l_\zeta}{l_\rho} \frac{\partial}{\partial \rho} A_{\rho\zeta} + \frac{l_\zeta}{l_\theta} \frac{\partial}{\partial \theta} A_{\theta\zeta} \right. \\
& \qquad + \frac{1}{l_\rho} \frac{\partial l_\zeta}{\partial \rho} A_{\zeta\rho} + \frac{1}{l_\theta} \frac{\partial l_\zeta}{\partial \theta} A_{\zeta\theta} - \frac{1}{l_\rho} \frac{\partial l_\rho}{\partial \zeta} A_{\rho\rho} - \frac{1}{l_\theta} \frac{\partial l_\theta}{\partial \zeta} A_{\theta\theta} \\
& \qquad + \left(\frac{1}{l_\rho} \frac{\partial l_\zeta}{\partial \rho} + \frac{l_\zeta}{l_\rho l_\theta} \frac{\partial l_\theta}{\partial \rho} \right) A_{\rho\zeta} + \left(\frac{1}{l_\theta} \frac{\partial l_\zeta}{\partial \theta} + \frac{l_\zeta}{l_\rho l_\theta} \frac{\partial l_\rho}{\partial \theta} \right) A_{\theta\zeta} \\
& \qquad \left. + \left(\frac{1}{l_\rho} \frac{\partial l_\rho}{\partial \zeta} + \frac{1}{l_\theta} \frac{\partial l_\theta}{\partial \zeta} + \frac{1}{l_\zeta} \frac{\partial l_\zeta}{\partial \zeta} \right) A_{\zeta\zeta} - \frac{1}{l_\zeta} \frac{\partial l_\zeta}{\partial \zeta} A_{\zeta\zeta} \right] \\
& = \frac{1}{l_\rho l_\theta l_\zeta} \left[\frac{\partial}{\partial \zeta} \left(A_{\zeta\zeta} l_\rho l_\theta l_\zeta \right) + \frac{\partial}{\partial \rho} \left(A_{\rho\zeta} l_\theta l_\zeta^2 \right) + \frac{\partial}{\partial \theta} \left(A_{\theta\zeta} l_\rho l_\zeta^2 \right) \right] \\
& \quad - \frac{1}{l_\zeta} \frac{\partial l_\zeta}{\partial \zeta} A_{\zeta\zeta} - \frac{1}{l_\rho} \frac{\partial l_\rho}{\partial \zeta} A_{\rho\rho} - \frac{1}{l_\theta} \frac{\partial l_\theta}{\partial \zeta} A_{\theta\theta}.
\end{aligned}
$$

Because of the assumed *axisymmetry*, the last three terms are identically null. Hence

$$
\begin{aligned}
& l_\zeta \boldsymbol{e}_\zeta \cdot (\nabla \cdot \boldsymbol{A}) \\
& \quad = \frac{1}{l_\rho l_\theta l_\zeta} \left[\frac{\partial}{\partial \zeta} \left(A_{\zeta\zeta} l_\zeta \cdot l_\rho l_\theta \right) + \frac{\partial}{\partial \rho} \left(A_{\rho\zeta} l_\zeta \cdot l_\theta l_\zeta \right) + \frac{\partial}{\partial \theta} \left(A_{\theta\zeta} l_\zeta \cdot l_\zeta l_\rho \right) \right] \\
& \quad = \nabla \cdot \left(\boldsymbol{A} \cdot \boldsymbol{e}_\zeta l_\zeta \right).
\end{aligned}
$$

The last step follows from eq. (**G**2.2.16). The identity is thus proven.

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13

The classical fluxes and the Pfirsch–Schlüter effect

13.1. The electric drift fluxes and the modified drift fluxes

The average radial fluxes were decomposed in section **12.9** into a certain number of contributions. We now begin the systematic calculation of these fluxes in order to obtain specific *transport equations*, i.e. relations between the fluxes and the thermodynamic forces. As will be seen, some of these fluxes can be evaluated by a pure moment method. This operation will be done in the present chapter. For the remaining ones, which involve the generalized stresses, we need to return to the kinetic equation and solve it under the appropriate (long mean free path) conditions.

We start with the simplest of the expressions of table **12.8.1**: the *electric drift flux*. This contribution concerns only the particle flux; there is no corresponding heat flux. When written in dimensional form (using table **3.2.1**), eq. (**12.9.10**) reads

$$\left\langle \Gamma_\rho^\alpha \right\rangle_{\mathrm{DR}} = -n_\alpha c \left\langle B^{-1} E_\wedge^{(A)} \right\rangle . \tag{1.1}$$

In this form, the interpretation of this contribution is obvious. Consider, indeed, the *electric drift velocity* of charged particles of species α, which is given by eq. (**1.5.22**). Evaluating it with the fields present in our toroidal configuration, we find that it has a radial component which is easily calculated by using eqs. (**12.5.2**)–(**12.5.7**) as well as (**12.4.11**),

$$\begin{aligned} w_{\mathrm{E},\rho} &= \frac{c}{B^2} (\boldsymbol{E} \wedge \boldsymbol{B}) \cdot \boldsymbol{e}_\rho \\ &= \frac{c}{B^2 l_\zeta} \left[\boldsymbol{E} \wedge \left(\mathscr{I} \boldsymbol{e}_\zeta + R_0 \mathscr{B}_{\mathrm{P}} l_\rho^{-1} \boldsymbol{e}_\theta \right) \right] \cdot \boldsymbol{e}_\rho \\ &= \frac{c}{B^2 l_\zeta} \left[\mathscr{I} (\boldsymbol{e}_\zeta \wedge \boldsymbol{e}_\rho) + R_0 \mathscr{B}_{\mathrm{P}} l_\rho^{-1} (\boldsymbol{e}_\theta \wedge \boldsymbol{e}_\rho) \right] \cdot \boldsymbol{E} \end{aligned}$$

$$= \frac{c}{B^2 l_\zeta}\left(\mathscr{I}\boldsymbol{e}_\theta - R_0 \mathscr{B}_\mathrm{p} l_\rho^{-1}\boldsymbol{e}_\zeta\right)\cdot\left[-l_\rho^{-1}\Phi'(\rho)\boldsymbol{e}_\rho + \boldsymbol{E}^{(A)}\right]$$

$$= -\frac{c}{B}\boldsymbol{e}_\wedge \cdot \boldsymbol{E}^{(A)}. \tag{1.2}$$

We note that the scalar potential does not contribute to the drift motion in the radial direction, because (to zeroth order in ϵ) its gradient is directed precisely in this direction. Only the externally induced field can produce a radial drift. Equation (1.1) can thus be rewritten as

$$\left\langle \Gamma_\rho^\alpha \right\rangle_{\mathrm{DR}} = n_\alpha \left\langle \boldsymbol{e}_\rho \cdot c \frac{\boldsymbol{E}^{(A)} \wedge \boldsymbol{B}}{B^2} \right\rangle. \tag{1.3}$$

Thus, the electric drift flux is simply the average flux of electrons and of ions drifting under the action of the electric and magnetic fields. This is a purely mechanical, non-dissipative motion: there is no transport coefficient associated with these fluxes. It is clearly an ambipolar motion, as we already know from chapter 1: the electrons and the ions drift in the same direction, with the same velocity, hence, the electric drift fluxes produce no radial electric current. It may be noted that these fluxes exist even in a straight, homogeneous magnetic field. *

In order to make order-of-magnitude estimates, we scale the electric drift velocity by a typical macroscopic velocity u, which we define, for convenience, as the average electron velocity in the parallel direction (the latter measures, indeed, the thermal motion of the particles, unimpeded by the magnetic field). It then follows from the formulae in table 1.9.1 that the dimensionless electric drift velocity $\tilde{w}_\mathrm{E} \equiv w_\mathrm{E}/u$ is given by

$$\tilde{w}_\mathrm{E} = \epsilon \frac{c}{u}\frac{\boldsymbol{E} \wedge \boldsymbol{b}}{B}. \tag{1.4}$$

The reader should not be mistaken by the huge factor c/u; the guiding centres do, of course, not move at the speed of the light! Indeed, it should not be forgotten that, using the Maxwell equations, the ratio of the electric field to the magnetic field may be estimated by eq. (4.1.3), from which follows

$$\frac{E}{B} \approx \frac{u}{c}. \tag{1.5}$$

* The reason why we did not find them in the classical treatment in chapter 5 is that we did not look for them. We were interested there in the electric current (to which the drift fluxes contribute strictly zero), not in the individual particle fluxes.

This leaves us with an order of magnitude, $\tilde{w}_E \sim \epsilon$. This, however, is only a maximum estimate. Indeed, as follows from table 1.9.1, with the orderings (1.9.9)–(1.9.11), only the potential part of the electric field, $\boldsymbol{E}^{(\Phi)} = -\nabla\Phi$ (see eq. 1.9.11) produces a drift or order ϵ. As the potential field (to leading order) points in the radial direction, the corresponding drift velocity is in the perp–tangential direction, hence it cannot contribute to the radial particle flux. In order to find such a flux from the electric drift contribution, we must consider the external field $\boldsymbol{E}^{(A)} = -c^{-1}\,\partial_t \boldsymbol{A}$, which is (globally) of order ϵ^2 (see eq. 1.9.11). This field possesses a perp–tangential component which, by eq. (1.4), can produce a radial drift. However, in the tokamak configuration, the field $\boldsymbol{E}^{(A)}$ (produced by the external coils) is oriented in the toroidal direction, i.e. "almost" parallel to the magnetic field. As a result, the perp–tangential component of the (toroidally directed) electric field is by an order of ϵ smaller than its parallel component. Thus, if $|\boldsymbol{E}^{(A)}| = O(\epsilon^2)$, then $E_\wedge^{(A)} = O(\epsilon^3)$. As we are interested in this book in the fluxes through order ϵ^2, *the electric drift flux is negligible in the tokamak configuration*. This point will be explicitly verified in the standard model in section 13.4.

The electric field also appears in the contribution called the "*modified drift fluxes*:

$$\left\langle \Gamma_\rho^\alpha \right\rangle_{\mathrm{MDR}} = -n_\alpha \frac{\mathscr{I} l_\rho c}{R_0 \mathscr{B}_P} \left\langle B\left(\frac{1}{B^2} - \frac{1}{\mathscr{B}_0^2}\right) E_\parallel^{(A)} \right\rangle. \tag{1.6}$$

This term appears under various equivalent guises in the literature, but (1.6) is certainly its simplest form. Its expression is, at least dimensionally, rather similar to the electric drift term; but the components of $\boldsymbol{E}$ and $\boldsymbol{B}$ involved here are different.

The main point about this contribution is its obvious dependence on the details of the confining magnetic field geometry. These enter through the factor $\mathscr{I}$ and mainly through the combinations $(B^{-2} - \mathscr{B}_0^{-2})$. The latter factor is directly related to the magnetic field line curvature, as will be shown in section 13.4. From this point of view, these fluxes are similar to the Pfirsch–Schlüter fluxes and many authors classify them as such. Without going into "taxonomical" considerations, it may be said that these fluxes are electric drift contributions, modified by a Pfirsch–Schlüter curvature effect.

It will be shown in section 13.4 that, in the standard model, these modified fluxes are by one order of ϵ smaller than the electric drift fluxes: they are of order ϵ^4.

In conclusion, for all toroidal configurations of the tokamak type, the two fluxes discussed here are very small compared to other contributions to be presently studied. This does not mean, however, that they should be com-

pletely forgotten. They actually represent a special type of diffusion, entirely controlled by the electromagnetic fields. Their study cannot be separated from other types of slow motions of the confinement system. In a general situation, the magnetic surfaces are not strictly stationary. Their evolution in time – which was hitherto neglected, because of its slowness – turns out to be controlled by the same factors as the electric drift fluxes. In a general treatment, these two types of motion must be discussed together.

In the forthcoming text, these *electrodynamic effects* will be disregarded, because of their smallness, and because they do not influence the other transport phenomena. In chapter 18, they will, however, be integrated in the general framework of transport in a toroidally confined plasma.

13.2. The classical fluxes

We now proceed to the analysis of the truly dissipative fluxes, beginning with those denoted as *classical* in table **12**.9.1 and in eq. (**12**.10.4):

$$\left\langle h_\rho^{\mathrm{e}(1)} \right\rangle_{\mathrm{CL}} = -\left\langle \frac{1}{\Omega_{\mathrm{e}}} Q_\wedge^{\mathrm{e}(1)} \right\rangle,$$

$$\left\langle h_\rho^{\alpha(3)} \right\rangle_{\mathrm{CL}} = -\left\langle \frac{1}{\Omega_\alpha} Q_\wedge^{\alpha(3)} \right\rangle,$$

$$\left\langle \frac{B}{\mathscr{B}_0} h_\parallel^{(1)} \right\rangle = \tilde{\sigma}_\parallel \hat{g}_\parallel^{(1)A}. \tag{2.1}$$

We first note that there is no need to write two separate equations for the particle fluxes $\langle h_\rho^{\alpha(1)} \rangle_{\mathrm{CL}}$ and $\langle h_\rho^{\mathrm{i}(1)} \rangle_{\mathrm{CL}}$. Indeed, we know from (**12**.9.21) that these fluxes are ambipolar, thus the two quantities are not independent: a single equation for, say, the electron flux, is sufficient.

The evaluation of these expressions is very easy. Consider, for instance, the electron flux. Using eq. (**4**.6.35) for the generalized friction, and eq. (**12**.7.12) for the perp–tangential fluxes, we find

$$\left\langle h_\rho^{\mathrm{e}(1)} \right\rangle_{\mathrm{CL}} = -\left\langle \frac{1}{\Omega_{\mathrm{e}}\tau_{\mathrm{e}}} \left(c_{11}^{\mathrm{e}} h_\wedge^{(1)} + c_{13}^{\mathrm{e}} h_\wedge^{\mathrm{e}(3)} + c_{15}^{\mathrm{e}} h_\wedge^{\mathrm{e}(5)} \right) \right\rangle$$

$$= \left\langle \frac{c_{11}^{\mathrm{e}}}{(\Omega_{\mathrm{e}}\tau_{\mathrm{e}})^2} g_\rho^{(1)P} \right\rangle - \left\langle \frac{c_{13}^{\mathrm{e}}}{(\Omega_{\mathrm{e}}/\tau_{\mathrm{e}})^2} g_\rho^{\mathrm{e}(3)} \right\rangle. \tag{2.2}$$

We note that the pressure, the temperatures and the densities are, to leading order ϵ^0, independent of the poloidal angle, and so is the scale factor l_ρ, by our assumption (**12**.8.19). It then follows that *the source terms* $g_\rho^{(1)P}$, $g_\rho^{\alpha(3)}$ *are surface quantities*, which can be taken out of the average sign. We thus obtain

$$\left\langle h_\rho^{e(1)} \right\rangle_{CL} = \left\langle \frac{1}{(\Omega_e \tau_e)^2} \right\rangle \left(c_{11}^e \, g_\rho^{(1)P} - c_{13}^e \, g_\rho^{e(3)} \right). \tag{2.3}$$

This is a typical *transport equation* relating linearly the average electron flux to the thermodynamic forces. The remaining equations of (2.1) yield similar relations,

$$\left\langle h_\rho^{e(3)} \right\rangle_{CL} = \left\langle \frac{1}{(\Omega_e \tau_e)^2} \right\rangle \left(-c_{13}^e \, g_\rho^{(1)P} + c_{33}^e \, g_\rho^{e(3)} \right), \tag{2.4}$$

$$\left\langle h_\rho^{i(3)} \right\rangle_{CL} = \left\langle \frac{1}{(\Omega_i \tau_i)^2} \right\rangle c_{33}^i \, g_\rho^{i(3)}. \tag{2.5}$$

Before discussing these relations, we also write the corresponding dimensional equations. For this purpose, we introduce *two* equivalent systems of notation for the thermodynamic forces, for the classical fluxes and for the transport coefficients.

In a first system, the thermodynamic forces are denoted by the set (X_1^e, X_3^e, X_3^i, X_E), the fluxes by (J_1^e, J_3^e, J_3^i, J_E) and the transport coefficients by symbols such as L_{11}^{ee}, L_{13}^{ei}, ... In this system, the nature of the fluxes and forces is clearly apparent; it is therefore appropriate for discussions of specific processes.

If more global structural properties are under discussion, a more compact system, avoiding the accumulation of subscripts and superscripts, is desirable. We then denote the four forces by the letter X with a single index, running from 1 to 4, and similarly for the fluxes. The two sets of symbols are defined as follows.

(a) *Thermodynamic forces*:

$$X_1^e \equiv X_1 = -(n_e T_e)^{-1} \nabla_\rho P, \qquad X_3^e \equiv X_2 = -T_e^{-1} \nabla_\rho T_e,$$

$$X_3^i \equiv X_3 = -T_i^{-1} \nabla_\rho T_i, \qquad X_E \equiv X_4 = \mathscr{B}_0^{-1} \left\langle B E_\parallel^{(A)} \right\rangle. \tag{2.6}$$

(b) *Classical fluxes*:

$$J_1^e \equiv J_1^{CL} = \langle \Gamma_\rho^e \rangle_{CL}, \qquad J_3^e \equiv J_2^{CL} = T_e^{-1} \langle q_\rho^e \rangle_{CL},$$

$$J_3^i \equiv J_2^{CL} = T_i^{-1} \langle q_\rho^i \rangle_{CL}, \qquad J_E \equiv J_4^{CL} = \mathscr{B}_0^{-1} \langle B j_\parallel \rangle_{CL}. \tag{2.7}$$

In the first system of notations, the transport equations are written in the form

$$J_1^e = L_{11}^{ee} X_1^e + L_{13}^{ee} X_3^e, \qquad J_3^e = L_{31}^{ee} X_1^e + L_{33}^{ee} X_3^e,$$

$$J_3^i = L_{33}^{ii} X_3^i, \qquad J_E = L_{EE} X_E. \tag{2.8}$$

Note that, for completeness, both the flux symbols J and the transport coefficients L should actually carry an additional index "CL"; the latter is, however, suppressed merely for avoiding a very heavy notation.

In the second system of notations, the same equations are simply written as

$$J_\mu^{CL} = \sum_{\nu=1}^{4} \mathscr{L}_{\mu\nu}^{CL} X_\nu. \tag{2.9}$$

The correspondence between transport coefficients in the two notation systems is obtained by comparing the elements in the transport matrices:

$$\mathscr{L}^{CL} = \begin{pmatrix} L_{11}^{ee} & L_{13}^{ee} & 0 & 0 \\ L_{31}^{ee} & L_{33}^{ee} & 0 & 0 \\ 0 & 0 & L_{33}^{ii} & 0 \\ 0 & 0 & 0 & L^{EE} \end{pmatrix} = \begin{pmatrix} \mathscr{L}_{11}^{CL} & \mathscr{L}_{12}^{CL} & \mathscr{L}_{13}^{CL} & \mathscr{L}_{14}^{CL} \\ \mathscr{L}_{21}^{CL} & \mathscr{L}_{22}^{CL} & \mathscr{L}_{23}^{CL} & \mathscr{L}_{24}^{CL} \\ \mathscr{L}_{31}^{CL} & \mathscr{L}_{32}^{CL} & \mathscr{L}_{33}^{CL} & \mathscr{L}_{34}^{CL} \\ \mathscr{L}_{41}^{CL} & \mathscr{L}_{42}^{CL} & \mathscr{L}_{43}^{CL} & \mathscr{L}_{44}^{CL} \end{pmatrix}. \tag{2.10}$$

It is not difficult to see why the present contributions to the fluxes are called "classical". The following facts are manifest from the form of the transport matrix:

- the average electron fluxes and the average ion fluxes in the same (radial) direction are uncoupled;
- the average fluxes in different (radial and parallel) directions are uncoupled.

These characteristics are the same as in classical transport.

Turning now to the values of the transport coefficients, we start the discussion with the *parallel electric current. The coefficient* L_{EE} *appearing in* (2.8) *is precisely equal to the (exact) parallel conductivity appearing in the classical theory* (see table 5.3.3). In practice, of course, we use an approximate

value, consistent with the approximation of the other fluxes. Here, we use the 21M value of $\tilde{\sigma}$ given in table 5.3.2 (the symbols are defined in eq. 5.3.26),

$$L_{EE} = \sigma_{\|} = \frac{D_{35}^{e}}{F_{135}^{e}} \frac{e^2 n_e}{m_e} \tau_e . \tag{2.11}$$

As we know, this 21M approximation is excellent.

Many authors (Hinton and Hazeltine 1976, Hirshman and Sigmar 1981) call coefficient (2.11) the *Spitzer conductivity* for the obvious reason that the earliest work on this problem was due to Spitzer and his co-workers. We have discussed Spitzer's work in section 5.7A. It was shown there that the Spitzer conductivity was only defined numerically.

Consider now the *ion heat flux* equation. The transport coefficient appearing here is

$$L_{33}^{ii} = \frac{5}{2} \frac{n_i T_i}{m_i} \tau_i c_{33}^{i} \left\langle \frac{1}{(\Omega_i \tau_i)^2} \right\rangle . \tag{2.12}$$

Recalling eq. (5.6.2) and table 5.3.3, we see that this is nothing other than the *classical asymptotic value of the perpendicular ion thermal conductivity, averaged over a magnetic surface*,

$$L_{33}^{ii} = \left\langle \kappa_{\perp}^{i\infty} \right\rangle . \tag{2.13}$$

The appearance of the *asymptotic* value of the thermal conductivity is consistent with our systematic use of the drift approximation in the study of magnetically confined plasmas.

The same conclusion holds for the average electron fluxes, but here the relations appear in a form different from the classical ones, because we are now working with the radial electron flux (rather than the radial electric current), which is conjugate to a new thermodynamic force: the radial pressure gradient (rather than the modified electric field).

The connection with the classical asymptotic perpendicular transport coefficients (5.6.2) is again manifest, but the interpretation of the coefficients is different.

In the absence of a temperature gradient, there exists an average radial electron flux, driven by the presence of a total pressure gradient *. This

* The role of the pressure gradient as a driving force for the diffusion will be discussed in section 16 5 from the thermodynamic point of view

phenomenon is a typical *diffusion process*. Therefore, L_{11}^{ee} is to be interpreted as the *electron diffusion coefficient*,

$$L_{11}^{ee} = \frac{n_e T_e}{m_e} \tau_e c_{11}^e \left\langle \frac{1}{(\Omega_e \tau_e)^2} \right\rangle. \tag{2.14}$$

The corresponding dimensionless coefficient is simply the surface-average of the asymptotic classical coefficient $\langle \tilde{\sigma}_\perp^\infty \rangle$; but the dimensional form is different because of the new interpretation of this quantity.

Fo the same reason, the cross-coefficients L_{13}^{ee}, L_{31}^{ee} are no longer interpreted as thermoelectric coefficients. Instead, L_{13}^{ee} measures the electron flux produced by the presence of a temperature gradient, i.e. the *Soret effect*, well-known in non-equilibrium thermodynamics (de Groot and Mazur 1984). Conversely, L_{31}^{ee} determines the electron heat flux produced by a pressure gradient, i.e. the *Dufour effect*. The *Onsager symmetry* is manifest,

$$L_{13}^{ee} = L_{31}^{ee} = -\sqrt{\frac{5}{2}} \frac{n_e T_e}{m_e} \tau_e c_{13}^e \left\langle \frac{1}{(\Omega_e \tau_e)^2} \right\rangle. \tag{2.15}$$

L_{13}^{ee} will be called the *thermodiffusion coefficient*.

Finally, the coefficient L_{33}^{ee} is simply the surface-averaged classical asymptotic *electron thermal conductivity*,

$$L_{33}^{ee} = \langle \kappa_\perp^{e\infty} \rangle = \frac{5}{2} \frac{n_e T_e}{m_e} \tau_e c_{33}^e \left\langle \frac{1}{(\Omega_e \tau_e)^2} \right\rangle. \tag{2.16}$$

The properties of the asymptotic classical transport coefficients were discussed in detail in section **5.6**. We therefore only briefly recall them here:

(a) The values of the coefficients L_{11}^{ee}, L_{13}^{ee}, L_{33}^{ee}, L_{33}^{ii} given in eqs. (2.12)–(2.16) are *universal*, i.e. independent of the level of approximation chosen for the calculation of the moments.

(b) The classical radial transport coefficients are *inversely proportional to the square of the magnetic field intensity*. In our drift scaling, this means that they are of order ϵ^2, in agreement with our general discussion in section **12.7**.

(c) The classical radial transport coefficients are *proportional to the collision frequency* τ_α^{-1}.

(d) The sign of the classical transport coefficients is in agreement with the second law of thermodynamics. In particular,

$$L_{11}^{ee} > 0. \qquad L_{33}^{\alpha\alpha} > 0, \qquad L_{EE} > 0. \tag{2.17}$$

As for the non-diagonal thermodiffusion coefficient (for which thermodynamics requires no definite sign), it is *negative*:

$$L_{13}^{ee} < 0. \tag{2.18}$$

This implies the following transport phenomena. At constant total pressure, an electron temperature gradient produces an average electron flux directed towards the hot side. On the other hand, at constant electron temperature, a pressure gradient produces an average electron heat flux directed toward the high pressure region.

In a tokamak configuration, it is clear that the hottest region, as well as the highest pressures are found near the magnetic axis. Thus *both thermodiffusion fluxes are directed inwards*. This is in contrast to the *diagonal diffusion and heat conduction fluxes, which are directed outwards*. Thus, *the thermodiffusive phenomena tend to partially compensate the outward leakage of matter and energy in a tokamak*.

(e) When the classical transport coefficients taken from eqs. (2.3)–(2.5), together with the numerical values of the collision matrix elements taken from table **4**.6.3, are inserted into (2.8), one obtains pretty simple equations,

$$\langle \Gamma_\rho^e \rangle_{CL} = \frac{1}{m_e \tau_e l_\rho} \langle \Omega_e^{-2} \rangle \left(-\frac{\partial P}{\partial \rho} + \tfrac{3}{2} n_e \frac{\partial T_e}{\partial \rho} \right),$$

$$\langle q_\rho^e \rangle_{CL} = \frac{T_e}{m_e \tau_e l_\rho} \langle \Omega_e^{-2} \rangle \left[\frac{3}{2} \frac{\partial P}{\partial \rho} - \left(\frac{13}{4} + \frac{\sqrt{2}}{Z} \right) n_e \frac{\partial T_e}{\partial \rho} \right],$$

$$\langle q_\rho^i \rangle_{CL} = -\frac{n_i T_i}{m_i \tau_i l_\rho} \langle \Omega_i^{-2} \rangle \sqrt{2} \frac{\partial T_i}{\partial \rho}. \tag{2.19}$$

These relations were obtained by Rosenbluth and Kaufman (1958) (see also Kaufman 1966, Braginskii 1965).

(f) We also present the classical transport coefficients in a form very frequently used in the neoclassical literature. This form is analogous to the one given in table **5**.6.1 and involves the Larmor radius of the species α. The only difference with (**5**.6.10) is that we need to define the latter in terms of a *constant* reference magnetic field which, quite naturally, is taken as the average field $\mathscr{B}_0$, eq. (**12**.8.16),

$$\rho_{\alpha 0}^2 = \frac{V_{T\alpha}^2}{\Omega_{\alpha 0}^2} = \left(\frac{m_\alpha c}{e_\alpha \mathscr{B}_0} \right)^2 \cdot 2 \frac{T_\alpha}{m_\alpha}. \tag{2.20}$$

Table 2.1
The classical transport coefficients.
The two systems of notation are defined in eqs (2.7) and (2.10)

L_{11}^{ee}	$\mathscr{L}_{11}^{CL}$	$\frac{1}{2}\frac{n_e\rho_{e0}^2}{\tau_e}\mathscr{G}c_{11}^e$
L_{13}^{ee}	$\mathscr{L}_{12}^{CL}$	$-\frac{1}{2}\sqrt{\frac{5}{2}}\frac{n_e\rho_{e0}^2}{\tau_e}\mathscr{G}c_{13}^e$
L_{33}^{ee}	$\mathscr{L}_{22}^{CL}$	$\frac{5}{4}\frac{n_e\rho_{e0}^2}{\tau_e}\mathscr{G}c_{33}^e$
L_{33}^{ii}	$\mathscr{L}_{33}^{CL}$	$\frac{5}{4}\frac{n_i\rho_{i0}^2}{\tau_i}\mathscr{G}c_{33}^i$
L_{EE}	$\mathscr{L}_{44}^{CL}$	$\frac{e^2n_e}{m_e}\tau_e\tilde{\sigma}_{\|}$

We also introduce the symbol $\mathscr{G}$ for denoting a surface quantity appearing frequently in the forthcoming formulae,

$$\mathscr{G} \equiv \left\langle \frac{\mathscr{B}_0^2}{B^2} \right\rangle. \tag{2.21}$$

We then obtain the values of the *classical transport coefficients* collected in table 2.1.

13.3. The Pfirsch–Schlüter fluxes

We now come to what might be conceived as the most surprising feature of the transport phenomena in a confined plasma. Indeed, it is easily accepted that if a new element, viz. the generalized stresses, is introduced into the moment equations in order to take into account the long mean free path effects, we should find a corresponding new contribution to the fluxes, viz. the banana fluxes. On the contrary, in a strongly collisional regime, where the generalized stresses play no significant role, it is difficult to imagine, at a quick glance, that the transport laws in a confined plasma should differ from the classical laws.

We have seen, however, that a new contribution arises, which does not depend (directly, at least) on the generalized stresses. The *Pfirsch-Schlüter fluxes* of table **12**.9.1. are non-zero quantities whenever

$$\frac{1}{\langle B^2 \rangle} \neq \left\langle \frac{1}{B^2} \right\rangle .$$

The difference between these two surface averages reflects the purely geometrical characteristics due to the curvature of the confining magnetic field lines.

We now evaluate the Pfirsch–Schlüter fluxes in order to, hopefully, obtain a set of transport equations relating the fluxes to the thermodynamical forces. The expression in table **12**.9.1 are first combined with the expressions of the parallel generalized frictions given in eqs. (**4**.6.34), (**4**.6.35) in the 21M approximation. We thus find, for the electron flux,

$$\left\langle h_\rho^{e(1)} \right\rangle_{PS} = -\mathscr{K}_e \left\langle \left(\frac{\mathscr{B}_0^2}{B^2} - 1 \right) \frac{B}{\mathscr{B}_0} \left(c_{11}^e h_\parallel^{(1)} + c_{13}^e h_\parallel^{e(3)} + c_{15}^e h_\parallel^{e(5)} \right) \right\rangle . \tag{3.1}$$

This can also be written in the following, more symmetrical form

$$\left\langle h_\rho^{e(1)} \right\rangle_{PS} = -\mathscr{K}_e \left[c_{11}^e \left\langle \left(\frac{\mathscr{B}_0^2}{B^2} - 1 \right) \frac{B}{\mathscr{B}_0} h_\parallel^{(1)} \right\rangle + c_{13}^e \left\langle \left(\frac{\mathscr{B}_0^2}{B^2} - 1 \right) \frac{B}{\mathscr{B}_0} h_\parallel^{c(3)} \right\rangle \right.$$

$$\left. + c_{15}^e \left\langle \left(\frac{\mathscr{B}_0^2}{B^2} - 1 \right) \frac{B}{\mathscr{B}_0} h_\parallel^{e(5)} \right\rangle \right] . \tag{3.2}$$

We recall here the definition of the surface quantity $\mathscr{K}_\alpha$ that was introduced in (**12**.8.18) [together with assumption (**12**.8.19)],

$$\mathscr{K}_\alpha = \frac{\mathscr{J} l_\rho}{R_0 \mathscr{B}_P \Omega_{\alpha 0} \tau_\alpha} . \tag{3.3}$$

We now recall that we obtained in section **12**.8 a set of relations between the parallel fluxes and the *radial source terms*, i.e. the thermodynamic forces of

interest to us. We thus substitute eqs. (**12**.8.23) and (**12**.8.25) into (3.2) and find

$$\left\langle h_\rho^{e(1)} \right\rangle_{PS} = -\mathscr{K}_e \times \left\{ c_{11}^e \left\langle \left(\frac{\mathscr{B}_0^2}{B^2} - 1 \right) \frac{B}{\mathscr{B}_0} \left[-\mathscr{K}_e \frac{\mathscr{B}_0}{B} g_\rho^{(1)P} + \frac{B}{\mathscr{B}_0} \left(a\omega_1^i - \omega_1^e \right) \right] \right\rangle + c_{13}^e \left\langle \left(\frac{\mathscr{B}_0^2}{B^2} - 1 \right) \frac{B}{\mathscr{B}_0} \left[\mathscr{K}_e \frac{\mathscr{B}_0}{B} g_\rho^{e(3)} + \frac{B}{\mathscr{B}_0} \omega_3^e \right] \right\rangle + c_{15}^e \left\langle \left(\frac{\mathscr{B}_0^2}{B^2} - 1 \right) \frac{B}{\mathscr{B}_0} \cdot \frac{B}{\mathscr{B}_0} \omega_5^e \right\rangle \right\}. \tag{3.4}$$

The fifth-order moment $h_\parallel^{e(5)}$ only contributes to (3.4) through a "poloidal flux" ω_5^e: there is no source term $g_\rho^{e(5)}$ associated with this moment (or with any higher-order moment), because it is a *non-privileged moment* (see section **4**.4).

We note that, in the expressions written under the average sign, the field intensity B is the only quantity depending on the poloidal angle: all the others are surface quantities. Equation (3.4) can thus be rewritten as

$$\left\langle h_\rho^{e(1)} \right\rangle_{PS} = \mathscr{K}_e^2 \left\langle \left(\frac{\mathscr{B}_0^2}{B^2} - 1 \right) \right\rangle \left[c_{11}^e g_\rho^{1(P)} - c_{13}^e g_\rho^{e(3)} \right] - \mathscr{K}_e \left\langle \left(\frac{\mathscr{B}_0^2}{B^2} - 1 \right) \frac{B^2}{\mathscr{B}_0^2} \right\rangle \left[c_{11}^e \left(a\omega_1^i - \omega_1^e \right) + c_{13}^e \omega_3^e + c_{15}^e \omega_5^e \right]. \tag{3.5}$$

We see here two quite distinct terms. The first one ($\sim \mathscr{K}_e^2$) is expressed in terms of the thermodynamic forces and has already the form of a transport relation. The second term ($\sim \mathscr{K}_e$) involves the "poloidal fluxes" which are, in turn, determined by eqs. (**12**.8.30) as functions of the thermodynamic forces, but *also* of the generalized stresses. The Pfirsch–Schlüter fluxes would therefore have different expressions in the short mean free path regime (where

$\bar{g}_{\|}^{\alpha(P)} \approx 0$) and in the long mean free path regime. This is, however, an illusory conclusion. Indeed, keeping in mind (8.16), we note that

$$\left\langle \left(\frac{\mathscr{B}_0^2}{B^2} - 1 \right) \frac{B^2}{\mathscr{B}_0^2} \right\rangle = \left\langle 1 - \frac{B^2}{\mathscr{B}_0^2} \right\rangle = 1 - \frac{\langle B^2 \rangle}{\mathscr{B}_0^2} = 0.$$

Thus, *the "poloidal fluxes" do not contribute at all to the PS fluxes.* *

The remaining PS fluxes are treated in the same way as above and lead to the following final formulae, in which we use the quantity $\mathscr{G}$, defined in (2.21),

$$\left\langle h_\rho^{\mathrm{e}(1)} \right\rangle_{\mathrm{PS}} = \mathscr{K}_{\mathrm{e}}^2 (\mathscr{G} - 1) \left(c_{11}^{\mathrm{e}} \, g_\rho^{(1)P} - c_{13}^{\mathrm{e}} \, g_\rho^{\mathrm{e}(3)} \right), \tag{3.6}$$

$$\left\langle h_\rho^{\mathrm{e}(3)} \right\rangle_{\mathrm{PS}} = \mathscr{K}_{\mathrm{e}}^2 (\mathscr{G} - 1) \left(- c_{13}^{\mathrm{e}} \, g_\rho^{(1)P} + c_{33}^{\mathrm{e}} \, g_\rho^{\mathrm{e}(3)} \right), \tag{3.7}$$

$$\left\langle h_\rho^{\mathrm{i}(3)} \right\rangle_{\mathrm{PS}} = \mathscr{K}_{\mathrm{i}}^2 (\mathscr{G} - 1) \, c_{33}^{\mathrm{i}} \, g_\rho^{\mathrm{i}(3)}. \tag{3.8}$$

These expressions are in the desired form of a set of *transport equations*, relating the average radial fluxes to the thermodynamic forces. The analogy with the classical transport equations (2.3)–(2.5) is quite striking. We insist on the fact that these relations are *universal* in two respects:

(1) *They are valid in all collisionality regimes.* Indeed, the only feature which introduce a difference between short- and long mean free path regimes is the presence of the "poloidal fluxes" $\omega_{\mathrm{n}}^{\alpha}$. But we have seen that these quantities give rigorously no contribution to the PS fluxes.

(2) *They are independent of the truncation level.* The difference between the higher approximation levels (21M, 29M, ...) and the lowest one (13M) is the inclusion of the higher moments ($h_{\|}^{\alpha(5)}$, $h_{\|}^{\alpha(7)}$, ...) in the expression of the parallel friction. But, as shown in eq. (3.4), these *non-privileged moments* contribute no source term, but only "poloidal fluxes" to the PS radial fluxes; the latter contributions, however, have a rigorously zero coefficient.

The physical mechanism underlying the Pfirsch–Schlüter effect is the build-up, in a quasi-steady state, of pressure and temperature inhomogeneities on the magnetic surfaces due to the geometrical toroidicity constraints. Such gradients (in the perp–tangential direction), combined with the magnetic field,

* It appears that this very important conclusion has not been noticed by earlier investigators: it comes out quite directly when the calculation is done in the way presented here; see Appendix 13A.1.

give rise to radially directed diamagnetic drifts, which can be precisely identified with the PS-fluxes.

It is interesting to perform the evaluation of these fluxes by using an alternative method which exhibits these effects explicitly; this is done in Appendix 13A.1.

This may be the appropriate point for a short historical discussion. The *PS effect* was discovered by Pfirsch and Schlüter (1962) but was, unfortunately, never published as a formal paper. On the basis of a purely macroscopic model (resistive MHD) they showed that the quasi-stationary state in toroidal geometry imposes a specific form for the parallel electric current (determined by the condition $\nabla \cdot \boldsymbol{j} = 0$) which, in turn, produces an enhanced radial diffusion. It was then shown by Shafranov (1966) that the heat fluxes are enhanced by the same mechanism. Whereas these results were obtained in the large aspect ratio (standard) model, the formulae were generalized by Maschke (1971, 1972) for an arbitrary geometry. Rutherford (1970) treated the problem by solving the drift kinetic equation.

A much simpler and more systematic approach to the problem was developed by Hazeltine and Hinton (1973). Their elegant solution was considered a classic for a long time: it was taken over in their celebrated review paper (Hinton and Hazeltine 1976). Their method is briefly discussed in Appendix 13A.1. A refinement of the Pfirsch–Schlüter theory for the short mean free path regime was performed by Engelmann and Nocentini (1976, 1977) and by Nocentini and Engelmann (1977).

Subsequently, the problem was taken over again by Hirshman (1976, 1978) with the main emphasis laid on the transport in a multicomponent plasma. These results are collected and reviewed by Hirshman and Sigmar (1981). In the latter paper, the decomposition of the radial fluxes discussed in section 12.9 was derived and used as a starting point for the calculation of the various fluxes. *

It appears that none of the authors quoted above noted that the PS fluxes can be written in a *universal form* (3.6)–(3.8). In particular, the Hazeltine–Hinton result was obtained specifically for a *collision-dominated plasma* and in the *13M approximation*. It appears that their method, which is reviewed in Appendix 13A.1, cannot be easily extended to higher approximations and to a long mean free path regime without considerable complications. Although their result coincides exactly with our eqs. (3.6)–(3.8) in the short mean free path 13M approximation, the universality of those equations is concealed in their approach.

* A similar decomposition appears also in the article by Hinton and Hazeltine (1976), but is not systematically exploited in the calculations.

We now come back to our transport equations and write them in dimensional form. In our first system of notations they read

$$J_1^e = L_{11}^{ee} X_1^e + L_{13}^{ee} X_3^e, \qquad J_3^e = L_{31}^{ee} X_1^e + L_{33}^{ee} X_3^e, \qquad J_3^i = L_{33}^{ii} X_3^i. \quad (3.9)$$

In the second system of notations we write

$$J_\mu^{PS} = \sum_{\mu=1}^{4} \mathscr{L}_{\mu\nu}^{PS} X_\nu. \quad (3.10)$$

The PS transport matrix has the form

$$\mathscr{L} = \begin{pmatrix} \mathscr{L}_{11}^{PS} & \mathscr{L}_{12}^{PS} & 0 & 0 \\ \mathscr{L}_{21}^{PS} & \mathscr{L}_{22}^{PS} & 0 & 0 \\ 0 & 0 & \mathscr{L}_{33}^{PS} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (3.11)$$

It has the same structure as the classical one, except that $\mathscr{L}_{44}^{PS} = 0$: *there is no PS contribution to the average parallel electric current* (this was already noted in section **12**.10).

The values of the PS transport coefficients are collected in table 3.1, using notations (2.10), (2.21). The following useful symbol was introduced in these formulae:

$$\mathscr{J} = \frac{\mathscr{I} l_p}{R_0 \mathscr{B}_p}. \quad (3.12)$$

Table 3 1
The Pfirsch–Schlüter transport coefficients
The two systems of notation are defined in eqs. (2 7) and (2 10).

L_{11}^{ee}	$\mathscr{L}_{11}^{PS}$	$\frac{1}{2}\frac{n_e \rho_{e0}^2}{\tau_e} \mathscr{J}^2 (\mathscr{G} - 1) c_{11}^e$
L_{13}^{ee}	$\mathscr{L}_{12}^{PS}$	$-\frac{1}{2}\sqrt{\frac{5}{2}}\frac{n_e \rho_{e0}^2}{\tau_e} \mathscr{J}^2 (\mathscr{G} - 1) c_{13}^e$
L_{33}^{ee}	$\mathscr{L}_{22}^{PS}$	$\frac{5}{4}\frac{n_e \rho_{e0}^2}{\tau_e} \mathscr{J}^2 (\mathscr{G} - 1) c_{33}^e$
L_{33}^{ii}	$\mathscr{L}_{33}^{PS}$	$\frac{5}{4}\frac{n_i \rho_{i0}^2}{\tau_i} \mathscr{J}^2 (\mathscr{G} - 1) c_{33}^i$
L_{bE}	$\mathscr{L}_{44}^{PS}$	0

The remarkable feature which results from a comparison between tables 2.1 and 3.1 is the following:

Each PS transport coefficient is exactly proportional to the corresponding classical transport coefficient.

The proportionality coefficient is the same for all the transport coefficients: it is a purely geometrical quantity, independent of the temperature and of the density,

$$\mathscr{L}_{\mu\nu}^{\mathrm{PS}} = \mathscr{J}^2 \frac{\mathscr{G}-1}{\mathscr{G}} \mathscr{L}_{\mu\nu}^{\mathrm{CL}}, \quad \mu, \nu = 1, 2, 3, 4. \tag{3.13}$$

This property makes the discussion of the PS transport coefficients very simple. It immediately implies that the PS transport matrix possesses the *Onsager symmetry*,

$$L_{13}^{\mathrm{ee}} = L_{31}^{\mathrm{ee}}, \tag{3.14a}$$

or

$$\mathscr{L}_{\mu\nu}^{\mathrm{PS}} = \mathscr{L}_{\nu\mu}^{\mathrm{PS}}. \tag{3.14b}$$

The only point which is not immediately manifest is the *sign* of the transport coefficients, which depends on the sign of the quantity $(\mathscr{G}-1)$. It is, however easily shown (Hinton and Hazeltine 1976) that

$$0 \leqslant \left\langle \left(\frac{1}{B} - \frac{B}{\mathscr{B}_0^2} \right)^2 \right\rangle = \left\langle \left(\frac{1}{B^2} - 2\frac{1}{\mathscr{B}_0^2} + \frac{B^2}{\mathscr{B}_0^4} \right) \right\rangle$$

$$= \left\langle \frac{1}{B^2} \right\rangle - \frac{1}{\mathscr{B}_0^2},$$

from which follows

$$\mathscr{G} - 1 \geqslant 0. \tag{3.15}$$

As a result, all of the properties of the classical transport coefficients listed under (a)–(d) in section 13.2 are shared by the PS transport coefficients.

The geometrical factor $(\mathscr{G}-1)$ vanishes trivially when the magnetic field is uniform. This is the reason why the PS effect was not found in the classical theory of chapter 5. Less evident is the fact that $(\mathscr{G}-1)=0$ also when the magnetic field is *straight*, though inhomogeneous. This will be shown in the next section. The PS effect is thus clearly related to the *curvature of the magnetic field lines*.

We now collect all the results obtained in the present chapter. The *total average fluxes* (in absence of generalized stresses) are simply obtained from (2.9) and (3.10) in the following form:

$$J_\mu = \sum_{\nu=1}^{4} \left(\mathscr{L}_{\mu\nu}^{\mathrm{CL}} + \mathscr{L}_{\mu\nu}^{\mathrm{PS}} \right) X_\nu . \tag{3.16}$$

(We recall that the electric drift and modified drift fluxes are consistently neglected.)

We note the following important facts:

– Whenever the generalized stresses $\bar{g}_{\|}^{\alpha(p)}$ are negligible, i.e. in the *collision-dominated, short mean free path regime*, eq. (3.16) gives the *complete* average fluxes.

– In the *long mean free path regime*, these expressions need no modification; the contribution of the banana fluxes must simply be added to the right-hand side of (3.16).

– In the short mean free path regime, the average fluxes obey transport equations having all the required thermodynamic properties. (The non-equilibrium thermodynamics of the transport phenomena in magnetically confined plasmas will be studied in detail in chapter 17.)

– The total transport coefficients are simply the *sum* of the classical and the PS coefficients.

– The only point which is not yet obvious is the *relative order of magnitude* of the classical and the PS transport coefficients. It will be shown in the next section that, in realistic situations, *the PS transport coefficients are definitely larger than the classical ones*. The PS effect thus represents an *enhancement* of the outward radial transport of the matter and energy. Thus, the confinement properties of the toroidal configurations are less favourable than predicted in the classical transport theory.

13.4. Classical and Pfirsch–Schlüter transport

The transport coefficients derived in the present chapter involve characteristic geometrical factors whose value depends on the detailed form of the confining magnetic field. Their evaluation for a real system is a very complicated problem which can only be handled numerically. One may get, however, an idea about the size of the effects in the limiting case of a toroidal configuration of very large aspect ratio, by approximating the magnetic field by the "standard model" described in detail in section **8.9**. In this case, we know from (**8.9.6**) that:

$$\eta(r) \equiv \frac{r}{R_0} \leqslant \bar{\eta} \ll 1.$$

We shall assume here that the inverse aspect ratio is of the same order as the drift parameter ϵ,

$$\bar{\eta} \approx \epsilon. \tag{4.1}$$

We now apply the results of section **8**.9 to the calculation of the surface averages appearing in the formulae of the previous section, through order η^2.

We recall the representation of the magnetic field in the standard model (**8**.9.5),

$$\boldsymbol{B}(r, \theta) = \frac{B_0}{q(r)} \eta(r)\, \boldsymbol{e}_\theta + \frac{B_0}{1 + \eta(r) \cos\theta}\, \boldsymbol{e}_\zeta, \tag{4.2}$$

where $q(r)$ is the safety factor, a function of r alone. [One should carefully distinguish between the standard model parameter B_0 and the average magnetic field $\mathscr{B}_0$ of (**12**.8.16)]. We recall eq. (**8**.9.17) for the scale factor $l_\rho = 1$. Next, we recall eq. (**8**.9.14) for the effective poloidal field,

$$\mathscr{B}_{\mathrm{P}}(r) = \frac{\mathscr{B}_0}{q(r)} \frac{r}{R_0}. \tag{4.3}$$

We also need the result (**8**.9.9),

$$\mathscr{I} = B_0 R_0. \tag{4.4}$$

Finally, we find from (**12**.5.8) the magnetic field intensity,

$$B(r, \theta) = \frac{B_0}{1 + \eta(r) \cos\theta} \left(1 + q^{-2}\eta^2\right)^{1/2}. \tag{4.5}$$

We now use (**8**.9.12) in order to calculate the surface-averages. For instance, using (4.5), we find

$$\begin{aligned} \mathscr{B}_0^2 &\equiv \langle B^2 \rangle \\ &= \frac{B_0^2}{2\pi} \int_0^{2\pi} \mathrm{d}\theta (1 + \cos\theta)\left[1 - 2\eta \cos\theta + \eta^2 (3\cos^2\theta + q^{-2})\right] \\ &= \frac{B_0^2}{2\pi} \int_0^{2\pi} \mathrm{d}\theta \left[1 - \eta \cos\theta + \eta^2 (\cos^2\theta + q^{-2})\right], \end{aligned}$$

and finally

$$\mathscr{B}_0^2 \equiv \langle B^2 \rangle = B_0^2 \left[1 + \left(\tfrac{1}{2} + q^{-2}\right)\eta^2\right]. \tag{4.6}$$

Similarly, we find

$$\langle B^{-2} \rangle = B_0^{-2} \left[1 + \left(\tfrac{3}{2} - q^{-2}\right)\eta^2\right]. \tag{4.7}$$

We now calculate the averages involving the external electric field. To do so, we must complete the standard model by an assumption about the latter field. We choose a purely toroidal electric field defined as

$$\boldsymbol{E}^{(A)} = \frac{E_0(r)}{1 + \eta(r)\cos\theta}\, \boldsymbol{e}_\zeta. \tag{4.8}$$

This field can also be resolved along the *physical basis* $(\boldsymbol{b}, \boldsymbol{e}_\wedge, \boldsymbol{e}_\rho)$ by using (**12**.5.7) as well as (4.2)–(4.4),

$$\boldsymbol{E}^{(A)} = \frac{E_0(r)}{1 + \eta(r)\cos\theta}\left(\boldsymbol{b} + q^{-1}(r)\ \eta(r)\ \boldsymbol{e}_\wedge\right). \tag{4.9}$$

In order to understand this choice, we calculate

$$\begin{aligned} &\boldsymbol{e}_\rho \cdot (\nabla \wedge \boldsymbol{E}^{(A)}) \\ &\quad = \frac{1}{l_\theta l_\zeta} \frac{\partial}{\partial\theta}\, l_\zeta E_\zeta^{(A)} \\ &\quad = \frac{1}{r(R_0 + r\cos\theta)} \frac{\partial}{\partial\theta}\left((R_0 + r\cos\theta)\frac{E_0(r)}{1 + (r/R_0)\cos\theta}\right) = 0. \end{aligned} \tag{4.10}$$

But, by the Maxwell equations we have

$$\boldsymbol{e}_\rho \cdot (\nabla \wedge \boldsymbol{E}^{(A)}) = -c^{-1} \boldsymbol{e}_\rho \cdot \partial_t \boldsymbol{B}. \tag{4.11}$$

Thus, the choice (4.8) ensures the stationarity of the magnetic field in the radial direction. In other words, *the magnetic surfaces are stationary in this model*: they neither expand nor contract.

An important point is the order of magnitude of the external field intensity. As was repeatedly stated before (section **10**.6), in the tokamak ordering, the

external electric field is by an order ϵ smaller than the potential field. This estimate must be completed with the estimate (1.5), which actually follows from (4.11). Thus, we assume that (E_0/B_0) *is of order* $\epsilon^2(u/c)$ (we recall that u is the average electron parallel velocity).

We collect all these results and substitute them into the transport equations. As a result of eqs. (**8**.9.17), (**8**.9.18), we simply have, for arbitrary F,

$$\nabla_\rho F = \frac{\partial F}{\partial r}. \tag{4.12}$$

We also easily find

$$\langle \boldsymbol{B} \cdot \boldsymbol{E}^{(A)} \rangle = B_0 E_0 \left[1 + \tfrac{1}{2}\left(1 + q^{-2}\right)\eta^2\right]. \tag{4.13}$$

Thus, to leading order in the standard model, the thermodynamic forces are

$$X_1^e = X_1 = -\frac{1}{n_e T_e}\frac{\partial P}{\partial r}, \qquad X_3^e = X_2 = -\frac{1}{T_e}\frac{\partial T_e}{\partial r},$$

$$X_3^i = X_3 = -\frac{1}{T_i}\frac{\partial T_i}{\partial r}, \qquad X_E = X_4 = E_0. \tag{4.14}$$

We now calculate the *classical transport coefficients*. We only need the value of the geometrical factor $\mathcal{G}$ which, from the combination of (4.6) and (4.7) is given by

$$\mathcal{G} = 1 + 2\eta^2. \tag{4.15}$$

Hence, the classical transport coefficients of table 2.1 reduce (through order ϵ^2) to those of table 4.1.

The *Pfirsch–Schlüter transport coefficients* involve a different geometrical factor, which is found from (4.3)–(4.7),

$$\mathcal{J}^2(\mathcal{G} - 1) = B_0^2 R_0^2 \frac{q^2}{B_0^2 r^2} 2\frac{r^2}{R_0^2} = 2q^2. \tag{4.16}$$

These coefficients are also collected in table 4.1.

Table 4.1
Transport coefficients in the standard model

Classical transport coefficients			**Pfirsch–Schlüter transport coefficients**		
L_{11}^{ee}	$\mathscr{L}_{11}^{CL}$	$\frac{1}{2}\frac{n_e\rho_{e0}^2}{\tau_e}c_{11}^e$	L_{11}^{ee}	$\mathscr{L}_{11}^{PS}$	$q^2\frac{n_e\rho_{e0}^2}{\tau_e}c_{11}^e$
L_{13}^{ee}	$\mathscr{L}_{12}^{CL}$	$-\frac{1}{2}\sqrt{\frac{5}{2}}\frac{n_e\rho_{e0}^2}{\tau_e}c_{13}^e$	L_{13}^{ee}	$\mathscr{L}_{12}^{PS}$	$-\sqrt{\frac{5}{2}}q^2\frac{n_e\rho_{e0}^2}{\tau_e}c_{13}^e$
L_{33}^{ee}	$\mathscr{L}_{22}^{CL}$	$\frac{5}{4}\frac{n_e\rho_{e0}^2}{\tau_e}c_{33}^e$	L_{33}^{ee}	$\mathscr{L}_{22}^{PS}$	$\frac{5}{2}q^2\frac{n_e\rho_{e0}^2}{\tau_e}c_{33}^e$
L_{33}^{ii}	$\mathscr{L}_{33}^{CL}$	$\frac{5}{4}\frac{n_i\rho_{i0}^2}{\tau_i}c_{33}^i$	L_{33}^{ii}	$\mathscr{L}_{33}^{PS}$	$\frac{5}{2}q^2\frac{n_i\rho_{i0}^2}{\tau_i}c_{33}^i$
L_{EE}	$\mathscr{L}_{44}^{CL}$	$\frac{e^2n_e}{m_e}\tau_e\bar{\sigma}_{\|}$	L_{EE}	$\mathscr{L}_{44}^{PS}$	0

The most important point here is the ratio between the Pfirsch–Schlüter and the classical transport coefficients,

$$\mathscr{L}_{\mu\nu}^{PS} = 2q^2\,\mathscr{L}_{\mu\nu}^{CL}, \quad \mu, \nu = 1, 2, 3. \tag{4.17}$$

We note that in a real confinement configuration such as a tokamak, the safety factor *must* exceed unity in order to ensure the MHD stability. Realistic values of q range between 2 and 3. The main conclusion of the present analysis is thus:

The Pfirsch–Schlüter transport coefficients exceed the classical ones by a factor $2q^2$, *which may be of order* 10.

We now also evaluate the "electric" fluxes of section 13.1 (through order η^2). The electric drift electron flux is

$$\left\langle \Gamma_\rho^e \right\rangle_{DR} = n_e c \left\langle \frac{E_\wedge^{(A)}}{B} \right\rangle = n_e c \eta(r) \frac{E_0}{q(r)\,B_0}. \tag{4.18}$$

Recalling that (E_0/B_0) is of order $\epsilon^2(u/c)$, we find that *the electric drift flux is of order* ϵ^3 (compared to $n_e u$), hence negligible by our standards. The reason of the extra factor $\eta \sim \epsilon$ is easily understood, as explained in section 13.1. Indeed, it is seen from (4.9) that the perp–tangential part of $\boldsymbol{E}$ (which drives the electric drift flux) is by a factor η smaller than the parallel part.

We now calculate the modified electric drift, defined by (1.6), using (4.3), (4.5) and (4.9),

$$\left\langle \Gamma_\rho^e \right\rangle_{\mathrm{MDR}} = -n_e c \frac{qR_0}{B_0 r}$$

$$\times \left\langle \left(\frac{1+\eta\cos\theta}{(1+q^{-2}\eta^2)^{1/2}} - \frac{(1+q^{-2}\eta^2)^{1/2}}{(1+\eta\cos\theta)\left(1+\frac{1}{2}(1+q^{-2})\eta^2\right)} \right) \times \frac{E_0}{1+\eta\cos\theta} \right\rangle$$

$$= -n_e c \frac{qE_0}{\eta B_0} \left\langle 2\eta\cos\theta + \tfrac{1}{2}\eta^2 - 3\eta^2\cos^2\theta \right\rangle. \tag{4.19}$$

We now evaluate the average by using (**8**.9.12),

$$\left\langle 2\eta\cos\theta + \tfrac{1}{2}\eta^2 - 3\eta^2\cos^2\theta \right\rangle$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta (1+\eta\cos\theta)\left(2\eta\cos\theta + \tfrac{1}{2}\eta^2 - 3\eta^2\cos^2\theta\right)$$

$$= 0 + \mathrm{O}(\eta^3). \tag{4.20}$$

Recalling again that $E_0/B_0 = \mathrm{O}(\eta^2 u/c)$, we find that *the modified drift electron flux is of order* ϵ^4. It is thus by one order of magnitude smaller than the electric drift flux and therefore safely negligible, at least in the tokamak geometry.

Finally, we briefly consider the limiting form of the theory in a *cylindrically symmetric* (rather than toroidal) configuration. We then adopt the usual metric

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + \mathrm{d}z^2 \tag{4.21}$$

for which

$$l_r = 1, \qquad l_\theta = r, \qquad l_z = 1. \tag{4.22}$$

The condition of "toroidal axisymmetry" reduces in this case to the condition of independence of the z coordinate (see fig. 4.1). Equation (4.2) is still an acceptable representation of a magnetic field in this case. However, it does *not* possess a symmetry around the *new* axis ("cylindrical axisymmetry").

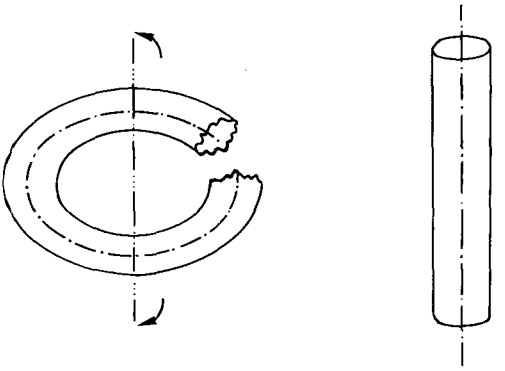

Fig. 4 1. Passage from toroidal symmetry to cylindrical symmetry. The new symmetry axis evolves from the old magnetic axis.

A quantity which is cylindrically symmetric must be independent of the "poloidal" angle θ as well. Thus, *the standard model for a cylindrically symmetric system* is

$$\boldsymbol{B} = B_0\left[\boldsymbol{e}_z + \beta(r)\boldsymbol{e}_\theta\right]. \tag{4.23}$$

On the other hand, eq. (**8**.8.21) for the surface average reduces to the simple form

$$\langle A\rangle = (2\pi)^{-1}\int_0^{2\pi} \mathrm{d}\theta\, A(r,\theta). \tag{4.24}$$

It then immediately follows from the cylindrical symmetry that

$$\langle B^2\rangle = B^2, \qquad \langle B^{-2}\rangle = B^{-2},$$

and we arrive at our conclusion, announced in sec. 13.3:

In a cylindrically symmetric configuration, i.e. for a straight, though inhomogeneous magnetic field, there is no Pfirsch-Schlüter effect:

$$\mathscr{G} - 1 = 0. \tag{4.25}$$

Appendix 13A.1. Alternative derivation of the Pfirsch–Schlüter fluxes

We briefly sketch here a derivation of the PS effect which is physically rather illuminating, but which also hides some important features. The calculation is very close to the original treatment by Hazeltine and Hinton (1973) [and also

to Maschke's (1972) less straightforward method]. It will be done only for the simpler case of the ion heat flux.

We start with the equations analogous to (**12**.6.11), (**12**.7.2) and (**12**.7.3). If we restrict ourselves to the *short mean free path regime* and to the *13M approximation* (just like Hazeltine and Hinton), the generalized stresses are neglected, and we find

$$h_{\|}^{i(3)} = \tilde{\kappa}_{\|}^{i} g_{\|}^{i(3)}, \qquad h_{\wedge}^{i(3)} = \tilde{\kappa}_{\perp}^{i} g_{\wedge}^{i(3)} - \tilde{\kappa}_{\wedge}^{i} g_{\rho}^{i(3)}, \qquad h_{\rho}^{i(3)} = \tilde{\kappa}_{\wedge}^{i} g_{\wedge}^{i(3)} + \tilde{\kappa}_{\perp}^{i} g_{\rho}^{i(3)}. \tag{A1.1}$$

If we consider the orderings (**12**.7.4), (**12**.7.5), we find that the parallel and the perp-tangential fluxes are of order ϵ, whereas the radial flux is of order ϵ^2 (in agreement with the conclusions of section **12**.7) and that the leading terms are

$$h_{\|}^{i(3)} = \tilde{\kappa}_{\|}^{i} g_{\|}^{i(3)}, \qquad h_{\wedge}^{i(3)} = -\tilde{\kappa}_{\wedge}^{i} g_{\rho}^{i(3)}, \qquad h_{\rho}^{i(3)} = \tilde{\kappa}_{\wedge}^{i} g_{\wedge}^{i(3)} + \tilde{\kappa}_{\perp}^{i} g_{\rho}^{i(3)}. \tag{A1.2}$$

The interesting fact here is that the radial heat flux has two contributions of quite distinct nature, but of the same order ϵ^2. The second term is a product of a large source term $g_{\rho}^{i(3)}$ (of order ϵ^0) by a very small perpendicular conductivity $\tilde{\kappa}_{\perp}^{i}$ (of order ϵ^2). Indeed, we know that in the confined plasma the zeroth order temperature must be a surface quantity, hence its gradient necessarily points in the radial direction. This term gives rise to the *classical flux*.

The really new feature is the first term on the right-hand side, driven by a perp–tangential temperature gradient Such a *non-uniformity of the temperature on the magnetic surface* is necessarily a smaller effect (of order ϵ) and results from a purely geometrical cause (very clearly described by Engelmann and Nocentini 1976). Indeed, the radial temperature gradient, combined with the magnetic field, produces a *diamagnetic drift* which tends to accumulate heat in the upper half of the toroidal configuration. This heat is redistributed mainly by conduction in the *parallel direction* (because $\tilde{\kappa}_{\|}^{i} \gg \tilde{\kappa}_{\wedge}^{i}, \tilde{\kappa}_{\perp}^{i}$). As a result, a self-consistent regime sets in, in which the temperature gradients in the three directions ($\|$, $\wedge$, ρ) are intimately connected. But the additional [small, $O(\epsilon)$] perp–tangential temperature gradient, combined with the magnetic field, produces an *additional diamagnetic drift, pointing in the radial direction*: this is precisely the *Pfirsch–Schlüter effect*, expressed by the first term on the right-hand side of eq. (A1.2).

The remaining task is the self-consistent determination of the relation between the various components of the temperature gradient. This goal is reached by exploiting the *zero-divergence constraint* (**12**.8.1),

$$\nabla \cdot \boldsymbol{q}_i = 0. \tag{A1.3}$$

This equation is solved for the parallel heat flux as in section **12**.8, with the result (**12**.8.23),

$$h_{\|}^{i(3)} = \frac{\mathscr{B}_0}{B} \mathscr{K}_i \, g_\rho^{i(3)} + \frac{B}{\mathscr{B}_0} \omega_3^i. \tag{A1.4}$$

The poloidal flux is determined as in section **12**.8 by identifying this expression of the parallel heat flux with the first equation of (A1.2),

$$\tilde{\kappa}_{\|}^i \, g_{\|}^{i(3)} = \frac{\mathscr{B}_0}{B} \mathscr{K}_i \, g_\rho^{i(3)} + \frac{B}{\mathscr{B}_0} \omega_3^i. \tag{A1.5}$$

Multiplying this equation by $B/\mathscr{B}_0$ and taking the surface average, we obtain the equation for ω_3^i in (**12**.8.30) for the *short mean free path regime*, i.e. for $\bar{g}_{\|}^{i(3)}$, $\bar{g}_{\|}^{i(5)}$ set equal to zero. As both l_ρ and $g_\rho^{i(3)}$ are surface quantities, the latter equation reduces to

$$\omega_3^i = -\frac{\mathscr{B}_0}{B} \mathscr{K}_i \, g_\rho^{i(3)}, \quad [SMFP]. \tag{A1.6}$$

When this result is substituted back into (A1.5) we find a relation between the parallel and the radial components of the self-consistent temperature gradient,

$$g_{\|}^{i(3)} = \mathscr{K}_i \frac{1}{\tilde{\kappa}_{\|}^i} \frac{\mathscr{B}_0}{B} \left(1 - \frac{B^2}{\mathscr{B}_0^2}\right) g_\rho^{i(3)}. \tag{A1.7}$$

In order to determine the perp–tangential temperature gradient, we go over to the "geometrical reference frame" and obtain the following relations by using the transformation formulae (**12**.5.6):

$$g_{\|}^{i(3)} = \frac{1}{B l_\zeta} \left(\frac{R_0 \mathscr{B}_P}{l_\rho} g_\theta^{i(3)} + \mathscr{I} g_\zeta^{i(3)} \right),$$

$$g_{\wedge}^{i(3)} = \frac{1}{B l_\zeta} \left(-\mathscr{I} g_\theta^{i(3)} + \frac{R_0 \mathscr{B}_P}{l_\rho} g_\zeta^{i(3)} \right). \tag{A1.8}$$

But, because of the assumed *axisymmetry*, $g_\zeta^{i(3)} \equiv 0$; hence, $g_{\|}^{i(3)}$ and $g_{\wedge}^{i(3)}$ are simply proportional to each other,

$$g_{\wedge}^{i(3)} = -\frac{\mathscr{I} l_\rho}{R_0 \mathscr{B}_P} g_{\|}^{i(3)}. \tag{A1.9}$$

We now combine this result with (A1.2) and with the value $\tilde{\kappa}^{i}_{\wedge} = -(\Omega_i \tau_i)^{-1}$ taken from (5.6.2), and find

$$h^{i(3)}_{\rho PS} \equiv \tilde{\kappa}^{i}_{\wedge} g^{i(3)}_{\wedge} = \left(\frac{\mathscr{I} l_\rho}{R_0 \mathscr{B}_P \Omega_{i0} \tau_i} \frac{\mathscr{B}_0}{B} \right)^2 \frac{1}{\tilde{\kappa}^{i}_{\|}} \left(1 - \frac{B^2}{\mathscr{B}_0^2} \right) g^{i(3)}_{\rho}. \tag{A1.10}$$

Surface-averaging this equation, we find the final expression of Hazeltine and Hinton (1973),

$$\left\langle h^{i(3)}_{\rho} \right\rangle_{PS} = \mathscr{K}_i^2 \left\langle \frac{\mathscr{B}_0^2}{B^2} - 1 \right\rangle \frac{1}{\tilde{\kappa}^{i}_{\|}} g^{i(3)}_{\rho}. \tag{A1.11}$$

Recalling (table 5.3.2) that, *in the 13M approximation,*

$$\tilde{\kappa}^{i}_{\|} = \left(c^{i}_{33} \right)^{-1}, \tag{A1.12}$$

we see that this result is precisely equivalent to eq. (3.8) of the main text.

A similar treatment of the electron vector fluxes comes out with the result:

$$\left\langle h^{e(1)}_{\rho} \right\rangle_{PS} = \mathscr{K}_e^2 \left\langle \frac{\mathscr{B}_0^2}{B^2} - 1 \right\rangle \frac{1}{\Delta_{\|}} \left[\tilde{\kappa}^{e}_{\|} g^{(1)P}_{\rho} + \tilde{\alpha}_{\|} g^{e(3)}_{\rho} \right],$$

$$\left\langle h^{e(3)}_{\rho} \right\rangle_{PS} = \mathscr{K}_e^2 \left\langle \frac{\mathscr{B}_0^2}{B^2} - 1 \right\rangle \frac{1}{\Delta_{\|}} \left[\tilde{\alpha}_{\|} g^{(1)P}_{\rho} + \tilde{\sigma}_{\|} g^{e(3)}_{\rho} \right], \tag{A1.13}$$

with

$$\Delta_{\|} = \tilde{\sigma}_{\|} \tilde{\kappa}^{e}_{\|} - \tilde{\alpha}^2_{\|}.$$

Using again the expressions in table 5.3.2, we find the following relations, *valid in the 13M approximation*:

$$\frac{\tilde{\kappa}^{e}_{\|}}{\Delta_{\|}} = c^{e}_{11}, \qquad \frac{\tilde{\alpha}_{\|}}{\Delta_{\|}} = -c^{e}_{13}, \qquad \frac{\tilde{\sigma}_{\|}}{\Delta_{\|}} = c^{e}_{33}. \tag{A1.14}$$

Hence, the Hazeltine–Hinton form (A1.12) is exactly equivalent to (3.6), (3.7).

The derivation given in this appendix has the advantage of clarifying the *physical mechanism of the PS effect as an interplay of diamagnetic drifts caused by the toroidal geometry.*

On the other hand, this method has serious limitations. First, it is restricted to the 13M approximation. If the formulae (A1.11), (A1.13) were extrapolated by using the 21M, 29M,... values of the parallel transport coefficients, the result would be wrong, because the specific relations (A1.12), (A1.14) break down for the higher truncation levels.

Second, the "poloidal fluxes" ω_n^α play a quite central role in the present derivation. As can be seen in the passage from (A1.5) to (A1.7), it is the form of these "poloidal fluxes" that introduces the Pfirsch–Schlüter amplification factor $(\mathcal{I}l_\rho/B_0\mathcal{B}_P)^2(\mathcal{G}-1)$. If one tried to extend this method to the calculation of the PS fluxes in the *long mean free path regime*, the simple eq. (A1.6) would no longer be valid: the full set of coupled equations (**12**.8.30) should be used for determining the "poloidal fluxes". This would lead to considerable complications in the calculation.

On the contrary, in the derivation of sec. 13.3, the initial grouping of the terms is such that *the "poloidal fluxes" simply drop out of the expressions of the PS fluxes* and we find a form valid for all collisionality regimes and independent of the truncation level.

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14

Solution of the drift kinetic equation in the long mean free path regime

14.1 Expansion of the distribution function according to the collision frequency

It was repeatedly stated in chapter 12 that the evaluation of the *banana fluxes*, which are characteristic of the transport in a long mean free path regime, requires a solution of the drift kinetic equation appropriate to that regime. The present chapter is devoted to this problem.

The starting point is the Landau kinetic equation (**10**.2.1), which is the basic description of our electron–ion plasma in an axisymmetric confining magnetic field. We have made an assumption about the latter, which is formulated as the smallness of the drift parameter ϵ,

$$\epsilon \ll 1.$$

With this assumption, we expanded the distribution functions according to their order in ϵ [see eq. (**10**.3.1); we now no longer write explicitly the formal parameter ϵ],

$$f^\alpha = f_0^\alpha + f_1^\alpha + \cdots . \tag{1.1}$$

It is assumed here that the distribution functions are expressed in terms of the NGC variables ($\mathscr{E}$, M, ϕ, $\boldsymbol{Y}$), and that the position $\boldsymbol{Y}$ of the guiding centre is measured in toroidal coordinate (ρ, θ, ζ).

The analysis in chapter 10 can be summarized in two results:

(A) The zeroth-order ($=\epsilon^0$) distribution function f_0^α depends only on the energy $\mathscr{E}$ and on the radial coordinate ρ; it is of Maxwell–Boltzmann form (**10**.5.14),

$$f_0^\alpha(\mathscr{E};\, \rho) = \left(\frac{m_\alpha}{2\pi T_\alpha(\rho)}\right)^{3/2} n_\alpha(\rho) \exp\left(\frac{e_\alpha}{T_\alpha(\rho)}\, \Phi(\rho)\right) \exp\left(-\frac{1}{T_\alpha(\rho)}\, \mathscr{E}\right). \tag{1.2}$$

In order to unburden the notations, we no longer write explicitly the slow time dependence argument (actually, we shall be interested in solutions which are quasi-stationary, as discussed in chapter 12). Moreover, we drop the subscript zero on the densities $n_\alpha \equiv n_{\alpha 0}$, the temperatures $T_\alpha \equiv T_{\alpha 0}$ and the potential $\Phi \equiv \Phi_0$. We recall that this local equilibrium form came out of a competition between the effects of the magnetic field and the collisions, expressed by eq. (**10**.5.1). We thus emphasize that the *collisions play a fundamental role, even in the long mean free path regime* (see also the discussion in section **12**.1). We shall therefore never use the extremely misleading name of "collisionless regime" given to the latter in many published works.

(B) The first-order ($=\epsilon^1$) gyro-averaged distribution function $\bar{f}_1^\alpha$ obeys the drift kinetic equation (**11**.5.14), which we rewrite here in a slightly more condensed form,

$$U_\alpha \boldsymbol{b}\cdot\frac{\partial}{\partial \boldsymbol{Y}}\bar{f}_1^\alpha + \boldsymbol{V}_{\mathrm{D}}^\alpha\cdot\frac{\partial}{\partial \boldsymbol{Y}} f_0^\alpha = (\nu^{\alpha e}+\nu^{\alpha i})\mathscr{L}_\alpha \bar{f}_1^\alpha + U_\alpha \mathscr{N}_\parallel^\alpha f_0^\alpha$$

$$- U_\alpha e_\alpha E_\parallel^{(A)}\frac{\partial}{\partial \mathscr{E}} f_0^\alpha, \tag{1.3}$$

with the gyro-averaged pitch-angle scattering operator defined as follows (for simplicity, we now drop the overbar on $\mathscr{L}_\alpha$)

$$\mathscr{L}_\alpha = \frac{m_\alpha}{B} U_\alpha \frac{\partial}{\partial M} U_\alpha M \frac{\partial}{\partial M}. \tag{1.4}$$

The first-order deviation from the reference state may depend on the energy $\mathscr{E}$ and on the magnetic moment M, but not on the gyrophase ϕ (because, by definition, $\bar{f}_1^\alpha$ is a gyro-averaged function). As regards the position variables, $\bar{f}_1^\alpha$ may depend on the radial coordinate ρ and on the poloidal angle θ, but not on the toroidal angle ζ (because the configuration is axisymmetric); thus

$$\bar{f}_1^\alpha = \bar{f}_1^\alpha(\mathscr{E}, M; \theta, \rho). \tag{1.5}$$

In order to solve the drift kinetic equation (1.3), we must, as usual, introduce additional simplifying assumptions. *It is at this point that we radically depart from our former treatments, valid in the collision-dominated regime* (notably, chapters 4 and 5). In the long mean free path regime, we *invert assumption* (**5**.1.3):

$$\delta \equiv \frac{1}{\lambda_{\mathrm{H}}} \ll 1. \tag{1.6}$$

Thus, *the relaxation time is assumed to be much longer than the hydrodynamical time.*

This definition of the long mean free path regime agrees with the one given by Hinton and Hazeltine (1976). Hirshman and Sigmar (1981) (as well as other authors) prefer to use the ratio between collision frequency and "bounce frequency" as a small parameter. The latter frequency, however, is not a well-defined concept. We know from chapter 9 that the time it takes for a trapped particle to bounce between its turning points (or for a passing particle to go around the poloidal direction) depends on the value of its trapping parameter; it *tends to infinity* for a barely trapped (or barely passing) particle. Therefore, the "bounce frequency" used by these authors is a conventional concept. In practice, however, there is no difference between their treatment and ours.

As we now have a second smallness parameter, we expand the unknown function $\bar{f}_1^\alpha$, *using superscripts between parentheses to denote the order in* δ:

$$\bar{f}_1^\alpha = \bar{f}_1^{\alpha(0)} + \bar{f}_1^{\alpha(1)} + \cdots \tag{1.7}$$

It is immediately recognized that, under the present assumption, the leading terms in (1.3) will be provided by those on the left-hand side, which are proportional to the gradients ($\partial/\partial \boldsymbol{Y} \sim \tau_H^{-1}$), whereas the collision term ($\sim \nu^{\alpha\beta}$) must be considered as a first-order correction in the present regime. We must still decide something about the last term on the right-hand side. The external, induced electric field $E^{(A)} = c^{-1}\, \partial \boldsymbol{A}/\partial t$ is usually small in a tokamak; a safe assumption is to consider the electric field term of the same order as the collision term.

The drift kinetic equation is now expanded in a straightforward manner,

Order δ^0:

$$U_\alpha \boldsymbol{b} \cdot \nabla \bar{f}_1^{\alpha(0)} + \boldsymbol{V}_D^\alpha \cdot \nabla f_0^\alpha = 0. \tag{1.8}$$

Order δ:

$$U_\alpha \boldsymbol{b} \cdot \nabla \bar{f}_1^{\alpha(1)} = (\nu^{\alpha e} + \nu^{\alpha i})\, \mathscr{L}_\alpha \bar{f}_1^{\alpha(0)} + U_\alpha \mathscr{N}_\parallel^{\alpha(0)} f_0^\alpha + U_\alpha \frac{e_\alpha}{T_\alpha} E_\parallel^{(A)} f_0^\alpha. \tag{1.9}$$

Here $\mathscr{N}_\parallel^{\alpha(0)}$ denotes the value of the functional $\mathscr{N}_\parallel^\alpha[f]$ evaluated with the δ^0-order distribution function $\bar{f}_1^{\alpha(0)}$ (see eq. **11**.4.18); the derivative $\partial f_0^\alpha/\partial \mathscr{E}$ in the electric field term was evaluated, using (1.2). From here on we shall systematically use the gradient notation

$$\nabla \equiv \frac{\partial}{\partial \boldsymbol{Y}}.$$

In order to manipulate more compact notations, we introduce the following abbreviations. The global collision frequency ν_α is defined as

$$\nu_\alpha = \nu^{\alpha e} + \nu^{\alpha i}. \tag{1.10}$$

The following symbol allows us to group together the last two terms,

$$\mathscr{M}_\alpha \equiv \mathscr{N}_\parallel^{\alpha(0)} + \frac{e_\alpha}{T_\alpha} E_\parallel^{(A)}. \tag{1.11}$$

Equation (1.9) thus takes the simpler form

$$U_\alpha \boldsymbol{b} \cdot \nabla \bar{f}_1^{\alpha(1)} = \nu_\alpha \mathscr{L}_\alpha \bar{f}_1^{\alpha(0)} + U_\alpha \mathscr{M}_\alpha f_0^\alpha. \tag{1.12}$$

14.2 Integration of the zeroth-order drift kinetic equation

We rewrite eq. (1.8), which determines the (order ϵ) deviation $\bar{f}_1^{\alpha(0)}$ to zeroth order in δ,

$$U_\alpha \boldsymbol{b} \cdot \nabla \bar{f}_1^{\alpha(0)} = -\boldsymbol{V}_D^\alpha \cdot \nabla f_0^\alpha. \tag{2.1}$$

Thus, the unknown function is determined by a balance between the effects of the *parallel motion* of the particles and the (slow) *perpendicular drift* of their guiding centres. The term describing the second effect (the right-hand side) is a *source term*, involving the known ϵ^0-part of f^α, i.e. the Maxwell–Boltzmann distribution (1.2). As the latter depends only on the radial position coordinate ρ, we have

$$\boldsymbol{V}_D^\alpha \cdot \nabla f_0^\alpha = (\boldsymbol{V}_D^\alpha \cdot \nabla \rho) \frac{\partial}{\partial \rho} f_0^\alpha. \tag{2.2}$$

The radial derivative is readily evaluated from (1.2),

$$\frac{\partial}{\partial \rho} f_0^\alpha = \left[\left(\frac{\mathscr{E} - e_\alpha \Phi}{T_\alpha} - \frac{5}{2} \right) \frac{\mathrm{d} \ln T_\alpha}{\mathrm{d}\rho} + \frac{\mathrm{d} \ln n_\alpha T_\alpha}{\mathrm{d}\rho} + \frac{e_\alpha}{T_\alpha} \frac{\mathrm{d}\Phi}{\mathrm{d}\rho} \right] f_0^\alpha. \tag{2.3}$$

The coefficients appearing here are proportional to the radial components of the *source terms* appearing in the Hermitian moment equations of chapter 5. Indeed, from eqs. (**5**.1.20), (**5**.1.13) and (**5**.1.15), we obtain

$$\frac{\mathrm{d} \ln n_\alpha T_\alpha}{\mathrm{d}\rho} + \frac{e_\alpha}{T_\alpha} \frac{\mathrm{d}\Phi}{\mathrm{d}\rho} = -\frac{1}{\tau_\alpha} \left(\frac{m_\alpha}{T_\alpha} \right)^{1/2} l_\rho \left(g_\rho^{\alpha(1)} - g_\rho^{\alpha(1)A} \right), \tag{2.4}$$

$$\frac{\mathrm{d} \ln T_\alpha}{\mathrm{d}\rho} = -\sqrt{\frac{2}{5}} \frac{1}{\tau_\alpha} \left(\frac{m_\alpha}{T_\alpha} \right)^{1/2} l_\rho \, g_\rho^{\alpha(3)}. \tag{2.5}$$

In connection with (2.4) we note that only the potential part of the electric field appears on the left-hand side; hence the subtraction of the external field contribution $g_\rho^{\alpha(1)A}$ (see eq. **12**.4.15). Equation (2.3) is thus rewritten in terms of the familiar source terms as

$$\frac{\partial f_0^\alpha}{\partial \rho} = -\frac{1}{\tau_\alpha}\left(\frac{m_\alpha}{T_\alpha}\right)^{1/2} l_\rho \left[g_\rho^{\alpha(1)} - g_\rho^{\alpha(1)A} + \sqrt{\frac{2}{5}}\left(\frac{\mathscr{E} - e_\alpha \Phi}{T_\alpha} - \frac{5}{2}\right) g_\rho^{\alpha(3)} \right] f_0^\alpha . \tag{2.6}$$

This factor is important, because it introduces the *thermodynamical forces* (gradients of temperature, density and electric potential) into the theory. Although we are in a long mean free path regime, these forces enter the equations in a way similar to the collision-dominated case, through a source term determined by the reference distribution function. The important point is that the latter is a local Maxwellian, the parameters of which are the hydrodynamical quantities (to order ϵ^0).

We now transform the factor multiplying $\partial f_0^\alpha / \partial \rho$ in (2.2), which is none other than the *contravariant* radial component of the drift velocity $\boldsymbol{V}_{\mathrm{D}}^\alpha$. The geometry of the magnetic field must now be specified. We adopt our usual representation (**12**.5.2) for an axisymmetric configuration,

$$\boldsymbol{B} = B_\theta \boldsymbol{e}_\theta + B_\zeta \boldsymbol{e}_\zeta , \tag{2.7}$$

with

$$B_\theta = \frac{R_0 \mathscr{B}_{\mathrm{P}}(\rho)}{l_\rho l_\zeta}, \qquad B_\zeta = \mathscr{I}(\rho)\frac{1}{l_\zeta} . \tag{2.8}$$

We give in Appendix 14A.1 the proof of a remarkable formula, due to Alfvén (1950), relating the radial component of the (*perpendicular*) guiding center drift velocity to the *parallel* velocity U_α and to the Larmor frequency Ω_α,

$$\boldsymbol{V}_{\mathrm{D}}^\alpha \cdot \nabla \rho = \frac{\mathscr{I}}{R_0 \mathscr{B}_{\mathrm{P}}} U_\alpha \boldsymbol{b} \cdot \nabla \frac{U_\alpha}{\Omega_\alpha} . \tag{2.9}$$

We thus see the same operator (parallel gradient) appearing on both sides of (2.1); this feature makes its integration trivial. Indeed, multiplying both sides by B/U_α we get

$$\boldsymbol{B} \cdot \nabla \bar{f}_1^{\alpha(0)} = -\frac{\mathscr{I}}{R_0 \mathscr{B}_{\mathrm{P}}} \frac{\partial f_0^\alpha}{\partial \rho} \boldsymbol{B} \cdot \nabla \frac{U_\alpha}{\Omega_\alpha} . \tag{2.10}$$

Next we note, using (2.7),

$$\boldsymbol{B}\cdot\nabla = \frac{B_\theta}{l_\theta}\frac{\partial}{\partial\theta} + \frac{B_\zeta}{l_\zeta}\frac{\partial}{\partial\zeta}. \tag{2.11}$$

In an axisymmetric configuration, the last term contributes nothing and we are left with

$$\frac{\partial}{\partial\theta}\tilde{f}_1^{\alpha(0)} = -\frac{\mathscr{I}}{R_0\mathscr{B}_\mathrm{P}}\frac{\partial f_0^\alpha}{\partial\rho}\frac{\partial}{\partial\theta}\frac{U_\alpha}{\Omega_\alpha}. \tag{2.12}$$

As the functions $\mathscr{I}$, $\mathscr{B}_\mathrm{P}$, f_0^α are independent of the poloidal angle, the integration is immediate,

$$\begin{aligned}\tilde{f}_1^{\alpha(0)}(\mathscr{E}, M; \theta, \rho) &= -\frac{\mathscr{I}}{R_0\mathscr{B}_\mathrm{P}}\frac{\partial f_0^\alpha(\mathscr{E}; \rho)}{\partial\rho}\frac{U_\alpha(\mathscr{E}, M; \theta, \rho)}{\Omega_\alpha(\theta; \rho)} \\ &\quad + \bar{G}_1^{\alpha(0)}(\mathscr{E}, M; \rho) \\ &\equiv \bar{F}_1^{\alpha(0)}(\mathscr{E}, M; \theta, \rho) + \bar{G}_1^{\alpha(0)}(\mathscr{E}, M; \rho),\end{aligned} \tag{2.13}$$

where we have written explicitly the arguments on which the various functions depend. $\bar{G}_1^{\alpha(0)}$ is an "integration constant", i.e. a function of $\mathscr{E}$, M, ρ, but *not* of θ. Its form is *arbitrary* at the present stage. It will be seen later that it plays an essential role in the neoclassical theory.

14.3. The integrability conditions of the first-order drift kinetic equation

Consider now eq. (1.12) for the first-order ($\sim\delta$) deviation of the distribution function $\tilde{f}_1^{\alpha(1)}$. After multiplication by B/U_α, it is written as

$$\boldsymbol{B}\cdot\nabla\tilde{f}_1^{\alpha(1)} = \nu_\alpha m_\alpha\frac{\partial}{\partial M}U_\alpha M\frac{\partial}{\partial M}\tilde{f}_1^{\alpha(0)} + B\mathscr{M}_\alpha f_0^\alpha. \tag{3.1}$$

This equation is very much reminiscent of a *magnetic differential equation* of the type studied in section **8**.5 (see also section **12**.8),

$$\boldsymbol{B}\cdot\nabla\tilde{f}_1^{\alpha(1)}(\theta, \rho; \mathscr{E}, M) = \sigma(\theta, \rho; \mathscr{E}, M). \tag{3.2}$$

There is, however, an important difference: *the unknown function $\tilde{f}_1^{\alpha(1)}$ and the source function σ are not macroscopic quantities, depending only on the*

spatial variables θ, ρ (as in section **8**.5), *but are rather microscopic quantities depending also on the "energy variables"* $\mathscr{E}$, M. In other words (3.2) may be called a *magnetic differential equation in phase space*. [A clear discussion of such equations can be found in the article by Hastie et al. (1967)].

Two intimately related questions must be asked about (3.2) before embarking on its solution:

(1) is the equation *soluble*?

(2) What *boundary conditions* must be specified for $\bar{f}_1^{\alpha(1)}$?

In the *macroscopic case*, the question can be settled very simply. We take the surface average of both sides:

$$\langle \boldsymbol{B} \cdot \nabla f \rangle = \langle \sigma \rangle. \tag{3.3}$$

Clearly, the physically reasonable boundary condition on $f(\theta)$ is that it must be single-valued, hence periodic in θ: $f(2\pi) = f(0)$. As a result (see eq. **8**.6.3), the left-hand side vanishes identically. Thus, the magnetic differential equation is devoid of contradiction only if

$$\langle \sigma \rangle = 0. \tag{3.4}$$

This is the simpler form of the *integrability condition*.

Considering now eq. (3.2) in *phase space*, the procedure described above is inapplicable. In a toroidal confinement configuration, *the surface average of a phase space function is not well defined*. At this precise point, the *topology* of the particle (or, more precisely, of the guiding centre) trajectories enters the problem in an essential way.

We know from the detailed study in chapter 9 that on each magnetic surface (i.e. for given ρ), according to the values of $\mathscr{E}$ and M, the particles can be divided into *passing* and *trapped* particles. The trajectories of the latter exist only in the domain $-\theta_* \leqslant \theta \leqslant \theta_*$ of the poloidal angle. At the turning point θ_* ($< \pi$), they are reflected by the local magnetic mirror, and the characteristic banana motion sets in.

On the other hand, the surface average, as defined by (**8**.8.21) for axisymmetric systems, is essentially an integration over the poloidal angle θ from $-\pi$ to π,

$$\langle \sigma(\theta, \rho; \mathscr{E}, M) \rangle \sim \int_{-\pi}^{\pi} \mathrm{d}\theta \, \frac{l_\theta}{B_\theta} \, \sigma(\theta, \rho; \mathscr{E}, M).$$

But a *trapped* particle has no access to values of θ exceeding θ_*: for $\theta_* < \theta < \pi$, some basic quantities, such as the parallel velocity U_α [which enters explicitly the source term (3.1)] become *imaginary*. Hence, *there exists*

no surface-averaging operation which can be uniformly defined in the whole phase space.

In order to tackle the integrability problem for (3.2), we divide the phase space into two regions corresponding, respectively, to the *passing particles* (P) and to the *trapped particles* (T). We recall that the distinction between the two classes is expressed as a relation between the values of $\mathscr{E}$, M and ρ [in the standard model, it is given by (**9**.5.9)] *. Corresponding to this division, the distribution function $\bar{f}_1^\alpha$, to each order in δ, can always be additively divided into a distribution function for passing particles and one for trapped particles,

$$\bar{f}_1^\alpha = \bar{f}_{1\mathrm{P}}^\alpha + \bar{f}_{1\mathrm{T}}^\alpha. \tag{3.5}$$

In the magnetic differential equation (3.2), the variables $\mathscr{E}$, M, ρ only enter as *parameters*; therefore, the equation can be split into two separate equations, one for each region ($\mathscr{E}$, M, ρ) of the phase space.

A. Passing particles

Equation (3.2) can be written as (see eq. 2.11)

$$\frac{\partial}{\partial\theta}\bar{f}_{1\mathrm{P}}^{\alpha(1)}(\theta;\ \rho,\ \mathscr{E},\ M) = \frac{I_\theta}{B_\theta}\sigma_{\mathrm{P}}(\theta;\ \rho,\ \mathscr{E},\ M). \tag{3.6}$$

As the passing particles can access all values of θ, this equation can be trivially integrated between $\theta = -\pi$ and $\theta = \pi$,

$$\bar{f}_{1\mathrm{P}}^{\alpha(1)}(\pi;\ \rho,\ \mathscr{E},\ M) - \bar{f}_{1\mathrm{P}}^{\alpha(1)}(-\pi;\ \rho,\ \mathscr{E},\ M) = \int_{-\pi}^{\pi} \mathrm{d}\theta \frac{I_\theta}{B_\theta}\sigma_{\mathrm{p}}(\theta;\ \rho,\ \mathscr{E},\ M). \tag{3.7}$$

The natural boundary condition for passing particles must ensure the one-valuedness of $\bar{f}_{1\mathrm{P}}^{\alpha(1)}$, hence $\bar{f}_{1\mathrm{P}}^{\alpha(1)}(\theta;\ \rho,\ \mathscr{E},\ M)$, *as a function of* θ, *must be periodic, of period* 2π.

As a result, the left-hand side of (3.7) vanishes identically, and we find

$$\int_{-\pi}^{\pi} \mathrm{d}\theta \frac{I_\theta}{B_\theta}\sigma_{\mathrm{P}}(\theta;\ \rho,\ \mathscr{E},\ M) = 0 \quad [\text{passing}]. \tag{3.8}$$

* In section 14.5 we shall introduce a set of NGC variables in terms of which this division is very simply expressed.

This is the *integrability condition in the passing domain*. Obviously, it has the same form as the macroscopic condition (3.4), because passing particles can access the whole range of poloidal angles.

B. Trapped particles

Here the discussion must be much more careful. The distribution functions are defined only in the domain

$$-\theta_* \leqslant \theta \leqslant \theta_*, \tag{3.9}$$

where $\theta_* \equiv \theta_*(\rho, \mathscr{E}, M)$ is the poloidal coordinate of the turning point.

We must also introduce a new quantity in the discussion: the *sign*, σ, *of the parallel velocity* (see eq. **11**.5.3). It was stressed in section **9**.5 that the passing particles maintain the sign of their parallel velocity during their whole motion. On the contrary, for trapped particles, the sign of U_α changes upon reflection at the turning points. Passing particles with $\sigma = +$ and with $\sigma = -$ are physically distinct and thus independent. There is no point in making a different treatment: eqs. (3.7) hold as they stand, for particles of both signs σ. For trapped particles it is useful to consider separately those with positive U_α and those with negative U_α; we thus write

$$\bar{f}^{\alpha(1)}_{1\mathrm{T}} = \sum_{\sigma = +,\,-} \bar{f}^{\alpha(1)}_{1\mathrm{T}\sigma} \equiv \bar{f}^{\alpha(1)}_{1\mathrm{T}+} + \bar{f}^{\alpha(1)}_{1\mathrm{T}-}. \tag{3.10}$$

It might also be said that the distribution function of the trapped particles is a *two-valued* function of θ. The two branches are not independent, as can be understood from the fact that each individual particle is counted alternatively in $\bar{f}^{\alpha(1)}_{1\mathrm{T}+}$ and in $\bar{f}^{\alpha(1)}_{1\mathrm{T}-}$ during its motion. In particular, *at the turning point*, $\theta = \theta_*$, the distinction between the two kinds of particles disappears and the two parts of $\bar{f}^{\alpha(1)}_{1\mathrm{T}}$ must have equal values. This discussion is summarized in fig. 3.1, which shows schematically the general shape of $\bar{f}^{\alpha(1)}_{1\mathrm{P}}$, $\bar{f}^{\alpha(1)}_{1\mathrm{T}+}$ and $\bar{f}^{\alpha(1)}_{1\mathrm{T}-}$ as functions of the poloidal angle.

We now show how this discussion can be exploited quantitatively for obtaining boundary conditions and integrability conditions. We write again the magnetic differential equation in the form (3.6) in the domain of trapped particles with positive and negative U, respectively, and integrate both sides over the whole *accessible domain* of the poloidal angle, i.e. from $-\theta_*$ to θ_*,

$$\bar{f}^{\alpha(1)}_{1\mathrm{T}\sigma}(\theta_*) - \bar{f}^{\alpha(1)}_{1\mathrm{T}\sigma}(-\theta_*) = \int_{-\theta_*}^{\theta_*} \mathrm{d}\theta \, \frac{l_\theta}{B_\theta} \, \sigma_{\mathrm{T}\sigma}(\theta). \tag{3.11}$$

(we do not write explicitly the parameters $\mathscr{E}$, M, ρ.)

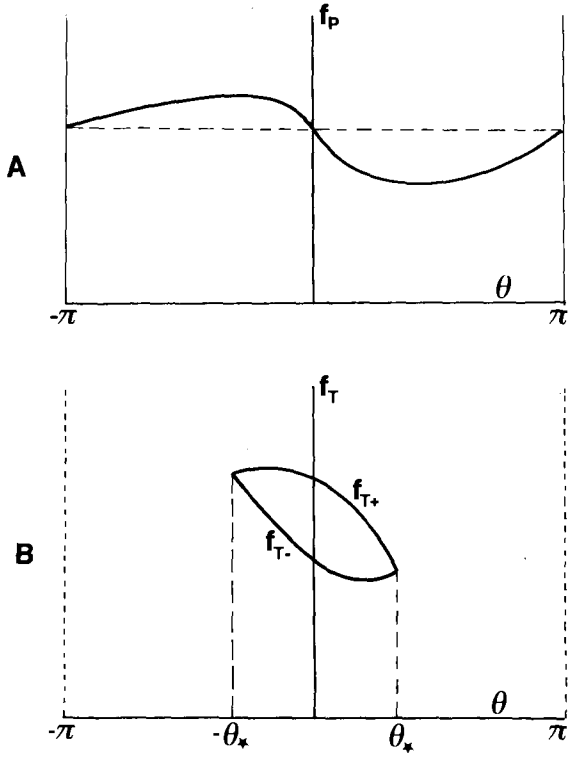

Fig. 3.1. Typical shapes of the distribution function as a function of the poloidal angle θ (A) passing particles, (B) trapped particles.

The situation is now very different from (3.7), in spite of the formal similarity. In (3.7) appears the difference between the values of $\tilde{f}_{1P}^{\alpha(1)}$ at $\theta = \pi$ and $\theta = -\pi$, which denote the *same* physical point; in (3.11) appears the difference between the values of $\tilde{f}_{1T\sigma}^{\alpha(1)}$ at $\theta = \theta_*$ and $\theta = -\theta_*$ which are physically *different* points. In a general inhomogeneous situation, these values depend on the initial preparation of the system. No reasonable argument allows us to interrelate the values of the distribution function at these distinct points. Thus, in general,

$$\tilde{f}_{1T\sigma}^{\alpha(1)}(\theta_*) \neq \tilde{f}_{1T\sigma}^{\alpha(1)}(-\theta_*). \tag{3.12}$$

This feature is stressed in fig. 3.1.

On the other hand, the distribution function of the trapped particles must be one-valued at the turning points; hence, the distribution functions of the

"plus" and "minus" particles must be equal at each of the turning points. This provides us with the following, somewhat unusual *boundary conditions for the trapped particles*:

$$\bar{f}^{\alpha(1)}_{1\mathrm{T}+}(\theta_*;\ \rho,\ \mathscr{E},\ M) = \bar{f}^{\alpha(1)}_{1\mathrm{T}-}(\theta_*;\ \rho,\ \mathscr{E},\ M),$$

$$\bar{f}^{\alpha(1)}_{1\mathrm{T}+}(-\theta_*;\ \rho,\ \mathscr{E},\ M) = \bar{f}^{\alpha(1)}_{1\mathrm{T}-}(-\theta_*;\ \rho,\ \mathscr{E},\ M). \tag{3.13}$$

In order to make use of these conditions, we write explicitly (3.11) for each value of σ and subtract them from each other,

$$\bar{f}^{\alpha(1)}_{1\mathrm{T}+}(\theta_*) - \bar{f}^{\alpha(1)}_{1\mathrm{T}+}(-\theta_*) - \bar{f}^{\alpha(1)}_{1\mathrm{T}-}(\theta_*) + \bar{f}^{\alpha(1)}_{1\mathrm{T}-}(-\theta_*)$$

$$= \int_{-\theta_*}^{\theta_*} \mathrm{d}\theta\, \frac{l_\theta}{B_\theta} \left[\sigma_{\mathrm{T}+}(\theta) - \sigma_{\mathrm{T}-}(\theta)\right].$$

The left-hand side of this equation vanishes, because of the boundary conditions (3.13), and we are left with

$$\int_{-\theta_*}^{\theta_*} \mathrm{d}\theta\, \frac{l_\theta}{B_\theta} \left[\sigma_{\mathrm{T}+}(\theta) - \sigma_{\mathrm{T}-}(\theta)\right] = 0 \quad [\text{trapped}]. \tag{3.14}$$

This is the *integrability condition for the trapped particles.*

If we now apply these results to our initial equation (3.1), the only problem left is the explicit decomposition of the source terms. The pitch-angle-scattering (PAS) term poses no serious difficulty, because its form, which involves explicitly U_α, including its sign, clearly shows that

$$\left(\nu_\alpha m_\alpha \frac{\partial}{\partial M} U_\alpha M \frac{\partial}{\partial M} \bar{f}^{\alpha(0)}_1\right)_{\mathrm{P}} = \nu_\alpha m_\alpha \frac{\partial}{\partial M} U_\alpha M \frac{\partial}{\partial M} \bar{f}^{\alpha(0)}_{1\mathrm{P}},$$

$$\left(\nu_\alpha m_\alpha \frac{\partial}{\partial M} U_\alpha M \frac{\partial}{\partial M} \bar{f}^{\alpha(0)}_1\right)_{\mathrm{T}\sigma} = \sigma\, \nu_\alpha m_\alpha \frac{\partial}{\partial M} |U_\alpha| M \frac{\partial}{\partial M} \bar{f}^{\alpha(0)}_{1\mathrm{T}\sigma}. \tag{3.15}$$

For the second term in the source, we note that the dependence on θ, contained in $\mathscr{M}_\alpha$, is factorized from the dependence on $\mathscr{E}$, contained in f_0^α. As $\mathscr{M}_\alpha(\theta, \rho)$ is a *macroscopic* quantity (made up of the electric field and of moments of the distribution function), it is well defined in the whole range of θ. By our criteria, this term is to be considered *entirely* as a "passing" contribution, thus $\mathscr{M}_{\alpha\mathrm{T}} \equiv 0$.

We now collect all our previous results and write explicitly and the *integrability conditions* (3.8) and (3.14) for eq. (3.1).

Passing particles:

$$\int_{-\pi}^{\pi} \mathrm{d}\theta \frac{l_\theta}{B_\theta} \left(v_\alpha m_\alpha \frac{\partial}{\partial M} U_\alpha M \frac{\partial}{\partial M} \bar{f}_1^{\alpha(0)} + B\mathscr{M}_\alpha f_0^\alpha \right) = 0. \tag{3.16}$$

Trapped particles:

$$\int_{-\theta_*}^{\theta_*} \mathrm{d}\theta \frac{l_\theta}{B_\theta} v_\alpha m_\alpha \frac{\partial}{\partial M} |U_\alpha| M \frac{\partial}{\partial M} \left(\bar{f}_{1\mathrm{T}+}^{\alpha(0)} + \bar{f}_{1\mathrm{T}-}^{\alpha(0)} \right) = 0. \tag{3.17}$$

14.4. Solution of the integrability constraints

We now come back to our problem, which is the solution of eq. (3.1). The result of section 14.3 shows that this solution only exists if the *integrability constraints* (3.16), (3.17) are satisfied. As will be presently shown, these constraints are *not automatically* met. Hence, before even trying to determine the unknown function $\bar{f}_1^{\alpha(1)}$, we must be sure that the constraints are satisfied. The latter, however, introduce the *source term*, which is independent of $\bar{f}_1^{\alpha(1)}$, and involves only the reference state f_0^α (which is completely determined) and the zeroth-order ($\sim \delta^0$) deviation $\bar{f}_1^{\alpha(0)}$. The only freedom left for satisfying the constraints is contained in the latter function which, fortunately, was only determined by the zeroth-order equation up to an arbitrary function $\bar{G}_1^{\alpha(0)}$ (eq. 2.13). Thus, the first result of our first-order analysis will be the complete determination of the *zeroth-order* function $\bar{f}_1^{\alpha(0)}$; only after the completion of this step can we solve for the first-order function $\bar{f}_1^{\alpha(1)}$. This state of affairs is not unfrequent in perturbation theory: a similar situation occurred in the analysis in chapter 10.

We now substitute $\bar{f}_1^{\alpha(0)}$ from (2.13) into (3.1). As a preparation for this calculation, we note the following useful result derived from (**11**.5.3):

$$\frac{\partial U_\alpha}{\partial M} = -\frac{B}{m_\alpha U_\alpha}. \tag{4.1}$$

Using this result, we easily get from (2.13)

$$U_\alpha \frac{\partial}{\partial M} \bar{f}_{1\mathrm{P}}^{\alpha(0)} = \frac{\mathscr{I}}{R_0 \mathscr{B}_\mathrm{P}} \frac{c}{e_\alpha} \frac{\partial}{\partial \rho} f_0^\alpha + U_\alpha \frac{\partial}{\partial M} \bar{G}_{1\mathrm{P}}^{\alpha(0)}. \tag{4.2}$$

We now consider the *solubility constraint* (3.16) *in the passing domain*. Using (3.1), (4.2) and (2.8), we obtain

$$\int_{-\pi}^{\pi} \mathrm{d}\theta\, l_\rho l_\theta l_\zeta \Bigg\{ v_\alpha m_\alpha \frac{\partial}{\partial M} \times M \Bigg[\frac{\mathscr{I}(\rho)}{R_0 \mathscr{B}_{\mathrm{P}}(\rho)} \frac{c}{e_\alpha} \frac{\partial f_0^\alpha(\rho, \mathscr{E})}{\partial \rho} + U_\alpha(\theta, \rho, \mathscr{E}, M) \frac{\partial}{\partial M} \bar{G}_{1\mathrm{P}}^{\alpha(0)}(\rho, \mathscr{E}, M) \Bigg] + B(\theta, \rho)\, \mathscr{M}_\alpha(\theta, \rho)\, f_0^\alpha(\rho, \mathscr{E}) \Bigg\} = 0. \tag{4.3}$$

(In the forthcoming arguments it is important to carefully keep in mind on which variables each factor depends.) We note that the last term in the integrand is independent of M; it can therefore be written as

$$B\mathscr{M}_\alpha f_0^\alpha = \frac{\partial}{\partial M} M B \mathscr{M}_\alpha f_0^\alpha. \tag{4.4}$$

Using now the definition of the surface average (**8**.8.35), (4.3) can be rewritten as

$$\frac{\mathrm{d}V}{\mathrm{d}\rho} \frac{\partial}{\partial M} M \Bigg(v_\alpha m_\alpha \frac{\mathscr{I}}{R_0 \mathscr{B}_{\mathrm{P}}} \frac{c}{e_\alpha} \frac{\partial f_0^\alpha}{\partial \rho} + v_\alpha m_\alpha \langle U_\alpha \rangle \frac{\partial}{\partial M} \bar{G}_{1\mathrm{P}}^{\alpha(0)} + \langle B\mathscr{M}_\alpha \rangle f_0^\alpha \Bigg) = 0. \tag{4.5}$$

(Let it be clear that the expression $\langle U_\alpha \rangle$ only makes sense in the *passing domain*; see section 14.3.)

This equation is satisfied by annulling the expression between parentheses *. We thus obtain

$$\frac{\partial}{\partial M} \bar{G}_{1\mathrm{P}}^{\alpha(0)} = \frac{1}{\langle U_\alpha \rangle} \Bigg(- \frac{\mathscr{I}}{R_0 \mathscr{B}_{\mathrm{P}}} \frac{c}{e_\alpha} \frac{\partial f_0^\alpha}{\partial \rho} - \frac{1}{m_\alpha v_\alpha} \langle B\mathscr{M}_\alpha \rangle f_0^\alpha \Bigg). \tag{4.6}$$

* A more general solution would be obtained by setting $M\{\cdots\} = \mathscr{K}$, where $\mathscr{K}$ is a function of $\mathscr{E}$, ρ but *not* of M It is easily seen that only $\mathscr{K} = 0$ provides a solution which is finite for all values of M (in particular, $M = 0$).

We now note that the only factor depending on the magnetic moment M is $\langle U_\alpha \rangle$. Hence, eq. (4.6) is easily integrated to yield the solution of the solubility constraint in the *passing domain*,

$$\bar{G}^{\alpha(0)}_{1P}(\rho, \mathscr{E}, M) = 2J_\alpha\left(-\frac{\mathscr{I}}{R_0 \mathscr{B}_P} \frac{c}{e_\alpha} \frac{\partial f_0^\alpha}{\partial \rho} - \frac{1}{m_\alpha \nu_\alpha} \langle B \mathscr{M}_\alpha \rangle f_0^\alpha \right), \tag{4.7}$$

where we defined

$$J_\alpha(\rho, \mathscr{E}, M) = \tfrac{1}{2} \int \mathrm{d}M \frac{1}{\langle U_\alpha \rangle}, \tag{4.8}$$

the integral being an indefinite one. This solution will be further discussed and transformed in the next section.

We now turn to the *integrability constraint in the trapped domain*. We first note that, writing $U_\alpha = \sigma |U_\alpha|$, multiplying both sides of (4.2) by σ and recalling (3.15), we have

$$|U_\alpha| \frac{\partial}{\partial M} \tilde{f}^{\alpha(0)}_{1T\sigma} = \sigma \frac{\mathscr{I}}{R_0 \mathscr{B}_P} \frac{c}{e_\alpha} \frac{\partial f_0^\alpha}{\partial \rho} + |U_\alpha| \frac{\partial}{\partial M} \bar{G}^{\alpha(0)}_{1T\sigma}. \tag{4.9}$$

Substituting this result into (3.17), the solubility condition reduces to

$$\nu_\alpha m_\alpha \int_{-\theta_*}^{\theta_*} \mathrm{d}\theta \, l_\rho l_\theta l_\zeta \frac{\partial}{\partial M} M |U_\alpha| \frac{\partial}{\partial M} \left(\bar{G}^{\alpha(0)}_{1T+} + \bar{G}^{\alpha(0)}_{1T-} \right) = 0. \tag{4.10}$$

We now note an important property of function G. From (2.13), considered in the trapping domain, we obtain

$$\tilde{f}^{\alpha(0)}_{1T\sigma}(\theta) = -\sigma \frac{\mathscr{I}}{R_0 \mathscr{B}_P} \frac{|U_\alpha(\theta)|}{\Omega_\alpha} \frac{\partial f_0^\alpha}{\partial \rho} + \bar{G}^{\alpha(0)}_{1T\sigma}, \tag{4.11}$$

where we wrote the argument θ in all functions depending effectively on the poloidal angle. Setting $\theta = \theta_*$ and recalling that, by definition, $U_\alpha(\theta_*) = 0$,

$$\tilde{f}^{\alpha(0)}_{1T\sigma}(\theta_*) = \bar{G}^{\alpha(0)}_{1T\sigma}. \tag{4.12}$$

Recalling the boundary condition (3.13) (which holds for $\tilde{f}^{\alpha(0)}_{1T\sigma}$ as well), we obtain

$$\bar{G}^{\alpha(0)}_{1T+}(\mathscr{E}, M, \rho) = \bar{G}^{\alpha(0)}_{1T-}(\mathscr{E}, M, \rho). \tag{4.13}$$

This relation holds everywhere, because G is independent of θ. Thus (4.10) reduces to the two separate equations

$$\int_{-\theta_*}^{\theta_*} \mathrm{d}\theta \, l_\rho l_\theta l_\zeta \frac{\partial}{\partial M} M \, | U_\alpha | \, \frac{\partial}{\partial M} \bar{G}_{1\mathrm{T}\sigma}^{\alpha(0)} = 0, \quad \sigma = +, - . \tag{4.14}$$

It is easily found that the only physically reasonable solution of these equations is the trivial one,

$$\bar{G}_{1\mathrm{T}\sigma}^{\alpha(0)} = 0, \quad \sigma = +, - . \tag{4.15}$$

Summarizing our results, we found that the zeroth-order (δ^0) deviation of the distribution function (2.13) must be completed by the following form of the "integration constant", in order to satisfy the integrability condition to order δ,

$$\bar{G}_1^{\alpha(0)}(\mathscr{E}, M, \rho) = \begin{cases} 2J_\alpha \left(- \dfrac{\mathscr{I}}{R_0 \mathscr{B}_\mathrm{P}} \dfrac{c}{e_\alpha} \dfrac{\partial f_0^\alpha}{\partial \rho} - m_\alpha v_\alpha \langle B \mathscr{M}_\alpha \rangle f_0^\alpha \right) & \text{[passing]}, \\ 0 & \text{[trapped]}. \end{cases} \tag{4.16}$$

14.5. The NGC variables: x, λ, φ

In order to proceed more clearly with the analysis of the results, we now choose to parametrize the phase space by a new set of NGC variables. Instead of the total energy $\mathscr{E}$, and the magnetic moment M, we shall use the following two variables: the *kinetic energy* (scaled by the thermal energy), x, and the *ratio of the magnetic moment to the kinetic energy*, λ. The gyrophase ϕ is unchanged in the transformation,

$$x = T_\alpha^{-1}(\mathscr{E} - e_\alpha \Phi), \qquad \lambda = \frac{M}{\mathscr{E} - e_\alpha \Phi}, \tag{5.1}$$

or, inversely,

$$\mathscr{E} = T_\alpha x + e_\alpha \Phi, \qquad M = T_\alpha \lambda x. \tag{5.2}$$

The local equilibrium distribution has a very simple form in these variables:

$$f_0^\alpha(x; \rho) = \left(\frac{m_\alpha}{2\pi T_\alpha(\rho)} \right)^{3/2} n_\alpha(\rho) \, \mathrm{e}^{-x}. \tag{5.3}$$

An important quantity is the parallel velocity, which is expressed as follows in terms of the new variables:

$$U_\alpha(x, \lambda; \theta, \rho) = \sigma \left(\frac{2T_\alpha}{m_\alpha}\right)^{1/2} \sqrt{x}\sqrt{1-\lambda B(\theta, \rho)}\,. \tag{5.4}$$

We also note the domain of variation of the variables,

$$0 \leqslant x \leqslant \infty, \qquad 0 \leqslant \lambda \leqslant \frac{1}{B(\theta, \rho)}. \tag{5.5}$$

Clearly, λ cannot exceed B^{-1}, otherwise U_α becomes imaginary. We underline the fact that this upper limit *depends on the spatial variables* θ, ρ. The transformation of the integrals over the entire velocity space introduces a Jacobian,

$$\begin{aligned}\int \mathrm{d}v\, F(v) &= \int_0^\infty \mathrm{d}\mathscr{E} \int_0^\infty \mathrm{d}M \int_0^{2\pi} \mathrm{d}\phi\, \frac{B}{m_\alpha^2 |U_\alpha|} F(\mathscr{E}, M, \phi) \\ &= \sum_{\sigma = +, -} \int_0^\infty \mathrm{d}x \int_0^{1/B} \mathrm{d}\lambda \int_0^{2\pi} \mathrm{d}\phi \left(\frac{T_\alpha^3}{2m_\alpha^3}\right)^{1/2} \\ &\quad \times \frac{\sqrt{x}\,B}{\sqrt{1-\lambda B}} F(x, \lambda, \phi; \sigma).\end{aligned} \tag{5.6}$$

A subtle, but important point is the summation over σ: without it, only positive directions of U_α would be covered in the integral. When $F(\mathscr{E}, M, \phi)$ is transformed to the new variables, the result $F(x, \lambda, \phi; \sigma)$ may depend explicitly on the sign σ of the parallel velocity. If it does not, the summation over σ just provides a factor 2. The reader is advised to check the normalization of f_0^α, eq. (5.3), as an exercise.

The main usefulness of the new variables is in providing a very simple form for the trapping criterion. We recall that, for given ρ, the magnetic field as a function of the poloidal angle θ has a maximum at $\theta = \pi$, as shown in fig. **9**.5.1,

$$B(\theta, \rho) \leqslant B(\pi, \rho) \equiv B_{\mathrm{M}}(\rho), \quad \forall\theta. \tag{5.7}$$

By the discussion of section **9**.5 we know that a particle is passing whenever its

parallel velocity does not vanish for any value of θ between $-\pi$ and $+\pi$. Given (5.7), it is easily seen that this happens whenever

$$0 \leqslant \lambda \leqslant \frac{1}{B_M(\rho)} \equiv \lambda_c.$$

On the contrary, when λ is comprised between B_M^{-1} and its natural upper limit B^{-1}, the quantity $1 - \lambda B(\theta, \rho)$ will vanish for a value θ_* of the poloidal angle, smaller than π; this value is defined implicitly by

$$B(\theta_*, \rho) = \lambda^{-1}. \tag{5.8}$$

Clearly, we see from (5.1) that a small value of λ corresponds to a small ratio between perpendicular energy and total energy; a large value of $\lambda B \approx 1$ corresponds to the perpendicular energy almost equal to the total energy, i.e. $U_\alpha^2 \approx 0$. The conditions found here thus agree with the discussion in section **9**.5. To sum up our discussion, we have the following criteria:

$$\begin{aligned} &0 \leqslant \lambda < \frac{1}{B_M(\rho)} \equiv \lambda_c: && \text{passing,} \\ &\lambda_c \equiv \frac{1}{B_M(\rho)} < \lambda \leqslant \frac{1}{B(\theta, \rho)}: && \text{trapped.} \end{aligned} \tag{5.9}$$

The main advantage here is that the division of the phase space into passing and trapped domains is provided by a single variable λ, unlike the trapping parameter κ of (**9**.5.6) which is a nonlinear combination of the two variables $\mathscr{E}$ and M. *

We now come back to our problem and transform the result (4.16) to the new variables. First we note that the condition $\bar{G}_1^{\alpha(0)} = 0$ in the trapped domain will now be very simply expressed in terms of a Heaviside step function $\Theta(\lambda_c - \lambda)$; we recall

$$\Theta(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases} \tag{5.10}$$

Consider now the function J_α defined by (4.8). Using (5.2) and (5.4), we obtain

$$J_\alpha(x, \lambda, \rho) = \frac{1}{2}\left(\frac{2T_\alpha}{m_\alpha}\right)^{1/2} \sqrt{x}\, \sigma \int_\lambda^{\lambda_c} d\lambda' \frac{1}{\langle \sqrt{1 - \lambda' B(\theta, \rho)} \rangle}. \tag{5.11}$$

* The parameter κ has its own advantages, which were exploited in chapter 9; in particular, it leads to a simple explicit solution in terms of elliptic functions, in the framework of the standard model.

The limits of integration should be clear from the fact that J_α is an indefinite integral, restricted to the passing domain. In order to get a feeling of its meaning, we consider the case of a *homogeneous magnetic field*, i.e. B independent of θ and ρ. In this case the surface average is deleted and a straightforward integration yields the result

$$J_\alpha(x, \lambda) = \frac{1}{2}\left(\frac{2T_\alpha}{m_\alpha}\right)^{1/2} \sqrt{x}\, \frac{1}{B}\, 2\sigma\sqrt{1-\lambda B} = \frac{1}{B} U_\alpha(x, \lambda). \tag{5.12}$$

Thus, in the homogeneous case, $J_\alpha B$ is simply the parallel velocity. In the general case, we may introduce a "*scaling magnetic field*" $\mathscr{B}_0$, and define $J_\alpha \mathscr{B}_0$ as a kind of "average parallel velocity", distorted by the magnetic field inhomogeneity. The natural quantity chosen for scaling any quantity having the dimensions of a magnetic field is the one defined in (**12**.8.16),

$$\mathscr{B}_0 \equiv \langle B^2 \rangle^{1/2} \tag{5.13}$$

It is a function of the radial coordinate ρ alone, and contains implicitly the effect of the geometry of the configuration. We now define the *average parallel velocity* as

$$\sqrt{x}\, V_\alpha(\lambda, \rho) = \mathscr{B}_0 J_\alpha(x, \lambda, \rho), \tag{5.14}$$

and, from (5.11), we get the explicit form

$$V_\alpha(\lambda, \rho) = \frac{1}{2}\left(\frac{2T_\alpha}{m_\alpha}\right)^{1/2} \mathscr{B}_0\, \sigma \int_\lambda^{\lambda_c} d\lambda' \frac{1}{\langle \sqrt{1-\lambda' B(\theta, \rho)} \rangle}. \tag{5.15}$$

The advantage of taking out the factor $\sqrt{x}$ is that V_α only depends on λ and ρ, not on x. Note also that V_α has the same sign σ as U_α.

As a final step, we recall that $\partial f_0^\alpha / \partial \rho$ is simply proportional to f_0^α, as follows from (2.6). Hence, the solution (4.16) of the integrability constraint may be written in the form

$$\bar{G}_1^{\alpha(0)}(x, \lambda, \rho) = 2\Theta(\lambda_c - \lambda)\, V_\alpha(\lambda, \rho) \sqrt{x}\, \mathscr{G}^\alpha(x, \rho)\, f_0^\alpha(x, \rho). \tag{5.16}$$

The function $\mathscr{G}^\alpha(x, \rho)$ can be readily obtained from (4.16) and will not be written down explicitly here.

The truly remarkable fact about result (5.16) (first obtained in this form and exploited by Hirshman and Sigmar 1976, 1981; see also Hirshman et al. 1976) is the following:

The dependence on λ *is completely factorized from the dependence on* x. Indeed, the dependence on λ is *entirely* contained in the factor $\Theta(\lambda_c - \lambda)$ $V_\alpha(\lambda, \rho)$. This will prove to be of major importance.

The next comment is about the function $\mathscr{G}^\alpha(x, \rho)$. If written explicitly in the form (4.16), it clearly depends on the collisional model chosen (through the moments $\mathscr{M}_\alpha$). A possible approach would be to choose a specific model and continue the calculations in that framework. This is essentially the philosophy of the older works (Kovrizhnykh 1970, Rosenbluth et al. 1972). The disadvantage is that, if one wishes to obtain analytical results, one is confined to very simple geometries (standard model) and rather primitive collision models (or else, one has to resort to variational techniques).

A radically different method, due to Hirshman and Sigmar (1976, 1981) is to use at this stage a short-circuit, reminiscent of a moment method. In this way, it will be shown that analytical results can be obtained very simply and in very general cases. Moreover (almost miraculously!) the results turn out to be independent of the particular collisional model chosen.

In this method one "forgets" the explicit, model-dependent expression of $\bar{G}_1^{\alpha(0)}$ and notes that $\sqrt{x}\,\mathscr{G}^\alpha(x)\, f_0^\alpha(x)$ is the factor that determines the dependence of $\bar{G}_1^{\alpha(0)}$ (hence, of the distribution function $\bar{f}_1^{\alpha(0)}$) on the kinetic energy x. The unknown part $\mathscr{G}^\alpha(x)$ can always be expanded on a basis of orthogonal polynomials. In general, a truncated series may prove to be a good approximation, as was the case in the Hermitian moment method of chapter 4.

The choice of a relevant basis of expansion is dictated by the following considerations:

– We have to expand a function of a *single scalar variable*, the dimensionless kinetic energy x. (This is a difference with the problem of chapter 4, where we had to expand functions of the *vector* variable c_r.)

– The reference function $f_0^\alpha(x)$ has the simple exponential form $\exp(-x)$. This suggests the choice of a set of Laguerre–Sonine polynomials $L_m^\beta(x)$. The definition and the main properties of these polynomials are given in general Appendix 1, section **G1**.3.

– The presence of the factor $\sqrt{x}$ suggests the value $\beta = \frac{3}{2}$. We thus decide to expand $\mathscr{G}^\alpha(x, \rho)$ as

$$\mathscr{G}^\alpha(x, \rho) = \sum_m b_m^\alpha(\rho)\; L_m^{3/2}(x). \tag{5.17}$$

We recall here that the polynomials are normalized as

$$\frac{2}{\sqrt{\pi}} \int_0^\infty \mathrm{d}x\; \mathrm{e}^{-x}\, x^{3/2}\, L_m^{3/2}(x)\; L_n^{3/2}(x) = \delta_{m,n}. \tag{5.18}$$

It will be shown in section 14.7 that the Laguerre–Sonine moments $b_m^\alpha(\rho)$ are related to macroscopic quantities of great interest.

14.6. Expansion of the zeroth-order distribution function

We first derive some important relations between the Laguerre–Sonine moments and the hydrodynamic quantities. A useful transformation formula is easily obtained by combining (5.6) and (5.3):

$$\int \mathrm{d}\boldsymbol{c}\, \phi^0(c)\, P(\boldsymbol{c}) = \frac{1}{8\pi} B \sum_\sigma \frac{2}{\sqrt{\pi}} \int_0^\infty \mathrm{d}x \sqrt{x}\, \mathrm{e}^{-x} \times \int_0^{B^{-1}} \mathrm{d}\lambda \int_0^{2\pi} \mathrm{d}\phi\, \frac{P(x, \lambda, \phi; \sigma)}{\sqrt{1-\lambda B}}, \tag{6.1}$$

where $P(\boldsymbol{c})$ is an arbitrary function of the dimensionless velocity $\boldsymbol{c} \equiv (m_\alpha/T_\alpha)^{1/2}\boldsymbol{v}$. The dependence on the spatial variables θ, ρ is not written explicitly in this formula [it comes in through $P(\theta, \rho)$, $B(\theta, \rho)$].

Consider now specifically the *parallel component of a Hermitian vector moment* $h_\parallel^{\alpha(2n+1)}$. From (4.3.12) we obtain

$$h_\parallel^{\alpha(2n+1)}(\theta, \rho) = \int \mathrm{d}\boldsymbol{c}\, \phi^0(c)\, \chi^\alpha(\boldsymbol{c}; \theta, \rho)\, b_r(\theta, \rho)\, H_r^{(2n+1)}(\boldsymbol{c}), \tag{6.2}$$

where $\phi^0\chi^\alpha$ is the deviation from the reference local equilibrium, and $b_r(\theta, \rho)$ is the r-component of the unit vector $\boldsymbol{b}$. We now make use of the relation between the Hermitian vector polynomials and the Laguerre–Sonine polynomials given in General Appendix G1, eq. (**G1**.4.4),

$$H_r^{(2n+1)}(\boldsymbol{c}) = \sqrt{\tfrac{3}{2}}\, c_r\, L_n^{3/2}\left(\tfrac{1}{2}c^2\right). \tag{6.3}$$

Noting that $b_r c_r$ is just the dimensionless parallel velocity $c_\parallel$, and that $c^2/2$ is the dimensionless kinetic energy x, we obtain the transformation formula

$$b_r H_r^{(2n+1)}(\boldsymbol{c}) = \sqrt{\tfrac{3}{2}}\, U_\alpha \left(\frac{m_\alpha}{T_\alpha}\right)^{1/2} L_n^{3/2}(x) = \sigma \sqrt{\tfrac{3}{2}} \sqrt{2x} \sqrt{1-\lambda B}\, L_n^{3/2}(x). \tag{6.4}$$

Hence, we get from (6.1) (omitting to write the variables θ, ρ),

$$h_\parallel^{\alpha(2n+1)} = \frac{\sqrt{3}}{8\pi} B \sum_\sigma \frac{2}{\sqrt{\pi}} \int_0^\infty \mathrm{d}x\, x\, \mathrm{e}^{-x} \int_0^{B^{-1}} \mathrm{d}\lambda \times \int_0^{2\pi} \mathrm{d}\phi\, \chi^\alpha(x, \lambda, \phi; \sigma)\, \sigma L_n^{3/2}(x). \tag{6.5}$$

We now introduce a quantity $\hat{\chi}^{\alpha}(x)$ depending only on x (and on the spatial variables θ, ρ),

$$\hat{\chi}^{\alpha}(x) = \frac{1}{8\pi} B \sum_{\sigma} \int_0^{B^{-1}} d\lambda \int_0^{2\pi} d\phi \, \sigma \chi^{\alpha}(x, \lambda, \phi; \sigma). \tag{6.6}$$

Then

$$h_{\|}^{\alpha(2n+1)} = \sqrt{3} \frac{2}{\sqrt{\pi}} \int_0^{\infty} dx \, e^{-x} \, x \, \hat{\chi}^{\alpha}(x) \, L_n^{3/2}(x). \tag{6.7}$$

We thus obtained *a simple formula expressing the parallel vector fluxes in terms of Laguerre–Sonine polynomials*. We now expand the deviation of the distribution function $\hat{\chi}^{\alpha}(x)$ as

$$\hat{\chi}^{\alpha}(x) = \sqrt{x} \sum_{m} a_m^{\alpha} L_m^{3/2}(x). \tag{6.8}$$

The orthogonality relations (5.18) tell us that

$$h_{\|}^{\alpha(2n+1)} = \sqrt{3} \, a_n^{\alpha}. \tag{6.9}$$

We note, in particular, the important particle fluxes $h_{\|}^{\alpha(1)}$ and heat fluxes $h_{\|}^{\alpha(3)}$,

$$h_{\|}^{\alpha(1)} = \sqrt{3} \, a_0^{\alpha}, \qquad h_{\|}^{\alpha(3)} = \sqrt{3} \, a_1^{\alpha}. \tag{6.10}$$

We now define the dimensionless deviations $\chi^{\alpha}(x, \lambda, \phi)$ in a more precise way in terms of the new NGC variables x, λ, ϕ. This will allow us to connect our previous results to the concepts discussed above. We introduce a scaling analogous to (4.3.5), but adapted to the new variables. The reference distribution function is written in the form

$$f_0^{\alpha}(x) = n_{\alpha} \left(\frac{m_{\alpha}}{2T_{\alpha}} \right)^{1/2} \frac{1}{2\pi} \phi^0(x), \tag{6.11}$$

with

$$\phi^0(x) = \frac{2}{\sqrt{\pi}} e^{-x}, \tag{6.12}$$

which is normalized as

$$\int_0^{\infty} dx \sqrt{x} \, \phi^0(x) = 1. \tag{6.13}$$

The ϵ-expansion (1.1) of the gyro-averaged distribution function is then written as

$$\bar{f}^{\alpha}(x, \lambda) = \frac{1}{2\pi}\left(\frac{m_\alpha}{2T_\alpha}\right)^{1/2} n_\alpha \phi^0(x)\left[1 + \bar{\chi}_1^{\alpha}(x, \lambda)\right]. \tag{6.14}$$

The deviation $\bar{\chi}_1^{\alpha}(x, \lambda)$ is, in turn, expanded in powers of δ, according to (1.7),

$$\bar{\chi}_1^{\alpha}(x, \lambda) = \bar{\chi}_1^{\alpha(0)}(x, \lambda) + \bar{\chi}_1^{\alpha(1)}(x, \lambda). \tag{6.15}$$

The dimensionless zeroth-order distribution function $\bar{\chi}_1^{\alpha(0)}$ can now be written explicitly, using (2.13), (2.6) and (5.16),

$$\begin{aligned}\bar{\chi}_1^{\alpha(0)}(x, \lambda) = {} & \frac{\mathscr{I}}{R_0 \mathscr{B}_{\mathrm{p}} \Omega_\alpha \tau_\alpha}\left(\frac{m_\alpha}{T_\alpha}\right)^{1/2} l_\rho U_\alpha(x, \lambda) \\ & \times\left[g_\rho^{\alpha(1)} - g_\rho^{\alpha(1)A} + \sqrt{\tfrac{2}{5}}\left(x - \tfrac{5}{2}\right) g_\rho^{\alpha(3)}\right] \\ & + 2\Theta(\lambda_c - \lambda)\, V_\alpha(x, \lambda)\sqrt{x}\, \mathscr{G}^{\alpha}(x) \\ \equiv {} & \bar{\chi}_{\mathrm{I}}^{\alpha(0)}(x, \lambda) + \bar{\chi}_{\mathrm{II}}^{\alpha(0)}(x, \lambda).\end{aligned} \tag{6.16}$$

We have a decomposition into two terms which, as will be seen, correspond to different physical properties. The integrations over ϕ and λ, according to (6.6) can now be done explicitly. Using (5.4), we find for the first term

$$\begin{aligned}\hat{\chi}_1^{\alpha(0)}(x) = {} & \frac{B}{8\pi}\sum_\sigma \int_0^{B^{-1}} \mathrm{d}\lambda\; 2\pi\sigma\left(\frac{2T_\alpha}{m_\alpha}\right)^{1/2} \sigma\sqrt{x}\sqrt{1 - \lambda B} \\ & \times \frac{\mathscr{B}_0}{B}\mathscr{K}_\alpha\left(\frac{m_\alpha}{T_\alpha}\right)^{1/2}\left[g_\rho^{\alpha(1)} - g_\rho^{\alpha(1)A} + \sqrt{\tfrac{2}{5}}\left(x - \tfrac{5}{2}\right) g_\rho^{\alpha(3)}\right].\end{aligned}$$

We introduced here the abbreviation $\mathscr{K}_\alpha$ defined in (**13**.3.3). The summation over σ gives a factor 2; the integration over λ is elementary. Recalling the definition of the Laguerre–Sonine polynomials $L_0^{3/2}$, $L_1^{3/2}$, we may write

$$\hat{\chi}_{\mathrm{I}}^{\alpha(0)} = \frac{1}{\sqrt{3}}\frac{\mathscr{B}_0}{B}\mathscr{K}_\alpha\sqrt{x}\left[\left(g_\rho^{\alpha(1)} - g_\rho^{\alpha(1)A}\right) L_0^{3/2}(x) + g_\rho^{\alpha(3)} L_1^{3/2}(x)\right]. \tag{6.17}$$

Thus, $\hat{\chi}_{\mathrm{I}}^{\alpha(0)}(x)$ is exactly a combination of two Laguerre–Sonine polynomials.

Consider now the second term in (6.16). Using (5.15), we get

$$\hat{\chi}_{\mathrm{II}}^{\alpha(0)} = \frac{B}{8\pi}\sum_{\sigma}\int_0^B {}^{1}\mathrm{d}\lambda\, 2\pi\sigma\,\Theta(\lambda_c-\lambda)\left(\frac{2T_\alpha}{m_\alpha}\right)^{1/2}\mathscr{B}_0$$

$$\times\sigma\int_\lambda^{\lambda_c}\mathrm{d}\lambda'\frac{1}{\langle\sqrt{1-\lambda' B}\,\rangle}\sqrt{x}\,\mathscr{G}^\alpha. \tag{6.18}$$

The integration over λ is now no longer trivial. Permuting the order of the integrations, we have

$$\int_0^{\lambda_c}\mathrm{d}\lambda\int_\lambda^{\lambda_c}\mathrm{d}\lambda'\frac{1}{\langle\sqrt{1-\lambda' B}\,\rangle} = \int_0^{\lambda_c}\mathrm{d}\lambda'\int_0^{\lambda'}\mathrm{d}\lambda\,\frac{1}{\langle\sqrt{1-\lambda' B}\,\rangle}$$

$$= \int_0^{\lambda_c}\mathrm{d}\lambda'\,\frac{\lambda'}{\langle\sqrt{1-\lambda' B}\,\rangle}. \tag{6.19}$$

We now introduce the following dimensionless quantity:

$$f_{\mathrm{P}} = \tfrac{3}{4}\mathscr{B}_0^2\int_0^{\lambda_c}\mathrm{d}\lambda\,\frac{\lambda}{\langle\sqrt{1-\lambda B}\,\rangle}, \tag{6.20}$$

where $\mathscr{B}_0$ is defined in (5.13). Hirshman and Sigmar (1981) call this quantity the "fraction of passing particles". We do not think that this name is appropriate. Given the importance of this quantity in the forthcoming treatment, we devote a special section (section 14.8) to its study.

The evaluation of $\hat{\chi}_{\mathrm{II}}^{\alpha(0)}$ from (6.18) is now straightforward:

$$\hat{\chi}_{\mathrm{II}}^{\alpha(0)}(x) = \frac{2}{3}\left(\frac{2T_\alpha}{m_\alpha}\right)^{1/2}\frac{B}{\mathscr{B}_0}f_{\mathrm{P}}\sqrt{x}\,\mathscr{G}^\alpha(x). \tag{6.21}$$

We recall that $\mathscr{G}(x)$ can be expanded in a series of Laguerre–Sonine polynomials, according to (5.17). But, in contrast to $\hat{\chi}_{\mathrm{I}}^{\alpha(0)}$, *the series contains an infinite number of terms*. We now state the basic approximation adopted in the forthcoming treatment: *the Laguerre-Sonine series* (5.17) *will be truncated at the level* $m = 2$. Thus, we retain three polynomials and write

$$\hat{\chi}_{\mathrm{II}}^{\alpha(0)}(x) = \frac{2}{3}\left(\frac{2T_\alpha}{m_\alpha}\right)^{1/2}\frac{B}{\mathscr{B}_0}f_{\mathrm{P}}\sqrt{x}\left(b_0^\alpha\, L_0^{3/2}(x) + b_1^\alpha\, L_1^{3/2}(x) + b_2^\alpha\, L_2^{3/2}(x)\right). \tag{6.22}$$

Given the relation between Hermite and Laguerre–Sonine moments, this approximation corresponds (in a sophisticated way!) to the *21M approximation* of section 4.3. Hirshman and Sigmar (1981) chose to truncate the series at $m = 1$, i.e. in the 13M approximation. By going one step further, we will obtain information about the rate of convergence of the approximations.

14.7 Relation between the zeroth-order distribution function and the macroscopic fluxes to order ϵ

We note, at this stage, that the problem has been reduced to the determination of the three moments b_0^α, b_1^α, b_2^α; these, in turn, are related to the physical fluxes. Indeed, because of the form (6.16), the parallel components of the vector moments can also be decomposed into two terms. Using (6.10), (6.17) and (6.22), we find that the zeroth-order (i.e. δ^0) deviation of the distribution function $\bar{\chi}_1^{\alpha(0)}$ gives rise to the following parallel fluxes:

$$
\begin{aligned}
h_{\|}^{\alpha(1)} &= h_{\|[\mathrm{I}]}^{\alpha(1)} + h_{\|[\mathrm{II}]}^{\alpha(1)}, \\
h_{\|}^{\alpha(3)} &= h_{\|[\mathrm{I}]}^{\alpha(3)} + h_{\|[\mathrm{II}]}^{\alpha(3)}, \\
h_{\|}^{\alpha(5)} &= \qquad\quad h_{\|[\mathrm{II}]}^{\alpha(5)},
\end{aligned}
\tag{7.1}
$$

with

$$
\begin{aligned}
h_{\|[\mathrm{I}]}^{\alpha(1)} &= \frac{\mathscr{B}_0}{B} \mathscr{K}_\alpha \left(g_\rho^{\alpha(1)} - g_\rho^{\alpha(1)A} \right), \\
h_{\|[\mathrm{I}]}^{\alpha(3)} &= \frac{\mathscr{B}_0}{B} \mathscr{K}_\alpha \, g_\rho^{\alpha(3)}, \\
h_{\|[\mathrm{I}]}^{\alpha(5)} &= 0,
\end{aligned}
\tag{7.2}
$$

and

$$
\begin{aligned}
h_{\|[\mathrm{II}]}^{\alpha(1)} &= 2\sqrt{\frac{2}{3}} \left(\frac{T_\alpha}{m_\alpha} \right)^{1/2} \frac{B}{\mathscr{B}_0} f_{\mathrm{P}} \, b_0^\alpha, \\
h_{\|[\mathrm{II}]}^{\alpha(3)} &= 2\sqrt{\frac{2}{3}} \left(\frac{T_\alpha}{m_\alpha} \right)^{1/2} \frac{B}{\mathscr{B}_0} f_{\mathrm{P}} \, b_1^\alpha, \\
h_{\|[\mathrm{II}]}^{\alpha(5)} &= 2\sqrt{\frac{2}{3}} \left(\frac{T_\alpha}{m_\alpha} \right)^{1/2} \frac{B}{\mathscr{B}_0} f_{\mathrm{P}} \, b_2^\alpha.
\end{aligned}
\tag{7.3}
$$

In order to understand the physical meaning of these terms, we recall some important earlier results. It was shown in section **12**.8 that, upon exploiting the condition of zero divergence of the fluxes to order ϵ, we obtained the following expressions for the parallel dimensionless fluxes (eq. **12**.8.23):

$$h_{\|}^{\alpha(1)} = \frac{\mathscr{B}_0}{B}\,\mathscr{K}_\alpha\left(g_\rho^{\alpha(1)} - g_\rho^{\alpha(1)A}\right) + \frac{B}{\mathscr{B}_0}\,\omega_1^\alpha,$$

$$h_{\|}^{\alpha(3)} = \frac{\mathscr{B}_0}{B}\,\mathscr{K}_\alpha\, g_\rho^{\alpha(3)} + \frac{B}{\mathscr{B}_0}\,\omega_3^\alpha,$$

$$h_{\|}^{\alpha(5)} = \frac{B}{\mathscr{B}_0}\,\omega_5^\alpha, \tag{7.4}$$

where the dimensionless surface functions ω_n^α, called the "*poloidal fluxes*" were defined in (**12**.8.24).

We now note that expressions (7.4) of the parallel components of the fluxes must be identical to those obtained from the distribution function to order ϵ and order δ^0, i.e. $\bar{\chi}_1^{\alpha(0)}$.

Comparing (7.4) to (7.1–7.3), we note the very pleasing fact that the first terms of the right-hand sides of the former are identical to the contributions $h_{\|[1]}^{\alpha(1)}$, $h_{\|[1]}^{\alpha(3)}$: they are the parts which are simply related to the thermodynamical forces.

As a result, *the Laguerre–Sonine moments* b_0^α, b_1^α, b_2^α of (7.3) *are directly proportional to the "poloidal fluxes"* ω_1^α, ω_3^α, ω_5^α, respectively. Comparing (7.4) to (7.3) we find

$$b_0^\alpha = \frac{1}{2}\sqrt{\frac{3}{2}}\left(\frac{m_\alpha}{T_\alpha}\right)^{1/2}\frac{1}{f_{\mathrm{P}}}\,\omega_1^\alpha,$$

$$b_1^\alpha = \frac{1}{2}\sqrt{\frac{3}{2}}\left(\frac{m_\alpha}{T_\alpha}\right)^{1/2}\frac{1}{f_{\mathrm{P}}}\,\omega_3^\alpha,$$

$$b_2^\alpha = \frac{1}{2}\sqrt{\frac{3}{2}}\left(\frac{m_\alpha}{T_\alpha}\right)^{1/2}\frac{1}{f_{\mathrm{P}}}\,\omega_5^\alpha. \tag{7.5}$$

We may rewrite the non-trivial x-depending part $\mathscr{G}^\alpha$ of the distribution function in terms of the quantities ω_n^α,

$$\mathscr{G}^\alpha(x) = \frac{1}{2f_{\mathrm{P}}}\left(\frac{m_\alpha}{T_\alpha}\right)^{1/2}\sqrt{\frac{3}{2}}\left(\omega_1^\alpha L_0^{3/2}(x) + \omega_3^\alpha L_1^{3/2}(x) + \omega_5^\alpha L_2^{3/2}(x)\right). \tag{7.6}$$

In order to sum up all the previous results, we rewrite the complete form of the zeroth-order deviation of the distribution function as

$$\tilde{f}_1^{\alpha(0)}(x, \lambda) = \frac{1}{2\pi} n_\alpha \left(\frac{m_\alpha}{2T_\alpha}\right)^{3/2} \phi^0(x)\, \bar{\chi}_1^{\alpha(0)}(x, \lambda), \tag{7.7}$$

$$\begin{aligned}\bar{\chi}_1^{\alpha(0)}(x, \lambda) = {} & \frac{\mathscr{B}_0}{B} \mathscr{K}_\alpha\, U_\alpha(x, \lambda) \\ & \times \frac{1}{\sqrt{3}} \left[\left(g_\rho^{\alpha(1)} - g_\rho^{\alpha(1)A}\right) L_0^{3/2}(x) + g_\rho^{\alpha(3)} L_1^{3/2}(x)\right] \\ & + \Theta(\lambda_c - \lambda) \frac{B}{\mathscr{B}_0} V_\alpha(x, \lambda) \left(\frac{m_\alpha}{T_\alpha}\right)^{1/2} \frac{1}{f_P} \sqrt{\tfrac{3}{2}} \sqrt{x} \\ & \times \left[\omega_1^\alpha L_0^{3/2}(x) + \omega_3^\alpha L_1^{3/2}(x) + \omega_5^\alpha L_2^{3/2}(x)\right].\end{aligned} \tag{7.8}$$

14.8. The function f_P and the neoclassical factor φ

We study here some properties of the surface function f_P:

$$f_P(\rho) = \tfrac{3}{4} \mathscr{B}_0^2 \int_0^{\lambda_c(\rho)} d\lambda \frac{\lambda}{\langle \sqrt{1 - \lambda B(\theta, \rho)} \rangle}. \tag{8.1}$$

Clearly, the value of this factor depends in an essential way on the geometry of the magnetic field. Indeed, the surface average in the integrand can only be calculated explicitly if the dependence of B on θ is known.

In the case of a *straight field*, i.e. when B is independent of θ, the average can be omitted; moreover, $B = \mathscr{B}_0 = \lambda_c^{-1}$, and the integral is easily evaluated,

$$f_p = 1 \quad \text{[straight field]}. \tag{8.2}$$

A property, stated without proof by Hirshman and Sigmar (1981) is the following:

$$0 \leqslant f_P \leqslant 1. \tag{8.3}$$

As will be seen in the next chapter, this property is essential for transport theory. One may therefore wonder whether this is a general statement. We shall look for an upper bound for f_P. First note the inequalities

$$\int_0^{\lambda_c} d\lambda \frac{\lambda}{\langle\sqrt{1-\lambda B(\theta,\rho)}\rangle} \leqslant \int_0^{\lambda_c} d\lambda \left\langle \frac{\lambda}{\sqrt{1-\lambda B(\theta,\rho)}} \right\rangle$$

$$= \left\langle \int_0^{\lambda_c} d\lambda \frac{\lambda}{\sqrt{1-\lambda B(\theta,\rho)}} \right\rangle. \tag{8.4}$$

Indeed, from

$$0 \leqslant \left\langle \left(\frac{1}{x} - \frac{x}{\langle x^2\rangle} \right)^2 \right\rangle = \left\langle \frac{1}{x^2} \right\rangle - \frac{1}{\langle x^2\rangle}$$

we obtain (8.4) by taking $x = (1-\lambda B)^{1/4}$. Thus,

$$f_P \leqslant f_P' = \tfrac{3}{4}\mathscr{B}_0^2 \left\langle \int_0^{\lambda_c} d\lambda \frac{\lambda}{\sqrt{1-\lambda B(\theta,\rho)}} \right\rangle$$

$$= \mathscr{B}_0^2 \left\langle \frac{1}{B^2}\left[1-\left(1-\tfrac{1}{2}\lambda_c B\right)\sqrt{1-\lambda_c B}\right] \right\rangle$$

$$= \frac{\mathscr{B}_0^2}{B_M^2} \left\langle \frac{1}{\chi^2}\left[1-\left(1-\tfrac{1}{2}\chi\right)\sqrt{1-\chi}\right] \right\rangle,$$

where we use the quantity

$$\chi(\theta) = \lambda_c B(\theta) \equiv \frac{B(\theta)}{B_M} \leqslant 1.$$

From this form it can be clearly seen that (8.3) is *not* an absolute identity, valid for all possible geometries. The argument under the average operation is

not uniformly small; it actually tends to infinity if $\chi \to 0$. In order to get a more precise estimate, we need some additional information about the geometry. A useful parameter in this connection is the *magnetic field modulation* (or the local mirror ratio), defined as *

$$\Xi = \frac{B_M - B_{min}}{B_M} \leqslant 1. \tag{8.5}$$

This quantity, which is intrinsically less than one, can be considered very small in a tokamak configuration (Freidberg 1982),

$$\Xi \ll 1. \tag{8.6}$$

In the standard model, this quantity is of order η. We then write

$$\chi(\theta) = 1 - \xi(\theta),$$

where

$$\xi(\theta) = \frac{B_M - B(\theta)}{B_M},$$

and we note the obvious properties

$$0 \leqslant \xi \leqslant \Xi \ll 1.$$

For such small values, the factor f_P is estimated, to dominant order, as

$$f_P \leqslant f_P' \approx 1 - 2\langle\sqrt{\xi}\rangle \leqslant 1.$$

This appears to be about the limit of the precision in our estimate, in absence of additional information about the geometry. A détailed calculation of f_P in the standard model will be given below.

In conclusion, we may say that *property* (8.3) is *true when the field modulation Ξ is sufficiently small.* In any real configuration, the validity of this

* This definition differs from the one by Hirshman and Sigmar (1981); they use $B_M + B_{min}$ in the denominator.

property must be checked, in order to make sure that the neoclassical theory is applicable.

Property (8.3) together with (8.2) suggested to Hirshman and Sigmar the name *fraction of passing particles* given to the number f_P: it is a true fraction ($\leqslant 1$) and, for a straight field (when there is no trapping) its value is 1. We do not want to adopt this terminology. Indeed, f_P is a purely geometrical factor, depending only on the shape of the magnetic field and not at all on the characteristics of the particle distribution function. The only quantity which may properly be called the fraction of passing particles is (see eq. (5.6)

$$p_P^\alpha = \sum_\sigma \int dx \int d\lambda \int d\phi \left(\frac{T_\alpha}{m_\alpha}\right)^{3/2} \frac{\sqrt{x}\,B}{\sqrt{1-\lambda B}} \Theta(\lambda_c - \lambda)\, f^\alpha(x, \lambda, \phi). \tag{8.7}$$

Moreover, as shown above, the number f_P is not necessarily less than one. For these reasons, we prefer not to use this name and interpretation of f_P.

It will appear in chapter 15 that a fundamental role is played in the theory by the number

$$\varphi = \frac{1 - f_P}{f_P}. \tag{8.8}$$

This number will be called the *neoclassical factor*. Whenever (8.3) is true, we have

$$\varphi \geqslant 0. \tag{8.9}$$

This property is a fundamental criterion of validity of the neoclassical theory. Indeed, it will appear (section **15**.3) that all the neoclassical banana fluxes are proportional to φ. A negative value for φ would therefore imply negative transport coefficients, and thus an inacceptable violation of thermodynamics.

Clearly, just like f_P, the neoclassical factor can only be evaluated when the geometry of the configuration is known. As usual, we shall get a useful estimate of this factor by using the *standard model* of section **8**.9. The latter corresponds to the large aspect ratio limit,

$$\bar{\eta} = \frac{a}{R_0} \ll 1.$$

Let us call η the ratio (r/R_0) $[\leqslant (a/R_0)]$, which is constant in our calculations. Using the formulae of section **8.9**, we find (setting $\mu = \lambda \mathscr{B}_0$)

$$\left\langle \left(1 - \mu \frac{B(\theta)}{\mathscr{B}_0}\right)^{1/2} \right\rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \, (1 + \eta \cos \theta) \left(1 - \frac{\mu}{1 + \eta \cos \theta}\right)^{1/2}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \, [(1 + \eta \cos \theta)(1 - \mu + \eta \cos \theta)]^{1/2}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \left[1 - \mu - \mu\eta \cos \theta + \eta^2 \cos^2 \theta\right]^{1/2}$$

$$\equiv F(\mu; \eta).$$

We now note

$$\mu_c = \frac{\mathscr{B}_0}{B_M} = 1 + \eta \cos \pi = 1 - \eta.$$

Hence, the factor f_P is given by

$$f_P = \tfrac{3}{4} \int_0^{1-\eta} d\mu \, \frac{\mu}{F(\mu; \eta)}.$$

The trouble with this quantity is that it is not an analytical function of η. We may, however, evaluate it by using a not very rigourous, but satisfactory method.

We expand the function F (which is bounded) in a power series in the small parameter η; the result is

$$F(\mu; \eta) = \sqrt{1 - \mu} - \frac{1}{16} \frac{\mu^2}{(1 - \mu)^{3/2}} \eta^2 + O(\eta^4).$$

Hence

$$f_P \approx \tfrac{3}{4} \int_0^{1-\eta} d\mu \, \frac{\mu}{\sqrt{1 - \mu}} \left(1 + \frac{1}{16} \frac{\mu^2}{(1 - \mu)^2} \eta^2 + \cdots\right).$$

The difficulty lies in the fact that the upper limit of integration depends on η; as a result, the η-ordering of the integral is not the same as for the

integrand. Using a standard integral table, we find

$$f_{\mathrm{P}}^{(0)} = \tfrac{3}{4}\int_0^{1-\eta} \mathrm{d}\mu \, \frac{\mu}{\sqrt{1-\mu}} = 1 - \tfrac{3}{2}\sqrt{\eta} + \mathrm{O}(\eta^{3/2}),$$

and

$$f_{\mathrm{P}}^{(1)} = \tfrac{3}{4}\tfrac{1}{16}\eta^2 \int_0^{1-\eta} \mathrm{d}\mu \, \frac{\mu^3}{(1-\mu)^{5/2}} = \tfrac{1}{32}\sqrt{\eta} + \mathrm{O}(\eta^{3/2}).$$

Thus, the two terms give contributions of the same order $\sqrt{\eta}$ but the numerical coefficient of the second term is much smaller. One may hope that this procedure "converges" in some sense. If we keep the two terms, we obtain

$$f_{\mathrm{P}} \approx 1 - (\tfrac{3}{2} - \tfrac{1}{32})\sqrt{\eta} + \mathrm{O}(\eta^{3/2}),$$

or

$$f_{\mathrm{P}} \approx 1 - 1.469\sqrt{\eta}\,. \tag{8.10}$$

This value agrees with the results of other authors [for instance Hirshman et al. (1976) give a coefficient 1.46]. From this result we find the approximate value for the *neoclassical factor in the standard model*:

$$\varphi \approx 1.469\sqrt{\eta}\,. \tag{8.11}$$

This value will be used in the discussion of the transport coefficients in chapter 15.

14.9. Strategy for the solution of the first-order drift kinetic equation

The previous sections were devoted to the solution of the drift kinetic equation to order 0 in the collisionality parameter, i.e. the function $\bar{f}_1^{\alpha(0)}$. In order to calculate the relevant radial fluxes required in transport theory, we need to know the next order contribution $\bar{f}_1^{\alpha(1)}$ to the distribution function. In other words, we must solve eq. (3.1). This equation determines $\bar{f}_1^{\alpha(1)}$ as a functional of $\bar{f}_1^{\alpha(0)}$, which enters the source term (i.e. the collision term). We have spent much effort in the previous sections in order to make sure that this equation is indeed soluble.

It turns out that eq. (3.1) is actually too difficult for an explicit analytic solution. However, for our specific purpose, we do not need to know the detailed form of function $\bar{f}_1^{\alpha(1)}$: only certain moments are required for the calculation of the banana fluxes. The object of chapter 12 was precisely to isolate the necessary information in eqs. (**12**.9.16) and (**12**.9.17); the quantities

required are the *surface-averaged divergences of the generalized stress tensor* $\boldsymbol{h}^{\alpha(2p)}$, for $p = 1, 2, 3$. The microscopic expressions of these quantities were given in eqs. (**12**.11.7). The form of these relations is extremely suggestive: comparing them with the drift kinetic equation (3.1), we immediately determine a strategy for *calculating the required quantities without solving the equation*. Indeed, let us perform the following operations on eq. (3.1):
- multiply both sides by $m_\alpha B U_\alpha$, by $m_\alpha B U_\alpha(x - \frac{5}{2})$, or by $m_\alpha B U_\alpha(x^2 - 7x + \frac{35}{4})$, respectively;
- integrate over the velocity variables x, λ, ϕ;
- take the surface average of the result.

The left-hand side of the transformed equation is precisely one of the three required moments (**12**.11.7). These three operations therefore yield an expression of the generalized stresses (hence of the banana fluxes) in terms of $\bar{f}_1^{\alpha(0)}$, hence in terms of the reference distribution function f_0^α, i.e., finally, in terms of the thermodynamic forces.

This very elegant and simple strategy was already pointed out in the original paper by Galeev and Sagdeev (1968) and was later used, in one form or another, by all the authors (e.g. Hinton and Hazeltine 1976, Hirshman and Sigmar 1981). A very clear treatment is given by Shaing and Callen (1983) (who treat the more general, non-axisymmetric case).

In order to prepare the solution, we derive in the next section a few properties of the drift kinetic equation, which lead to a considerable simplification of the calculations.

14.10. Three properties of the drift kinetic collision operator

The properties of the collision operator to be presently derived are best exhibited by combining the drift kinetic equation (1.12) with the solubility constraint (3.16) for the passing particles. Multiplying the latter constraint by $f_P^{-1} V_\alpha (\mathscr{B}_0/B^2) (dV/d\rho)^{-1}$ [where f_P was defined in (6.20), $\mathscr{B}_0$ in (5.13), V_α in (5.15) and V in (**8**.8.36)] and using definition (**8**.8.35), we may write

$$\Theta(\lambda_c - \lambda) \frac{1}{f_P} \frac{\mathscr{B}_0}{B^2} V_\alpha(\lambda) \sqrt{x} \left\langle \frac{B}{U_\alpha} \nu_\alpha \mathscr{L} \bar{f}_1^{\alpha(0)} + B \mathscr{M}_\alpha f_0^\alpha \right\rangle = 0. \tag{10.1}$$

We note that the pitch-angle scattering operator has the following form in the variables x, λ (in these variables, $\mathscr{L}$ is independent of the particle index α):

$$\nu_\alpha \mathscr{L} = \nu_\alpha(x) \frac{2}{B} \sqrt{1 - \lambda B} \frac{\partial}{\partial \lambda} \lambda \sqrt{1 - \lambda B} \frac{\partial}{\partial \lambda}. \tag{10.2}$$

We subtract the left-hand side of (10.1) (i.e. subtract zero!) from the right-hand side of (1.12). The result is written in the form

$$U_\alpha \boldsymbol{b} \cdot \nabla \tilde{f}_1^{\alpha(1)} = \mathbb{L}_{\mathrm{I}}^\alpha \tilde{f}_1^{\alpha(0)} + \mathbb{L}_{\mathrm{II}}^\alpha f_0^\alpha, \tag{10.3}$$

with

$$\mathbb{L}_{\mathrm{I}}^\alpha f(x, \lambda) = \nu_\alpha \mathscr{L} f - \Theta(\lambda_c - \lambda) \frac{1}{f_P} \frac{\mathscr{B}_0}{B^2} V_\alpha(\lambda) \sqrt{x} \left\langle \frac{B}{U_\alpha} \nu_\alpha \mathscr{L} f \right\rangle, \tag{10.4}$$

$$\mathbb{L}_{\mathrm{II}}^\alpha = U_\alpha \mathscr{M}_\alpha - \Theta(\lambda_c - \lambda) \frac{1}{f_P} \frac{\mathscr{B}_0}{B^2} V_\alpha(\lambda) \sqrt{x} \langle B \mathscr{M}_\alpha \rangle. \tag{10.5}$$

We not state three properties of the drift kinetic equation (10.3). Except for the first (whose proof is trivial), these lemmas are due to Hirshman and Sigmar (1981).

Lemma 1. *The parallel velocity* U_α *is an eigenfunction of the pitch-angle scattering operator, with eigenvalue* -1:

$$\mathscr{L} U_\alpha = - U_\alpha, \tag{10.6}$$

or, using the variables x, λ and (5.4),

$$\mathscr{L} \frac{\sqrt{1 - \lambda B}}{B} = - \frac{\sqrt{1 - \lambda B}}{B}. \tag{10.7}$$

The proof of this property is straightforward, by using (10.2).

Lemma 2. *The operator* $\mathbb{L}_{\mathrm{II}}^\alpha$ *has the following property, when acting on an arbitrary function* $\mathrm{A}(x)$ *of* x *alone:*

$$\left\langle B \int \mathrm{d}v \, U_\alpha \, \mathbb{L}_{\mathrm{II}}^\alpha f_0^\alpha \, A(x) \right\rangle = 0. \tag{10.8}$$

Proof. Using (10.5), we must show that

$$\left\langle B \int \mathrm{d}v \, U_\alpha^2 \, \mathscr{M}_\alpha \, f_0^\alpha \, A(x) \right\rangle$$

$$- \left\langle B \int \mathrm{d}v \, U_\alpha \, \Theta(\lambda_c - \lambda) \frac{1}{f_P} \frac{\mathscr{B}_0}{B^2} V_\alpha(\lambda) \sqrt{x} \left\langle B \mathscr{M}_\alpha \right\rangle f_0^\alpha \, A(x) \right\rangle$$

$$\equiv T_1 - T_2 = 0.$$

Using the transformation formula (6.1) and performing first the integration over x, then over λ, we evaluate T_1 as

$$T_1 = \left\langle \frac{B^2}{4} n_\alpha \mathscr{M}_\alpha \sum_\sigma \frac{2}{\sqrt{\pi}} \int_0^\infty \mathrm{d}x \sqrt{x}\, \mathrm{e}^{-x} A(x) \right.$$
$$\left. \times \int_0^{1/B} \mathrm{d}\lambda \frac{1}{\sqrt{1-\lambda B}} \frac{2T_\alpha}{m_\alpha} x(1-\lambda B) \right\rangle$$
$$= J \left\langle B^2 \mathscr{M}_\alpha \int_0^{1/B} \mathrm{d}\lambda \sqrt{1-\lambda B} \right\rangle = \tfrac{2}{3} J \langle B \mathscr{M}_\alpha \rangle,$$

where we have defined

$$J \equiv \frac{n_\alpha T_\alpha}{m_\alpha} \frac{2}{\sqrt{\pi}} \int_0^\infty \mathrm{d}x\, x^{3/2}\, \mathrm{e}^{-x} A(x).$$

The calculation of T_2 is somewhat more complicated; it involves using the explicit forms (6.20) of f_P and (5.15) of V_α, as well as (6.19):

$$T_2 = \left\langle \left(\frac{B^2 n_\alpha}{4} \sum_\sigma \frac{2}{\sqrt{\pi}} \int_0^\infty \mathrm{d}x \sqrt{x}\, \mathrm{e}^{-x} A(x) \right.\right.$$
$$\times \int_0^{1/B} \mathrm{d}\lambda \frac{1}{\sqrt{1-\lambda B}} \Theta(\lambda_\mathrm{c} - \lambda) \langle B \mathscr{M}_\alpha \rangle$$
$$\left.\left. \times \frac{2T_\alpha}{m_\alpha} x \sqrt{1-\lambda B} \frac{\mathscr{B}_0}{2} \int_\lambda^{\lambda_\mathrm{c}} \mathrm{d}\lambda' \frac{1}{\langle \sqrt{1-\lambda' B} \rangle} \frac{\mathscr{B}_0}{B^2} \frac{1}{f_\mathrm{P}} \right) \right\rangle$$
$$= \tfrac{1}{2} J \langle B \mathscr{M}_\alpha \rangle \frac{\displaystyle\int_0^{\lambda_\mathrm{c}} \mathrm{d}\lambda \int_\lambda^{\lambda_\mathrm{c}} \mathrm{d}\lambda' \frac{1}{\langle \sqrt{1-\lambda' B} \rangle}}{\displaystyle \tfrac{3}{4} \int_\lambda^{\lambda_\mathrm{c}} \mathrm{d}\lambda'' \frac{\lambda''}{\langle \sqrt{1-\lambda'' B} \rangle}} = \tfrac{2}{3} J \langle B \mathscr{M}_\alpha \rangle = T_1.$$

Lemma 3. *The operator* $\mathbb{L}_1^\alpha$ *has the following property, for an arbitrary function* $A(x)$ *of* x *alone*:

$$\int \mathrm{d}v\, U_\alpha \mathbb{L}_1^\alpha \frac{U_\alpha}{B} A(x) = 0. \tag{10.9}$$

Proof. Using (10.4) and (6.1), we must show that

$$\frac{BT_\alpha}{m_\alpha}\sum_\sigma \int_0^\infty \mathrm{d}x\, x^{3/2}\, \mathrm{e}^{-x}\, \nu_\alpha(x)\, A(x)$$

$$\times \left\{ \int_0^{1/B} \mathrm{d}\lambda \frac{1}{\sqrt{1-\lambda B}} \sqrt{1-\lambda B}\, \mathscr{L} \frac{\sqrt{1-\lambda B}}{B} \right.$$

$$- \int_0^{1/B} \mathrm{d}\lambda \frac{1}{\sqrt{1-\lambda B}} \Theta(\lambda_c - \lambda) \frac{1}{f_P} \frac{\mathscr{B}_0^2}{2B^2}$$

$$\left. \times \int_\lambda^{\lambda_c} \mathrm{d}\lambda' \frac{1}{\sqrt{1-\lambda' B}} \sqrt{1-\lambda B} \left\langle \frac{B}{\sqrt{1-\lambda B}} \mathscr{L} \frac{\sqrt{1-\lambda B}}{B} \right\rangle \right\}$$

$$\equiv \frac{BT_\alpha}{m_\alpha} \int_0^\infty \mathrm{d}x\, x^{3/2}\, \mathrm{e}^{-x}\, A(x)(T_1' - T_2') = 0.$$

Using Lemma 1, the first term is easily evaluated:

$$T_1' = \int_0^{1/B} \mathrm{d}\lambda\, \mathscr{L} \frac{\sqrt{1-\lambda B}}{B} = -\frac{1}{B} \int_0^{1/B} \mathrm{d}\lambda \sqrt{1-\lambda B} = -\frac{2}{3B^2}.$$

The same lemma implies that

$$\left\langle \frac{B}{\sqrt{1-\lambda B}} \mathscr{L} \frac{\sqrt{1-\lambda B}}{B} \right\rangle = -1.$$

Hence, using (6.19) and (6.20),

$$T_2' = -\frac{1}{2f_P} \frac{\mathscr{B}_0^2}{B^2} \int_0^{\lambda_c} \mathrm{d}\lambda \int_\lambda^{\lambda_c} \mathrm{d}\lambda' \frac{1}{\sqrt{1-\lambda' B}} = -\frac{2}{3B^2} = T_1'.$$

These properties will be used in the next section for the calculation of the generalized stresses.

14.11. Relation between generalized stresses and poloidal fluxes

We now come to a crucial point of our treatment. In order to calculate the surface-averaged divergence of the generalized stress tensors (**12**.11.7), we have

to perform the three operations listed in section 14.9. Let us consider in detail the evaluation of the term $\langle \boldsymbol{B}\cdot(\nabla\cdot\boldsymbol{h}^{\alpha(2)})\rangle$: all the arguments developed here are valid for the tensors $\boldsymbol{h}^{\alpha(4)}$, $\boldsymbol{h}^{\alpha(6)}$ as well, and the extension of the calculations is straightforward.

From the drift kinetic equation in the form (10.3), we obtain

$$\sqrt{2}\,\langle \boldsymbol{B}\cdot(\nabla\cdot\boldsymbol{h}^{\alpha(2)})\rangle = \frac{m_\alpha}{n_\alpha T_\alpha}\left\langle B\int \mathrm{d}\boldsymbol{v}\; U_\alpha U_\alpha\, \boldsymbol{b}\cdot\nabla f_1^{\alpha(1)}\right\rangle$$

$$= \frac{m_\alpha}{n_\alpha T_\alpha}\left\langle B\int \mathrm{d}\boldsymbol{v}\; U_\alpha\, \mathbb{L}_{\mathrm{I}}^{\alpha}\, \tilde{f}_1^{\alpha(0)}\right\rangle$$

$$+ \frac{m_\alpha}{n_\alpha T_\alpha}\left\langle B\int \mathrm{d}\boldsymbol{v}\; U_\alpha\, \mathbb{L}_{\mathrm{II}}^{\alpha}\, f_0^{\alpha}\right\rangle. \tag{11.1}$$

At this stage we see the considerable simplifications that occur in the calculation. Indeed, the second term on the right-hand side vanishes identically, as a result of Lemma 2. Consider now the first term, recalling that $\tilde{f}_1^{\alpha(0)}$ consists of two quite distinct contributions, (2.13). The first of these is of the form

$$\bar{F}_1^{\alpha(0)} = \frac{U_\alpha}{B} f(x;\ \theta,\ \rho).$$

Hence, using Lemma 3, we have

$$\frac{m_\alpha}{n_\alpha T_\alpha}\left\langle B\int \mathrm{d}\boldsymbol{v}\; U_\alpha\, \mathbb{L}_{\mathrm{I}}^{\alpha}\, \bar{F}_1^{\alpha(0)}\right\rangle = 0, \tag{11.2}$$

and we are left with the drastically simplified result

$$\sqrt{2}\,\left\langle \boldsymbol{B}\cdot(\nabla\cdot\boldsymbol{h}^{\alpha(2)})\right\rangle = \frac{m_\alpha}{n_\alpha T_\alpha}\left\langle B\int \mathrm{d}\boldsymbol{v}\; U_\alpha\, \mathbb{L}_{\mathrm{I}}^{\alpha}\, \bar{G}_1^{\alpha(0)}\right\rangle. \tag{11.3}$$

The most important fact here is that *the contribution of* $\mathbb{L}_{\mathrm{II}}^{\alpha}$ *vanishes exactly*. Hence, *the result we obtain is independent of the detailed modelling of the collision term*. In a sense, our long discussion in chapter 11 was a "luxury", because for the specific purpose sought here, the detailed results obtained there are irrelevant. Nevertheless, the discussion *was* important, because it showed that the correctly modelled equation has precisely the form (1.12), which makes Lemma 2 applicable.

We also stress the fact that the simplification only occurs in the calculation of the particular class of moments $\boldsymbol{h}^{\alpha(2p)}$ considered here, and will not hold for other types of moments

We note that the *particular dependence on the magnetic field* is also essential for the proof of Lemma 2. This shows that the decomposition of the neoclassical fluxes in eq. (**12**.9.13), which could appear rather arbitrary when it was first introduced, is the only one leading to the present simplifications.

Finally, we note that the *surface-averaging operation* is also essential in Lemma 2: *one should not expect any simple results (i.e. linear transport equations) involving non-averaged quantities.*

We now go over to the evaluation of eq. (11.3), which is a little bit delicate, but not difficult. Using (5.16), we get

$$\sqrt{2}\,\langle \boldsymbol{B}\cdot(\nabla\cdot\boldsymbol{h}^{\alpha(2)})\rangle = \frac{m_\alpha}{n_\alpha T_\alpha}\left\langle B\int \mathrm{d}\boldsymbol{v}\; U_\alpha\, \mathbb{L}_1^\alpha\, 2\Theta(\lambda_c-\lambda)\; V_\alpha\sqrt{x}\,\mathscr{G}^\alpha(x)\; f_0^\alpha\right\rangle$$

$$= f_P\frac{2}{\sqrt{\pi}}\int_0^\infty \mathrm{d}x\; x^{3/2}\,\mathrm{e}^{-x}\;\nu_\alpha(x)\mathscr{G}^\alpha(x)$$

$$\times\left\langle \int_0^{1/B}\mathrm{d}\lambda\left\{\mathscr{L}\Theta(\lambda_c-\lambda)\;\tilde{V}(\lambda)\frac{\mathscr{B}_0}{f_P}\right.\right.$$

$$\left.\left. -\Theta(\lambda_c-\lambda)\frac{1}{2f_P}\frac{\mathscr{B}_0^2}{B^2}\tilde{V}(\lambda)\Big\langle\frac{B}{\sqrt{1-\lambda B}}\mathscr{L}\tilde{V}(\lambda)\frac{\mathscr{B}_0}{f_P}\Big\rangle\right\}\right\rangle,$$

where we introduce the abbreviation

$$\tilde{V}(\lambda) = \int_\lambda^{\lambda_c}\mathrm{d}\lambda'\frac{1}{\langle\sqrt{1-\lambda' B}\,\rangle}.$$

The integral over x factors out, hence this result may be written in the form

$$\sqrt{2}\,\langle \boldsymbol{B}\cdot(\nabla\cdot\boldsymbol{h}^{\alpha(2)})\rangle = J(K_1-K_2), \tag{11.4}$$

with

$$J = f_P\frac{2}{\sqrt{\pi}}\int_0^\infty \mathrm{d}x\; x^{3/2}\,\mathrm{e}^{-x}\;\nu_\alpha(x)\;\mathscr{G}^\alpha(x). \tag{11.5}$$

We now use (10.2), perform two partial integrations and recall (6.20),

$$K_1 = \left\langle B^2 \int_0^{1/B} \mathrm{d}\lambda \; \mathscr{L}\Theta(\lambda_c - \lambda)\; \tilde{V}(\lambda) \right\rangle \frac{\mathscr{B}_0}{f_P}$$

$$= \left\langle 2B \int_0^{1/B} \mathrm{d}\lambda \sqrt{1-\lambda B}\; \frac{\partial}{\partial \lambda} \lambda \sqrt{1-\lambda B}\; \frac{\partial}{\partial \lambda} \Theta(\lambda_c - \lambda)\; \tilde{V}(\lambda) \right\rangle \frac{\mathscr{B}_0}{f_P}$$

$$= -\left\langle B^2 \int_0^{1/B} \mathrm{d}\lambda\; \Theta(\lambda_c - \lambda) \int_\lambda^{\lambda_c} \mathrm{d}\lambda' \frac{1}{\langle \sqrt{1-\lambda' B} \rangle} \right\rangle \frac{\mathscr{B}_0}{f_P}$$

$$= -\langle B^2 \rangle \frac{4}{3\mathscr{B}_0^2} f_P \frac{\mathscr{B}_0}{f_P} = -\tfrac{4}{3}\mathscr{B}_0. \tag{11.6}$$

The second term in (11.4) is

$$K_2 = \frac{\mathscr{B}_0^3}{f_P^2} \left\langle \int_0^{1/B} \mathrm{d}\lambda\; \Theta(\lambda_c - \lambda)\; \tilde{V}(\lambda) \left\langle \frac{\partial}{\partial \lambda} \lambda \sqrt{1-\lambda B}\; \frac{\partial}{\partial \lambda} \tilde{V}(\lambda) \right\rangle \right\rangle.$$

We first evaluate the inside average, noting that $\tilde{V}(\lambda)$ is independent of θ,

$$\left\langle \frac{\partial}{\partial \lambda} \lambda \sqrt{1-\lambda B}\; \frac{\partial}{\partial \lambda} \tilde{V}(\lambda) \right\rangle = \frac{\partial}{\partial \lambda} \lambda \langle \sqrt{1-\lambda B} \rangle \frac{\partial}{\partial \lambda} \int_\lambda^{\lambda_c} \mathrm{d}\lambda' \frac{1}{\langle \sqrt{1-\lambda' B} \rangle}$$

$$= -\frac{\partial}{\partial \lambda} \lambda \langle \sqrt{1-\lambda B} \rangle \frac{1}{\langle \sqrt{1-\lambda B} \rangle} = -1.$$

Hence

$$K_2 = -\frac{\mathscr{B}_0^3}{f_P^2} \int_0^{\lambda_c} \mathrm{d}\lambda\; \tilde{V}(\lambda) = -\frac{\mathscr{B}_0^3}{f_P^2} \frac{4}{3\mathscr{B}_0^2} f_P = -\frac{4}{3} \frac{\mathscr{B}_0}{f_P}. \tag{11.7}$$

Combining now (11.4), (11.6) and (11.7), we obtain

$$\sqrt{2}\, \langle \boldsymbol{B} \cdot (\nabla \cdot \boldsymbol{h}^{\alpha(2)}) \rangle = \tfrac{4}{3} \mathscr{B}_0 J\varphi, \tag{11.8}$$

where we introduced the *neoclassical factor* φ,

$$\varphi = \frac{1 - f_P}{f_P}. \tag{11.9}$$

The remarkable feature of formula (11.8) is that *the whole effect of the inhomogeneity, curvature, torsion of the magnetic field, briefly, the whole effect of the geometry is concentrated in the numerical factors* φ *and* $\mathscr{B}_0 \equiv \langle B^2 \rangle^{1/2}$. We note that no assumption was made about the specific geometry of the field (except, of course, the applicability of the drift approximation); from that point of view, (11.8) is quite general for all axisymmetric configurations. The properties of the neoclassical factor were discussed in section 14.8.

Finally, we make the integral J explicit, using (7.6),

$$\begin{aligned}\sqrt{2}\langle \boldsymbol{B}\cdot(\nabla\cdot\boldsymbol{h}^{\alpha(2)})\rangle = {} & \tfrac{4}{3}\varphi\frac{\mathscr{B}_0}{\tau_\alpha}\left(\frac{m_\alpha}{T_\alpha}\right)^{1/2} \\ & \times(1/\sqrt{\pi})\int_0^\infty \mathrm{d}x\, \mathrm{e}^{-x}\, x^{3/2}\,\tau_\alpha\nu_\alpha(x) \\ & \times\sqrt{\tfrac{3}{2}}\left(L_0^{3/2}(x)\,\omega_1^\alpha + L_1^{3/2}(x)\omega_3^\alpha + L_2^{3/2}(x)\,\omega_5^\alpha\right),\end{aligned} \tag{11.10}$$

where we introduced, for convenience, the relaxation time τ_α as a scaling factor (see eqs. **4**.6.4, **4**.6.10).

The corresponding formula for the *fourth-order stress tensor* is derived in a completely similar way, with the result

$$\begin{aligned}\langle \boldsymbol{B}\cdot(\nabla\cdot\overline{\boldsymbol{h}}^{\alpha(4)})\rangle = {} & \tfrac{4}{3}\varphi\frac{\mathscr{B}_0}{\tau_\alpha}\left(\frac{m_\alpha}{T_\alpha}\right)^{1/2} \\ & \times(1/\sqrt{\pi})\int_0^\infty \mathrm{d}x\, \mathrm{e}^{-x}\, x^{3/2}\,\tau_\alpha\nu_\alpha(x)\sqrt{\tfrac{15}{4}}\,L_1^{3/2}(x). \\ & \times\sqrt{\tfrac{3}{2}}\left(L_0^{3/2}(x)\,\omega_1^\alpha + L_1^{3/2}(x)\,\omega_3^\alpha + L_2^{3/2}(x)\,\omega_5^\alpha\right).\end{aligned} \tag{11.11}$$

Finally, we will also need the expression of the *sixth-order generalized stress tensor*,

$$\langle \boldsymbol{B}\cdot(\nabla\cdot\bar{\bar{\boldsymbol{h}}}^{\alpha(6)})\rangle = \tfrac{4}{3}\varphi\frac{\mathscr{B}_0}{\tau_\alpha}\left(\frac{m_\alpha}{T_\alpha}\right)^{1/2} \times(1/\sqrt{\pi})\int_0^\infty \mathrm{d}x\, \mathrm{e}^{-x}\, x^{3/2}\,\tau_\alpha\nu_\alpha(x)\sqrt{\tfrac{15}{4}}\,L_2^{3/2}(x) \times\sqrt{\tfrac{3}{2}}\left(L_0^{3/2}(x)\;\omega_1^\alpha + L_1^{3/2}(x)\;\omega_3^\alpha + L_2^{3/2}(x)\;\omega_5^\alpha\right). \tag{11.12}$$

The three relations (11.10–11.12) are remarkably similar. We stress again the fundamental fact that the effect of the geometry is wholly concentrated in the two factors φ and $\mathscr{B}_0$.

The other important feature of these formulae is that *the surface-averaged divergences of the generalized stresses appear as linear combinations of the poloidal fluxes* ω_1^α, ω_3^α, ω_5^α. These relations will be discussed in detail in the next section.

14.12. The pseudo-viscosity coefficients

The most important result of the last section is the proof of a *linear relationship between the generalized stress tensors and the "poloidal fluxes"*, which can be expressed in the form

$$\sqrt{2}\,\langle \boldsymbol{B}\cdot(\nabla\cdot\boldsymbol{h}^{\alpha(2)})\rangle = \frac{1}{\tau_\alpha}\left(\frac{m_\alpha}{T_\alpha}\right)^{1/2}\mathscr{B}_0\varphi\left(\mu_{11}^\alpha\,\omega_1^\alpha + \mu_{13}^\alpha\,\omega_3^\alpha + \mu_{15}^\alpha\,\omega_5^\alpha\right),$$

$$\langle \boldsymbol{B}\cdot(\nabla\cdot\bar{\bar{\boldsymbol{h}}}^{\alpha(4)})\rangle = \frac{1}{\tau_\alpha}\left(\frac{m_\alpha}{T_\alpha}\right)^{1/2}\mathscr{B}_0\varphi\sqrt{\tfrac{5}{2}}\left(\mu_{31}^\alpha\,\omega_1^\alpha + \mu_{33}^\alpha\,\omega_3^\alpha + \mu_{35}^\alpha\,\omega_5^\alpha\right),$$

$$\langle \boldsymbol{B}\cdot(\nabla\cdot\bar{\bar{\boldsymbol{h}}}^{\alpha(6)})\rangle = \frac{1}{\tau_\alpha}\left(\frac{m_\alpha}{T_\alpha}\right)^{1/2}\mathscr{B}_0\varphi\sqrt{\tfrac{5}{2}}\left(\mu_{51}^\alpha\,\omega_1^\alpha + \mu_{53}^\alpha\,\omega_3^\alpha + \mu_{55}^\alpha\,\omega_5^\alpha\right). \tag{12.1}$$

These relations are strongly reminiscent of a set of transport equations such as those of chapters 5 and 13. Indeed, they introduce a 3×3 matrix of coefficients μ_{ij}^α (for each species α); the diagonal coefficients are manifestly

Table 12 1
The pseudo-viscosity coefficients

Electron pseudo-viscosity coefficients

$$\mu_{11}^{e} = 1 + \frac{1}{Z}\{\sqrt{2} - \ln(1+\sqrt{2})\} = 1 + 0.5328\,Z^{-1}$$

$$\mu_{13}^{e} = -\frac{1}{\sqrt{10}}\left\{3 + \frac{1}{Z}\left[4\sqrt{2} - 5\ln(1+\sqrt{2})\right]\right\} = -0.9487 - 0.3953\,Z^{-1}$$

$$\mu_{15}^{e} = 4\frac{1}{\sqrt{70}}\left\{30 + \frac{1}{Z}\left[51\sqrt{2} - 70\ln(1+\sqrt{2})\right]\right\} = 0.8964 + 0.3116\,Z^{-1}$$

$$\mu_{33}^{e} = \frac{1}{20}\left\{26 + \frac{1}{Z}\left[39\sqrt{2} - 50\ln(1+\sqrt{2})\right]\right\} = 1.3000 + 0.5543\,Z^{-1}$$

$$\mu_{35}^{e} = -\frac{1}{80\sqrt{7}}\left\{276 + \frac{1}{Z}\left[507\sqrt{2} - 700\ln(1+\sqrt{2})\right]\right\} = -1.3040 - 0.4727\,Z^{-1}$$

$$\mu_{55}^{e} = \frac{1}{2240}\left\{3464 + \frac{1}{Z}\left[7007\sqrt{2} - 9800\ln(1+\sqrt{2})\right]\right\} = 1.5464 + 0.5678\,Z^{-1}$$

Ion pseudo-viscosity coefficients

$$\mu_{11}^{i} = \sqrt{2} - \ln(1+\sqrt{2}) = 0\ 5328$$

$$\mu_{13}^{i} = -\frac{1}{\sqrt{10}}\left[4\sqrt{2} - 5\ln(1+\sqrt{2})\right] = -0\ 3953$$

$$\mu_{15}^{i} = \frac{1}{4\sqrt{70}}\left[51\sqrt{2} - 70\ln(1+\sqrt{2})\right] = 0.3116$$

$$\mu_{33}^{i} = \tfrac{1}{20}\left[39\sqrt{2} - 50\ln(1+\sqrt{2})\right] = 0.5543$$

$$\mu_{35}^{i} = -\frac{1}{80\sqrt{7}}\left[507\sqrt{2} - 700\ln(1+\sqrt{2})\right] = -0.4727$$

$$\mu_{55}^{i} = \tfrac{1}{2240}\left[7007\sqrt{2} - 9800\ln(1+\sqrt{2})\right] = 0.5678$$

non-negative, and the non-diagonal ones are symmetric. Indeed, from (11.10)–(11.12), the following formula is easily obtained:

$$\mu_{2m+1,2n+1}^{\alpha} = \frac{2}{\sqrt{\pi}}\int_0^{\infty} \mathrm{d}x\, \mathrm{e}^{-x}\, x^{3/2} L_m^{3/2}(x)\ \tau_\alpha \nu_\alpha(x)\ L_n^{3/2}(x). \qquad (12.2)$$

Equations (12.1) are *not*, however, transport equations in the thermodynamic sense, such as those of chapters 5 and 13. Indeed, the left-hand sides are not thermodynamic fluxes, and the right-hand sides do not involve thermodynamic forces. Nevertheless, the symmetry properties of these coefficients will be at the origin of the Onsager symmetry properties of the final, true transport coefficients.

Hirshman and Sigmar (1981) * call the coefficients μ_{ij}^{α} the *neoclassical viscosity coefficients*. This name is improper. Indeed, a viscosity coefficient relates the pressure tensor to the gradient of the centre-of-mass velocity. In the present case, μ_{11}^{α} (for instance) relates the divergence of the pressure tensor to (a part of) the particle flux. Hence, these coefficients do not even have the dimensions of the viscosity coefficients. The name "viscosity" is only suggestive of a relation between stress tensors and fluxes. For this reason, we shall use the name *pseudo-viscosity coefficients* for the quantities μ_{ij}^{α}.

There is no difficulty in evaluating the integrals of eqs. (12.2). It is sufficient to substitute the expressions of the collision frequencies, expressed as functions of x, from (**11**.4.6), (**11**.4.7), (**11**.4.12), (**11**.4 13) and (**11**.5.5):

$$\nu_e(x) = \frac{3\sqrt{\pi}}{4\tau_e} x^{-3/2}\left[1 + Z^{-1}\mathscr{H}(\sqrt{x})\right],$$

$$\nu_i(x) = \frac{3\sqrt{\pi}}{4\tau_i} x^{-3/2}\mathscr{H}(\sqrt{x}). \tag{12.3}$$

With these functional forms and with the expressions of the Laguerre–Sonine polynomials of table **G1**.3.2 of General Appendix G1, all the integrals can be calculated analytically. A set of useful integral formulae given in appendix 14A.2 will help the reader in checking the formulae of table 12.1.

The results obtained in this chapter provide us with all the elements needed for the determination of the banana fluxes. The next chapter is devoted to the explicit evaluation and the detailed discussion of these fundamental quantities.

Appendix 14A.1. Proof of the Alfvén formula (2.9)

We first derive a useful lemma. Consider a tokamak configuration governed by the usual MHD equation (**8**.2.1). Combing this equation with the Ampère law (**8**.2.2), we obtain

$$\boldsymbol{B} \wedge (\nabla \wedge \boldsymbol{B}) = -4\pi \nabla P, \tag{A1.1}$$

* Their definition differs slightly from the one adopted here, which is closer to the definition given by Shaing and Callen (1983)

where P is the total plasma pressure. Using some standard vector identities, we find

$$\boldsymbol{B} \wedge (\nabla \wedge \boldsymbol{B}) = B^2 \boldsymbol{b} \wedge (\nabla \wedge \boldsymbol{b}) + B[\nabla B - \boldsymbol{b}(\boldsymbol{b} \cdot \nabla B)]. \tag{A1.2}$$

Hence, the equilibrium condition can be rewritten as

$$\boldsymbol{b} \wedge (\nabla \wedge \boldsymbol{b}) + \frac{1}{B} \nabla B - \frac{1}{B} \boldsymbol{b}(\boldsymbol{b} \cdot \nabla B) = -\frac{4\pi}{B^2} \nabla P. \tag{A1.3}$$

Consider now the well-known parameter β, defined as the ratio of the plasma pressure to the magnetic pressure,

$$\beta = \frac{P}{B^2/8\pi}. \tag{A1.4}$$

It is known that in a *tokamak configuration*, β is a small number, of order ϵ (see, e.g., Freidberg 1982),

$$\beta = \mathrm{O}(\epsilon) \ll 1. \tag{A1.5}$$

Thus, the right-hand side of (A1.3) is of order ϵ, and

$$\boldsymbol{b} \wedge (\nabla \wedge \boldsymbol{b}) \approx -\frac{1}{B} \nabla B + \frac{1}{B} \boldsymbol{b}(\boldsymbol{b} \cdot \nabla B) + \mathrm{O}(\epsilon). \tag{A1.6}$$

Noting also the identity

$$0 = \nabla(\boldsymbol{b} \cdot \boldsymbol{b}) = 2\boldsymbol{b} \wedge (\nabla \wedge \boldsymbol{b}) + 2(\boldsymbol{b} \cdot \nabla)\boldsymbol{b},$$

we get from (A1.6) (vector-multiplied by $\boldsymbol{b}$)

$$\boldsymbol{b} \wedge (\boldsymbol{b} \cdot \nabla)\boldsymbol{b} = \frac{1}{B} \boldsymbol{b} \wedge \nabla B + \mathrm{O}(\epsilon). \tag{A1.7}$$

This is the lemma announced above.

We now calculate the gradient of U_α/Ω_α, where we recall that U_α is a function of $\mathscr{E}$, M, $\boldsymbol{Y}$, defined by (**11**.5.3),

$$\nabla \frac{U_\alpha}{\Omega_\alpha} = \frac{1}{\Omega_\alpha} \left(-\frac{U_\alpha}{B} \nabla B + \nabla \sqrt{(2/m_\alpha)[\mathscr{E} - MB(\boldsymbol{Y}) - e_\alpha \Phi(\boldsymbol{Y})]} \right)$$

$$= \frac{1}{\Omega_\alpha} \left(-\frac{U_\alpha}{B} \nabla B - \frac{1}{m_\alpha U_\alpha} \nabla(MB + e_\alpha \Phi) \right).$$

As a result we get

$$\epsilon U_\alpha \boldsymbol{b} \wedge \nabla \frac{U_\alpha}{\Omega_\alpha} = -\frac{\epsilon}{\Omega_\alpha} \boldsymbol{b} \wedge \left[U_\alpha^2 \frac{1}{B} \nabla B + m_\alpha^{-1} \nabla (MB + e_\alpha \Phi) \right]$$

$$= -\frac{\epsilon}{\Omega_\alpha} \boldsymbol{b} \wedge \left[U_\alpha^2 (\boldsymbol{b} \cdot \nabla) \boldsymbol{b} + m_\alpha^{-1} \nabla (MB + e_\alpha \Phi) \right] + \mathrm{O}(\epsilon^2), \tag{A1.8}$$

where lemma (A1.7) was used. Comparing this result with the drift velocity, given in table **1**.8.3, we immediately find

$$\epsilon U_\alpha \boldsymbol{b} \wedge \nabla \frac{U_\alpha}{\Omega_\alpha} = -\epsilon \boldsymbol{V}_{\mathrm{D}}^\alpha + \mathrm{O}(\epsilon^2). \tag{A1.9}$$

In order to obtain the final form (2.9), we derive another lemma. Let $\Psi = \Psi(\rho, \theta)$ be an arbitrary function of ρ and θ, independent of ζ. Then

$$J \equiv (\nabla \rho) \cdot (\boldsymbol{b} \wedge \nabla \Psi) = \frac{1}{l_\rho} (\boldsymbol{e}_\rho \wedge \boldsymbol{b}) \cdot \nabla \Psi.$$

Using relations (**12**.5.6) between the physical and geometrical basis vectors, we get (using also eq. **12**.5.3)

$$J = \frac{1}{l_\rho l_\zeta B} \left(\frac{R_0 \mathscr{B}_{\mathrm{P}}}{l_\rho} (\boldsymbol{e}_\rho \wedge \boldsymbol{e}_\theta) + \mathscr{I} (\boldsymbol{e}_\rho \wedge \boldsymbol{e}_\zeta) \right) \cdot \nabla \Psi$$

$$= -\mathscr{I} \frac{1}{l_\rho l_\zeta B} \boldsymbol{e}_\theta \cdot \nabla \Psi = -\frac{\mathscr{I}}{R_0 \mathscr{B}_{\mathrm{P}}} \frac{B_\theta}{B} \boldsymbol{e}_\theta \cdot \nabla \Psi,$$

and finally

$$(\nabla \rho) \cdot (\boldsymbol{b} \wedge \nabla \Psi) = -\frac{\mathscr{I}}{R_0 \mathscr{B}_{\mathrm{P}}} \boldsymbol{b} \cdot \nabla \Psi. \tag{A1.10}$$

Applying this lemma to (A1.9), we find

$$(\nabla \rho) \cdot \boldsymbol{V}_{\mathrm{D}}^\alpha = \frac{\mathscr{I}}{R_0 \mathscr{B}_{\mathrm{P}}} U_\alpha \boldsymbol{b} \cdot \nabla \frac{U_\alpha}{\Omega_\alpha}.$$

Let us note again that this formula is not an exact identity, but is based on assumption (A1.5). The latter is quite well satisfied in a tokamak configuration, as well as in other magnetic confinement situations.

Appendix 14A.2. Integrals involving the Chandrasekhar function $\mathscr{H}(x)$

The following integrals are used in the calculation of the pseudo-viscosity coefficients:

$$N_p = \int_0^\infty \mathrm{d}y\, \mathrm{e}^{-y^2}\, y^p \mathscr{H}(y), \quad p = 1, 2, \ldots. \tag{A2.1}$$

The Chandrasekhar function $\mathscr{H}(y)$ was defined in eq. (**11**.4.8) in terms of the error function $\Phi(y)$.

It will be shown that, for *even values of* p,

$$N_2 = \frac{1}{2\sqrt{\pi}}, \qquad N_4 = \frac{\sqrt{\pi}}{2}\left(\frac{1}{4} + \frac{1}{\pi}\right),$$

$$N_6 = \frac{\sqrt{\pi}}{2}\left(\frac{3}{4} + \frac{5}{2\pi}\right), \qquad N_8 = \frac{\sqrt{\pi}}{2}\left(\frac{45}{16} + \frac{9}{\pi}\right), \tag{A2.2}$$

and, for *odd values of* p,

$$N_1 = \frac{1}{\sqrt{2}}\left(1 - \frac{1}{\sqrt{2}}\ln(1+\sqrt{2})\right),$$

$$N_3 = \frac{1}{2\sqrt{2}}, \qquad N_5 = \frac{9}{8\sqrt{2}},$$

$$N_7 = \frac{115}{32\sqrt{2}}, \qquad N_9 = \frac{1911}{128\sqrt{2}}. \tag{A2.3}$$

The method for calculating these integrals was given by Rosenbluth et al. (1972). In order to derive these results, we first consider the integrals

$$K_{2k} = \int_0^\infty \mathrm{d}x\, x^{2k}\, \Phi'^2(x),$$

$$K_{2k+1} = \int_0^\infty \mathrm{d}x\, x^{2k+1}\, \Phi'^2(x). \tag{A2.4}$$

These integrals are elementary:

$$K_{2k} = \frac{4}{\pi}\int_0^\infty \mathrm{d}x\, x^{2k}\, \mathrm{e}^{-2x^2} = \frac{(2k-1)!!}{2^{2k}}\sqrt{\frac{2}{\pi}}\,, \tag{A2.5}$$

$$K_{2k+1} = \frac{4}{\pi}\int_0^\infty \mathrm{d}x\, x^{2k+1}\, \mathrm{e}^{-2x^2} = \frac{k!}{2^k\pi}. \tag{A2.6}$$

For the evaluation of the integrals N_p we also need the integrals L_p defined as

$$L_p = \int_0^\infty \mathrm{d}x\, x^p\, \Phi(x)\, \Phi'(x), \quad p = -1, 0, 1, 2, \cdots. \tag{A2.7}$$

Consider first the case when p is even. We use the obvious property

$$\Phi''(x) = -2x\,\Phi'(x),$$

in order to rewrite the expression of L_{2k} and then integrate by parts:

$$L_{2k} = -\tfrac{1}{2}\int_0^\infty \mathrm{d}x\, x^{2k-1}\Phi\Phi'' = \tfrac{1}{2}\int_0^\infty \mathrm{d}x\, \Phi'\frac{\mathrm{d}}{\mathrm{d}x}(x^{2k-1}\Phi)$$

$$= \tfrac{1}{2}(2k-1)\int_0^\infty \mathrm{d}x\, x^{2k-2}\Phi\Phi' + \tfrac{1}{2}\int_0^\infty \mathrm{d}x\, x^{2k-1}\Phi'^2.$$

We thus obtained the recursive relation

$$L_{2k} = \left(k - \tfrac{1}{2}\right)L_{2k-2} + \tfrac{1}{2}K_{2k-1}. \tag{A2.8}$$

The first element in the recursion is easily evaluated:

$$L_0 = \int_0^\infty \mathrm{d}x\, \Phi\Phi' = \tfrac{1}{2}\int_0^\infty \mathrm{d}x \frac{\mathrm{d}}{\mathrm{d}x}\Phi^2 = \tfrac{1}{2}.$$

Thus, from (A2.8) and (A2.6), we get

$$L_0 = \tfrac{1}{2}, \qquad L_2 = \frac{1}{4} + \frac{1}{2\pi},$$

$$L_4 = \frac{3}{8} + \frac{1}{\pi}, \quad L_6 = \frac{15}{16} + \frac{11}{4\pi}. \tag{A2.9}$$

In a similar way, we derive a recursion formula for *odd values of p*:

$$L_{2k+1} = kL_{2k-1} + \tfrac{1}{2}K_{2k}. \tag{A2.10}$$

This formula holds for $k \geqslant 1$. We easily obtain

$$L_1 = \int_0^\infty \mathrm{d}x\, x\Phi\Phi' = -\tfrac{1}{2}\int_0^\infty \mathrm{d}x\, \Phi\Phi'' = \tfrac{1}{2}\int_0^\infty \mathrm{d}x\, \Phi'^2 = \tfrac{1}{2}K_0 = \frac{1}{2}\sqrt{\frac{2}{\pi}}\,.$$

From here on, we calculate all higher order integrals L_{2k+1}. But the lowest order integral N_1 in (A2.1) also involves the integral L_{-1}, which must be calculated separately,

$$\begin{aligned} L_{-1} &= \int_0^\infty \mathrm{d}x\, x^{-1}\Phi\Phi' = \int_0^\infty \mathrm{d}x\, x\Phi'\Phi x^{-2} \\ &= \int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y\, x\Phi'\Phi\, \mathrm{e}^{-yx^2} = \frac{2}{\sqrt{\pi}}\int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y\, x\Phi\, \mathrm{e}^{-(1+y)x^2} \\ &= \frac{2}{\sqrt{\pi}}\int_0^\infty \mathrm{d}y \int_0^\infty \mathrm{d}x\, \Phi\left(-\frac{1}{2(1+y)}\right)\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{-(1+y)x^2}. \end{aligned}$$

This result is integrated by parts:

$$\begin{aligned} L_{-1} &= \frac{1}{\sqrt{\pi}}\int_0^\infty \mathrm{d}y \int_0^\infty \mathrm{d}x \frac{1}{1+y}\Phi'\, \mathrm{e}^{-(1+y)x^2} = \frac{2}{\pi}\int_0^\infty \mathrm{d}y \int_0^\infty \mathrm{d}x \frac{1}{1+y}\mathrm{e}^{-(2+y)x^2} \\ &= \frac{1}{\sqrt{\pi}}\int_0^\infty \mathrm{d}y \frac{1}{(1+y)\sqrt{2+y}} = \frac{1}{\sqrt{\pi}}\int_2^\infty \mathrm{d}z \frac{1}{(z-1)\sqrt{z}}\,. \end{aligned}$$

This integral can be found in the standard tables,

$$\begin{aligned} L_{-1} &= -\frac{1}{\sqrt{\pi}}\ln\frac{1+z+2\sqrt{z}}{1-z}\bigg|_2^\infty = \frac{1}{\sqrt{\pi}}\ln(3+2\sqrt{2}\,) \\ &= \frac{1}{\sqrt{\pi}}\ln(1+\sqrt{2}\,)^2 = \frac{2}{\sqrt{\pi}}\ln(1+\sqrt{2}\,). \end{aligned}$$

We now complete the table:

$$\begin{aligned} &L_{-1} = \sqrt{2}\ln(1+\sqrt{2}\,)\cdot\sqrt{\frac{2}{\pi}}\,, \quad L_1 = \frac{1}{2}\sqrt{\frac{2}{\pi}}\,, \\ &L_3 = \frac{5}{8}\sqrt{\frac{2}{\pi}}\,, \qquad\qquad L_5 = \frac{43}{32}\sqrt{\frac{2}{\pi}}\,, \\ &L_7 = \frac{531}{128}\sqrt{\frac{2}{\pi}}\,, \qquad\qquad L_9 = \frac{8601}{512}\sqrt{\frac{2}{\pi}}\,. \end{aligned} \tag{A2.11}$$

It is easily seen that the integrals N_p, in which the Chandrasekhar function $\mathscr{H}(x)$ is explicitly expressed by (**11**.4.8), are simple combinations of the integrals L_p,

$$N_{2n+1} = \frac{\sqrt{\pi}}{2}\int_0^\infty \mathrm{d}x\, x^{2n+1}\Phi'(x)\,\mathscr{H}(x)$$

$$= \frac{\sqrt{\pi}}{2}\int_0^\infty \mathrm{d}x\, x^{2n+1}\Phi'\left(\Phi - \frac{1}{2x^2}(\Phi - x\Phi')\right).$$

Hence

$$N_{2n+1} = \tfrac{1}{2}\sqrt{\pi}\left(L_{2n+1} - \tfrac{1}{2}L_{2n-1} + \tfrac{1}{2}K_{2n}\right), \quad n = 0, 1, 2, \cdots. \tag{A2.12}$$

and similarly

$$N_{2n} = \tfrac{1}{2}\sqrt{\pi}\left(L_{2n} - \tfrac{1}{2}L_{2n-2} + \tfrac{1}{2}K_{2n-1}\right), \quad n = 1, 2, \cdots. \tag{A2.13}$$

Putting all these results together, we obtain tables (A2.2) and (A2.3).

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15

The banana transport equations

15.1. Strategy of the derivation of the banana transport equations

The last step towards a transport theory of plasmas in the long mean free path regime is the derivation of explicit transport equations relating the average radial particle and heat fluxes to the corresponding radial thermodynamic forces. The hitherto obtained solution of the drift kinetic equation provides us with all the necessary ingredients for this derivation. In particular, the linear relations between the generalized stresses and the "poloidal fluxes" (**14**.12.1) represent an important step towards the solution of our problem, but not yet the final step. Indeed, substituting these relations into definitions (**12**.9.16), (**12**.9.17) would yield expressions of the radial banana fluxes in terms of the "poloidal fluxes", which are as yet undetermined. We now take the key step of deriving the relation between the "poloidal fluxes" and the thermodynamic forces.

The solution of the banana transport problem was already prepared in section **12**.8. Indeed, we derived there a set of six linear algebraic equations (**12**.8.30), (**12**.8.31) relating the "poloidal fluxes" to the generalized stresses and to the thermodynamic forces. We now note that, in switching over to the notation of eq. (**12**.6.4), eqs. (**14**.12.1) become very simple,

$$\mathscr{B}_0^{-1}\left\langle B\bar{g}_{\|}^{\alpha(1)}\right\rangle = -\varphi\left(\mu_{11}^{\alpha}\,\omega_1^{\alpha} + \mu_{13}^{\alpha}\,\omega_3^{\alpha} + \mu_{15}^{\alpha}\,\omega_5^{\alpha}\right),$$

$$\mathscr{B}_0^{-1}\left\langle B\bar{g}_{\|}^{\alpha(3)}\right\rangle = -\varphi\left(\mu_{31}^{\alpha}\,\omega_1^{\alpha} + \mu_{33}^{\alpha}\,\omega_3^{\alpha} + \mu_{35}^{\alpha}\,\omega_5^{\alpha}\right),$$

$$\mathscr{B}_0^{-1}\left\langle B\bar{g}_{\|}^{\alpha(5)}\right\rangle = -\varphi\left(\mu_{51}^{\alpha}\,\omega_1^{\alpha} + \mu_{53}^{\alpha}\,\omega_3^{\alpha} + \mu_{55}^{\alpha}\,\omega_5^{\alpha}\right). \tag{1.1}$$

When these relations are substituted into (**12**.8.30), (**12**.8.31), we find a simple set of linear algebraic equations,

$$\sum_{\beta=\mathrm{e,i}}\sum_{s=1,3,5} a_{rs}^{\alpha\beta}\,\omega_s^{\beta} = G_r^{\alpha}, \quad \alpha = \mathrm{e,\ i}, \quad r = 1,\ 3,\ 5. \tag{1.2}$$

We do not write down the coefficients $a_{rs}^{\alpha\beta}$, which are rather cumbersome and can be easily derived by the reader. Let us just note that they are combinations involving the parallel (classical) transport coefficients and the pseudo-viscosity coefficients. As for the sources G_r^α, they involve the familiar four thermodynamic forces of chapter 13, i.e. the (dimensionless) radial pressure gradient $g_\rho^{(1)P}$ (eq. **12**.7.10), the radial temperature gradients $g_\rho^{\alpha(3)}$ (eq. **12**.4.6) *and* the parallel external electric field $\hat{g}_\parallel^{(1)A}$ (eq. **12**.10.3).

The actual way of solving eqs. (1.2) in the most transparent way will be discussed in the next section. Here, we rather focus on the general strategy. The solution of (1.2) can be expressed as a set of six linear relations between the "poloidal fluxes" and the four thermodynamic forces which, for convenience, are written in the following form [we use here the factor $\mathscr{K}_\alpha$ defined in eq. (**13**.3.3)]:

$$\omega_r^\alpha = m_{r1}^{\alpha e}\mathscr{K}_e\, g_\rho^{(1)P} + m_{r3}^{\alpha e}\mathscr{K}_e\, g_\rho^{e(3)} + m_{r3}^{\alpha i}\mathscr{K}_i\, g_\rho^{i(3)} + m_{rE}^{\alpha}\,\hat{g}_\parallel^{(1)A}. \tag{1.3}$$

Finally, the three privileged radial banana fluxes, defined in table **12**.9.1. are simply taken from (1.1),

$$\begin{aligned}\left\langle h_\rho^{e(1)}\right\rangle_B &= \mathscr{K}_e\,\mathscr{B}_0^{-1}\left\langle B\bar{g}_\parallel^{e(1)}\right\rangle \\ &= -\mathscr{K}_e\,\varphi\left(\mu_{11}^e\,\omega_1^e + \mu_{13}^e\,\omega_3^e + \mu_{15}^e\,\omega_5^e\right), \\ \left\langle h_\rho^{\alpha(3)}\right\rangle_B &= \mathscr{K}_\alpha\,\mathscr{B}_0^{-1}\left\langle B\bar{g}_\parallel^{\alpha(3)}\right\rangle \\ &= -\mathscr{K}_\alpha\,\varphi\left(\mu_{31}^\alpha\,\omega_1^\alpha + \mu_{33}^\alpha\,\omega_3^\alpha + \mu_{35}^\alpha\,\omega_5^\alpha\right).\end{aligned} \tag{1.4}$$

For completeness, we also consider the expression of the *non-privileged* fifth-order average radial fluxes, obtained from (1.1),

$$\begin{aligned}\left\langle h_\rho^{\alpha(5)}\right\rangle_B &= \mathscr{K}_\alpha\,\mathscr{B}_0^{-1}\left\langle B\bar{g}_\parallel^{\alpha(5)}\right\rangle \\ &= -\mathscr{K}_\alpha\,\varphi\left(\mu_{51}^\alpha\,\omega_1^\alpha + \mu_{53}^\alpha\,\omega_3^\alpha + \mu_{55}^\alpha\,\omega_5^\alpha\right).\end{aligned} \tag{1.5}$$

Upon substitution of the solution (1.3) into (1.4), we obtain a set of true *transport equations* relating the three radial fluxes to the four thermodynamic forces. The same substitution into (1.5) does not lead to a transport equation in the usual sense (because there is no corresponding "fifth-order thermodynamic force"). Nevertheless, the coefficients entering this expression ("*banana quasi-transport coefficients*") will be very useful in clarifying the structure of the ordinary transport coefficients, as shown in section 15.2.

Note that the parallel electric field now influences non-trivially the average radial particle and heat fluxes. Therefore, in order to obtain a complete set of transport equations, we must add an equation for its conjugate thermodynamic flux, as explained in section **12**.10. It was shown there that there exists, indeed, a banana contribution to the parallel electric current, given by eq. (**12**.10.5). This equation is combined with (1.4), (1.5) and yields

$$\mathscr{B}_0^{-1}\left\langle Bh_{\|}^{(1)}\right\rangle_B = \mathscr{K}_e^{-1}\left(-\tilde{\sigma}_{\|}\left\langle h_\rho^{e(1)}\right\rangle_B + \tilde{\alpha}_{\|}\left\langle h_\rho^{e(3)}\right\rangle_B + \tilde{\gamma}_{\|}\left\langle h_\rho^{e(5)}\right\rangle_B\right). \tag{1.6}$$

When this equation, combined with (1.3) is adjoined to (1.4), we obtain the *complete set of transport equations for the banana fluxes*. They will be discussed in detail in the forthcoming sections.

The peculiar form of eq. (1.6) is worth underscoring. It implies that the transport coefficients entering the electric current equation are expressed in terms of the previous ones. These important relations will be made explicit in the next section.

15.2. Derivation of the banana transport coefficients

The previous section presented the general strategy of the derivation of the banana transport equations. We now give some of the details of the calculation.

We shall use here some convenient abbreviations. The average fluxes will be denoted as

$$J_p^\alpha \equiv \left\langle h_\rho^{\alpha(p)}\right\rangle_B. \tag{2.1}$$

We also use systematically the symbols $\mathscr{K}_\alpha$ defined in (**13**.3.3). These quantities are related as follows to the quantity A, defined in (**12**.6.3),

$$\frac{\mathscr{K}_i}{\mathscr{K}_e} = -\frac{|\Omega_{e0}|\tau_e}{\Omega_{i0}\tau_i} = -A. \tag{2.2}$$

The quantity a was also defined in eq. (**12**.6.3). The symbols for all the (classical) dimensionless parallel quasi-transport coefficients (**12**.6.10), (**12**.6.11), will be written without the tilda and without the subscript $\|$. Thus

$$\tilde{\sigma}_{\|} \to \sigma, \qquad \tilde{\kappa}_{\|}^e \to \kappa^e, \quad \text{etc.}$$

Rather than solving directly the set of equations (1.2) – which originates from (**12**.8.30), (**12**.8.31) – we make some transformations that simplify the

calculations. By using relations (1.1), the first three equations can be recombined by taking as unknowns the three electronic radial banana fluxes [see eqs. (1.4), (1.5)] and keeping ω_1^i as a parameter. [Thus, the first three equations (**12**.8.30) are multiplied, respectively, by $\varphi\mu_{11}^3$, $\varphi\mu_{13}^e$, $\varphi\mu_{15}^3$ and added together: all the "poloidal fluxes", except ω_1^i can then be eliminated in favour of the fluxes J_p^α.] The result is

$$
\begin{aligned}
k_{11}^e J_1^e + k_{13}^e J_3^e + k_{15}^e J_5^e = \mathscr{K}_e\varphi\Big[& \mu_{11}^e \mathscr{K}_e g_\rho^{(1)P} + \mu_{13}^e \mathscr{K}_e g_\rho^{e(3)} \\
& + (\mu_{11}^e\sigma - \mu_{13}^e\alpha - \mu_{15}^e\gamma)\,\hat{g}_\parallel^{(1)A} - a\mu_{11}^e\omega_1^i\Big], \\
k_{31}^e J_1^e + k_{33}^e J_3^e + k_{35}^e J_5^e = \mathscr{K}_e\varphi\Big[& \mu_{31}^e \mathscr{K}_e g_\rho^{(1)P} + \mu_{33}^e \mathscr{K}_e g_\rho^{e(3)} \\
& + (\mu_{31}^e\sigma - \mu_{33}^e\alpha - \mu_{35}^e\gamma)\,\hat{g}_\parallel^{(1)A} - a\mu_{31}^e\omega_1^i\Big], \\
k_{51}^e J_1^e + k_{53}^e J_3^e + k_{55}^e J_5^e = \mathscr{K}_e\varphi\Big[& \mu_{51}^e \mathscr{K}_e g_\rho^{(1)P} + \mu_{53}^e \mathscr{K}_e g_\rho^{e(3)} \\
& + (\mu_{51}^e\sigma - \mu_{53}^e\alpha - \mu_{55}^e\gamma)\,\hat{g}_\parallel^{(1)A} - a\mu_{51}^e\omega_1^i\Big].
\end{aligned}
\tag{2.3}
$$

The coefficients k_{pq}^e can easily be calculated by the reader. [For instance: $k_{11}^e = 1 + \varphi(\mu_{11}^e\sigma - \mu_{13}^e\alpha - \mu_{15}^e\gamma)$, etc.]

Next, the last two equations (**12**.8.30) are combined with (1.1) and are considered as two equations for ω_3^i, ω_5^i,

$$
\begin{aligned}
& k_{33}^i\omega_3^i + k_{35}^i\omega_5^i = -\mathscr{K}_i g_\rho^{i(3)} + p_{31}^i\omega_1^i, \\
& k_{53}^i\omega_3^i + k_{55}^i\omega_5^i = p_{51}^i\omega_1^i,
\end{aligned}
\tag{2.4}
$$

where, for instance, $k_{33}^i = 1 + \varphi(\mu_{33}^i\kappa^i + \mu_{53}^i\delta^i)$, $p_{31}^i = -\varphi(\mu_{31}^i\kappa^i + \mu_{51}^i\delta^i)$, etc. Finally, the solubility constraint (**12**.8.31) becomes

$$
-\frac{1}{\mathscr{K}_e\varphi} J_1^e + aA\left[\mu_{11}^i\omega_1^i + \mu_{13}^i\omega_3^i + \mu_{15}^i\omega_5^i\right] = 0. \tag{2.5}
$$

We now solve eqs. (2.4) for ω_3^i, ω_5^i (in terms of $g_\rho^{i(3)}$ and ω_1^i) and substitute the result into (2.5). The latter provides us with an expression for ω_1^i which, after a short calculation reduces to the form

$$
\omega_1^i = m_3^i(\varphi)\mathscr{K}_i\, g_\rho^{i(3)} + \frac{1}{aA}\, p_i(\varphi)\frac{1}{\mathscr{K}_e\varphi} J_1^e. \tag{2.6}
$$

The explicit forms of the functions $m_3^i(\varphi)$ and $p_i(\varphi)$ need not bother us at this stage: they are combinations of parallel transport coefficients and pseudo-viscosities.

The forthcoming calculations are considerably simplified if we assume

$$A \gg 1. \tag{2.7}$$

The physical meaning of this reasonable assumption will be discussed in the next section.

Equation (2.6) is now substituted into (2.3). The second term in the right-hand side of (2.6) produces a correction of order A^{-1} to the coefficients k_{p1}^e; the latter is negligible when (2.7) is valid. Therefore, ω_1^i is simply replaced by its first term in the right-hand side of eqs. (2.3). The latter represent now a closed set of linear equations relating the fluxes J_p^e to the thermodynamic forces: their solution is straightforward.

The peculiar structure of the right-hand side of eqs. (2.3) is the origin of a very important set of relationships between the banana transport coefficients. Writing down the solution in terms of determinants, it is easily seen that the latter can be expressed as combinations of the following basic quantities:

$$l_{1q}^{ee}(\varphi) = M_e^{-1}(\varphi)\begin{vmatrix} \mu_{1q}^e & k_{13}^e & k_{15}^e \\ \mu_{3q}^e & k_{33}^e & k_{35}^e \\ \mu_{5q}^e & k_{53}^e & k_{55}^e \end{vmatrix},$$

$$l_{3q}^{ee}(\varphi) = M_e^{-1}(\varphi)\begin{vmatrix} k_{11}^e & \mu_{1q}^e & k_{15}^e \\ k_{31}^e & \mu_{3q}^e & k_{35}^e \\ k_{51}^e & \mu_{5q}^e & k_{55}^e \end{vmatrix},$$

$$l_{5q}^{ee}(\varphi) = M_e^{-1}(\varphi)\begin{vmatrix} k_{11}^e & k_{11}^e & \mu_{1q}^e \\ k_{31}^e & k_{33}^e & \mu_{3q}^e \\ k_{51}^e & k_{53}^e & \mu_{5q}^e \end{vmatrix}, \quad q = 1, 3, 5, \tag{2.8}$$

where $M_e(\varphi)$ is the determinant of the matrix k_{pq}^e. Upon evaluation, it turns out that these coefficients have the basic *symmetry* property

$$l_{pq}^{ee}(\varphi) = l_{qp}^{ee}(\varphi). \tag{2.9}$$

The solution of eqs. (2.3) (combined with eq. 2.6) is now written as

$$J_p^e = \mathscr{K}_e^2 \varphi \left[l_{p1}^{ee}(\varphi) g_\rho^{(1)P} + l_{p3}^{ee}(\varphi) g_\rho^{e(3)} + aA\, l_{p3}^{ei}(\varphi) g_\rho^{i(3)} \right]$$

$$+ \mathscr{K}_e \varphi \left[l_{pE}^{e}(\varphi) \hat{g}_\theta^{(1)A} \right], \quad p = 1, 3, 5. \tag{2.10}$$

The determinants (2.8) are thus closely connected to the *electron banana quasi-transport coefficients*:

- the coefficients $l_{p1}^{ee}(\varphi)$, $l_{p3}^{ee}(\varphi)$ are directly defined by (2.8);
- the "mixed coefficients" $l_{p3}^{ei}(\varphi)$ are expressed as

$$l_{p3}^{ei}(\varphi) = m_3^i(\varphi)\, l_{p1}^{ee}(\varphi), \quad p = 1, 3, 5; \tag{2.11}$$

- the "electrical coefficients" $l_{pE}^{e}(\varphi)$ are expressed as

$$l_{pE}^{e}(\varphi) = \sigma\, l_{p1}^{ee}(\varphi) - \alpha\, l_{p3}^{ee}(\varphi) - \gamma\, l_{p5}^{ee}(\varphi), \quad p = 1, 3, 5. \tag{2.12}$$

We now turn to the ion fluxes. The solution ω_3^i, ω_5^i of eqs. (2.4), together with (2.6), can be recombined into the ion fluxes,

$$J_q^i = -\mathscr{K}_i \varphi \left[\mu_{q1}^i\, \omega_1^i + \mu_{q3}^i\, \omega_3^i + \mu_{q5}^i\, \omega_5^i \right], \quad q = 3, 5. \tag{2.13}$$

Because of (2.6), these fluxes have the general form

$$J_q^i = \mathscr{K}_i^2 \varphi\, l_{q3}^{ii}(\varphi)\, g_\rho^{i(3)} - \frac{\mathscr{K}_i}{\mathscr{K}_e} \frac{1}{aA} m_q^i(\varphi)\, J_1^e. \tag{2.14}$$

[Note that $m_3^i(\varphi)$ is the same coefficient as in eq. (2.6).] The explicit form of $l_{q3}^{ii}(\varphi)$ and of $m_q^i(\varphi)$ will be given later. It should be noted that the second term in (2.6) can no longer be neglected here, because in (2.14) the factor A^{-1} is balanced by $(\mathscr{K}_i/\mathscr{K}_e)$ (see eq. (2.2). Equation (2.10) for $p = 1$ is now substituted into (2.14). The term proportional to $g_\rho^{i(3)}$ in (2.10) yields a correction of order A^{-1} to the first term on the right-hand side of (2.14) and can be neglected, because of (2.7). The ion fluxes can thus be written in the form

$$J_q^i = \mathscr{K}_e^2 a^{-1} \varphi \left[l_{q1}^{ie}(\varphi) g_\rho^{(1)P} + l_{q3}^{ie}(\varphi) g_\rho^{e(3)} \right] + \mathscr{K}_i^2 \varphi\, l_{33}^{ii}(\varphi) g_\rho^{i(3)}$$

$$+ \mathscr{K}_e a^{-1} \varphi\, l_{qE}^{i}(\varphi) \hat{g}_\parallel^{(1)A}, \quad q = 3, 5. \tag{2.15}$$

From (2.14) and (2.10) we deduce the relations

$$l^{ie}_{q1}(\varphi) = m^{i}_{q}(\varphi)\, l^{ee}_{11}(\varphi),$$

$$l^{ie}_{q3}(\varphi) = m^{i}_{q}(\varphi)\, l^{ee}_{13}(\varphi),$$

$$l^{i}_{qE}(\varphi) = m^{i}_{q}(\varphi)\, l^{e}_{1E}(\varphi). \tag{2.16}$$

Comparing with (2.11), we find the following relations, for $q = 3$,

$$l^{ei}_{13}(\varphi) = l^{ie}_{31}(\varphi), \qquad l^{ei}_{33}(\varphi) = l^{ie}_{33}(\varphi). \tag{2.17}$$

Thus, $l^{ii}_{33}(\varphi)$ is the only "new" coefficient entering (2.15).

Finally, we consider eq. (1.6) for the average parallel electric current. In the present notations, it is written as

$$\mathscr{B}_0^{-1}\left\langle Bh^{(1)}_{\|}\right\rangle_B = \mathscr{K}_e^{-1}\left(\sigma J^e_1 - \alpha J^e_3 - \gamma J^e_5\right).$$

Hence, using (2.10), it takes the form

$$\mathscr{B}_0^{-1}\left\langle Bh^{(1)}_{\|}\right\rangle_B = \mathscr{K}_e\ \varphi\left[l^{e}_{E1}(\varphi)\ g^{(1)P}_{\rho} + l^{e}_{E3}(\varphi)\ g^{e(3)}_{\rho} + aA\, l^{i}_{E3}(\varphi)\ g^{i(3)}_{\rho}\right]$$

$$+\varphi\, l_{EE}(\varphi)\ \hat{g}^{(1)A}_{\|}, \tag{2.18}$$

and we immediately obtain the relations

$$l^{e}_{E1}(\varphi) = -\sigma\, l^{ee}_{11}(\varphi) + \alpha\, l^{ee}_{31}(\varphi) + \gamma\, l^{ee}_{51}(\varphi) = -l^{e}_{1E}(\varphi),$$

$$l^{e}_{E3}(\varphi) = -\sigma\, l^{ee}_{13}(\varphi) + \alpha\, l^{ee}_{33}(\varphi) + \gamma\, l^{ee}_{53}(\varphi) = -l^{e}_{3E}(\varphi),$$

$$l^{i}_{E3}(\varphi) = -\sigma\, l^{ei}_{13}(\varphi) + \alpha\, l^{ei}_{33}(\varphi) + \gamma\, l^{ei}_{53}(\varphi) = -l^{i}_{3E}(\varphi). \tag{2.19}$$

We note that eq. (2.11) also implies

$$l^{i}_{E3}(\varphi) = m^{i}_{3}(\varphi)\, l^{e}_{E1}(\varphi). \tag{2.20}$$

The last coefficient in (2.18) is

$$l_{EE}(\varphi) = -\sigma\, l^{e}_{1E}(\varphi) + \alpha\, l^{e}_{3E}(\varphi) + \gamma\, l^{e}_{5E}(\varphi). \tag{2.21}$$

Combining this expression with (2.12), we find a nice form of the coefficient $l_{EE}(\varphi)$,

$$l_{FF}(\varphi) = -\sigma^2 l_{11}^{ee}(\varphi) + \alpha^2 l_{33}^{ee}(\varphi) + \gamma^2 l_{55}^{ee}(\varphi) + 2\alpha\sigma l_{13}^{ee}(\varphi) + 2\gamma\sigma l_{15}^{ee}(\varphi) - 2\alpha\gamma l_{35}^{ee}(\varphi). \tag{2.22}$$

We thus derived a host of relationships between the banana transport coefficients. To the best of our knowledge, most of these do not seem to have been noticed in previous works. [We have noted them in a short summary paper; see Balescu (1987).] The remarkable fact about them is that they are not accidental numerical coincidences, but they rather reveal the deep structure of the neoclassical theory.

To sum up the findings of the present section, we stress the fact that, in the 21M approximation, *all the 16 neoclassical coefficients entering the banana transport equations* (2.10) *for* $p = 1, 3$, (2.15) for $q = 3$ *and* (2.18), *are constructed in terms of seven independent basic banana quasi-transport coefficients*: the six elements of the symmetric matrix $l_{pq}^{ee}(\varphi)$ $[p, q = 1, 3, 5]$ and the additional coefficient $l_{33}^{ii}(\varphi)$.

15.3. The transport equations in the long mean free path regime

In the present section the final form of the transport equations and the expressions of the banana transport and quasi-transport equations are collected in a systematic manner. As in previous chapters (chs. 5 and 13) the results will be presented in two forms: dimensionless and dimensional.

A. Dimensionless transport equations

$$\left\langle h_\rho^{e(p)} \right\rangle_B = \mathscr{K}_e^2 \varphi \left[l_{p1}^{ee}(\varphi)\, g_\rho^{(1)P} + l_{p3}^{ee}(\varphi)\, g_\rho^{e(3)} + l_{p3}^{ei}(\varphi)\, aA\, g_\rho^{i(3)} \right] + \mathscr{K}_e\, \varphi\, l_{pE}^{e}(\varphi)\, \hat{g}_\parallel^{(1)A}, \quad p = 1, 3, 5, \tag{3.1}$$

$$\left\langle h_\rho^{i(p)} \right\rangle_B = \mathscr{K}_e^2 a^{-1} \varphi \left[l_{p1}^{ie}(\varphi)\, g_\rho^{(1)P} + l_{p3}^{ie}(\varphi)\, g_\rho^{e(3)} \right] + \mathscr{K}_i^2 \varphi\, l_{p3}^{ii}(\varphi)\, g_\rho^{i(3)} + \mathscr{K}_e a^{-1} \varphi\, l_{pF}^{i}(\varphi)\, \hat{g}_\parallel^{(1)A}, \quad p = 3, 5, \tag{3.2}$$

$$\mathscr{B}_0^{-1} \left\langle B h_\parallel^{(1)} \right\rangle_B = \mathscr{K}_e\, \varphi \left[l_{F1}^{e}(\varphi)\, g_\rho^{(1)P} + l_{F3}^{e}(\varphi)\, g_\rho^{e(3)} + l_{E3}^{i}(\varphi)\, aA\, g_\rho^{i(3)} \right] + \varphi\, l_{FF}(\varphi)\, \hat{g}_\parallel^{(1)A}. \tag{3.3}$$

We recall the notations

$$\mathscr{K}_\alpha = \frac{\mathscr{I} l_\rho}{R_0 \mathscr{B}_P \Omega_{\alpha 0} \tau_\alpha}, \quad [\text{Note:} \quad \mathscr{K}_e < 0!],$$

$$\mathscr{B}_0 = \langle B^2 \rangle^{1/2},$$

$$a = \left(\frac{T_i}{T_e} \frac{m_e}{m_i} \right)^{1/2},$$

$$A = \frac{m_i n_i}{\tau_i} \frac{\tau_e}{m_e n_e} = \frac{|\Omega_{e0}| \tau_e}{\Omega_{i0} \tau_i}. \tag{3.4}$$

The values of the coefficients $l_{pq}^{\alpha\beta}(\varphi)$ are collected at the end of this section. The transport and quasi-transport coefficients will be divided into three groups:

(1) *The pure diffusive coefficients*:

$$l_{11}^{ee}(\varphi),\ l_{13}^{ee}(\varphi),\ l_{15}^{ee}(\varphi);$$

$$l_{31}^{ee}(\varphi),\ l_{33}^{ee}(\varphi),\ l_{35}^{ee}(\varphi); \qquad l_{33}^{ii}(\varphi),\ l_{53}^{ii}(\varphi);$$

$$l_{51}^{ee}(\varphi),\ l_{53}^{ee}(\varphi),\ l_{55}^{ee}(\varphi). \tag{3.5}$$

(2) *The mixed diffusive coefficients*:

$$l_{13}^{ei}(\varphi),\ l_{33}^{ei}(\varphi),\ l_{53}^{ei}(\varphi);$$

$$l_{31}^{ie}(\varphi),\ l_{33}^{ie}(\varphi),\ l_{51}^{ie}(\varphi),\ l_{53}^{ie}(\varphi). \tag{3.6}$$

(3) *The electrical coefficients*:

$$\begin{aligned} &l_{1E}^{e}(\varphi),\ l_{3E}^{e}(\varphi),\ l_{5E}^{e}(\varphi); \qquad l_{3E}^{i}(\varphi),\ l_{5E}^{i}(\varphi); \\ &l_{E1}^{e}(\varphi),\ l_{E3}^{e}(\varphi); \qquad l_{E3}^{i}(\varphi); \\ &l_{EE}(\varphi). \end{aligned} \tag{3.7}$$

We note here the following symmetry properties, which were derived in section 15.2:

$$\begin{aligned} &l_{13}^{ee}(\varphi) = l_{31}^{ee}(\varphi), \quad l_{15}^{ee}(\varphi) = l_{51}^{ee}(\varphi), \quad l_{35}^{ee}(\varphi) = l_{53}^{ee}(\varphi), \\ &l_{13}^{ei}(\varphi) = l_{31}^{ie}(\varphi), \quad l_{33}^{ei}(\varphi) = l_{33}^{ie}(\varphi), \\ &l_{1F}^{e}(\varphi) = -l_{F1}^{e}(\varphi), \quad l_{3F}^{e}(\varphi) = -l_{E3}^{e}(\varphi), \quad l_{3F}^{i}(\varphi) = -l_{F3}^{i}(\varphi). \end{aligned} \tag{3.8}$$

B. Dimensional transport equations

We use here the two systems of notations defined in section **13**.2. The choice of one or the other depends on whether the problem at hand benefits from explicitness or from compactness. We recall here, for convenience, definitions (**13**.2.6)–(**13**.2.9).

(a) *Thermodynamic forces*:

$$X_1^e \equiv X_1 = -(n_e T_e)^{-1} \nabla_\rho P,$$

$$X_3^e \equiv X_2 = -T_e^{-1} \nabla_\rho T_e,$$

$$X_3^i \equiv X_3 = -T_i^{-1} \nabla_\rho T_i,$$

$$X_E \equiv X_4 = \mathscr{B}_0^{-1} \left\langle B E_\parallel^{(A)} \right\rangle. \tag{3.9}$$

(b) *Banana fluxes*:

$$J_1^e \equiv J_1^B = \left\langle \Gamma_\rho^e \right\rangle_B,$$

$$J_3^e \equiv J_2^B = T_e^{-1} \left\langle q_\rho^e \right\rangle_B,$$

$$J_3^i \equiv J_3^B = T_i^{-1} \left\langle q_\rho^i \right\rangle_B,$$

$$J_E \equiv J_4^B = \mathscr{B}_0^{-1} \langle B j_\parallel \rangle_B. \tag{3.10}$$

In the first system of notations, we write the dimensional banana transport equations as

$$\left\langle \Gamma_\rho^e \right\rangle_B = L_{11}^{ee} X_1^e + L_{13}^{ee} X_3^e + L_{13}^{ei} X_3^i + L_{1E}^e X_E,$$

$$T_e^{-1} \left\langle q_\rho^e \right\rangle_B = L_{31}^{ee} X_1^e + L_{33}^{ee} X_3^e + L_{33}^{ei} X_3^i + L_{3E}^e X_E,$$

$$T_i^{-1} \left\langle q_\rho^i \right\rangle_B = L_{31}^{ie} X_1^e + L_{33}^{ie} X_3^e + L_{33}^{ii} X_3^i + L_{3E}^i X_E,$$

$$\mathscr{B}_0^{-1} \langle B j_\parallel \rangle_B = L_{E1}^e X_1^e + L_{E3}^e X_3^e + L_{E3}^i X_3^i + L_{EE} X_E. \tag{3.11}$$

In the second system of notations, we write

$$J_\mu^B = \sum_{\nu=1}^{4} \mathscr{L}_{\mu\nu}^B X_\nu. \tag{3.12}$$

The correspondence between the two notations for the transport coefficients is obtained by comparing the matrix elements in the following two arrays:

$$\mathscr{L}^{\mathrm{B}} = \begin{pmatrix} L_{11}^{\mathrm{ee}} & L_{13}^{\mathrm{ee}} & L_{13}^{\mathrm{ei}} & L_{1\mathrm{E}}^{\mathrm{e}} \\ L_{31}^{\mathrm{ee}} & L_{33}^{\mathrm{ee}} & L_{33}^{\mathrm{ei}} & L_{3\mathrm{E}}^{\mathrm{e}} \\ L_{31}^{\mathrm{ie}} & L_{33}^{\mathrm{ie}} & L_{33}^{\mathrm{ii}} & L_{3\mathrm{E}}^{\mathrm{i}} \\ L_{\mathrm{E}1}^{\mathrm{e}} & L_{\mathrm{E}3}^{\mathrm{e}} & L_{\mathrm{E}3}^{\mathrm{i}} & L_{\mathrm{EE}} \end{pmatrix} = \begin{pmatrix} \mathscr{L}_{11}^{\mathrm{B}} & \mathscr{L}_{12}^{\mathrm{B}} & \mathscr{L}_{13}^{\mathrm{B}} & \mathscr{L}_{14}^{\mathrm{B}} \\ \mathscr{L}_{21}^{\mathrm{B}} & \mathscr{L}_{22}^{\mathrm{B}} & \mathscr{L}_{23}^{\mathrm{B}} & \mathscr{L}_{24}^{\mathrm{B}} \\ \mathscr{L}_{31}^{\mathrm{B}} & \mathscr{L}_{32}^{\mathrm{B}} & \mathscr{L}_{33}^{\mathrm{B}} & \mathscr{L}_{34}^{\mathrm{B}} \\ \mathscr{L}_{41}^{\mathrm{B}} & \mathscr{L}_{42}^{\mathrm{B}} & \mathscr{L}_{43}^{\mathrm{B}} & \mathscr{L}_{44}^{\mathrm{B}} \end{pmatrix}. \tag{3.13}$$

The following symmetries hold for the dimensional transport coefficients as a result of (3.8),

$$\begin{aligned} &L_{13}^{\mathrm{ee}} = L_{31}^{\mathrm{ee}}, \\ &T_{\mathrm{e}} L_{13}^{\mathrm{ei}} = T_{\mathrm{i}} L_{31}^{\mathrm{ie}}, \qquad T_{\mathrm{e}} L_{33}^{\mathrm{ei}} = T_{\mathrm{i}} L_{33}^{\mathrm{ie}} \\ &L_{1\mathrm{E}}^{\mathrm{e}} = -L_{\mathrm{E}1}^{\mathrm{e}}, \qquad L_{3\mathrm{E}}^{\mathrm{e}} = -L_{\mathrm{E}3}^{\mathrm{e}}, \qquad L_{3\mathrm{E}}^{\mathrm{i}} = -L_{\mathrm{E}3}^{\mathrm{i}}. \end{aligned} \tag{3.14}$$

The relations between dimensionless and dimensional banana transport coefficients are given in table 3.4.

C. Collected values of the banana transport coefficients

The expressions of the banana transport coefficients will be given both in the 13M and in the 21M approximations. In order to understand our system of abbreviations, it should be noted that the expressions of these coefficients are made up of combinations of elements of three matrices (for each species): the *collision matrix* c_{pq}^{α}, the *pseudo-viscosity matrix* μ_{pq}^{α} and the *parallel quasi-transport matrix* whose elements are denoted by the usual symbols $\tilde{\sigma}_{\|}$, $\tilde{\alpha}_{\|}, \ldots$. In order to unburden the notation, we suppress the tilda and the subscript $\|$ for the parallel quasi-transport coefficients in the forthcoming formulae. For easy reference, these coefficients have been recollected in table 3.1. From these coefficients we construct a set of quadratic expressions, which are partial second-order determinants, and cubic expressions, which are third-order determinants. Some of these expressions are the same as those which appeared in chapter 5 in the definition of the classical transport coefficients (see eq. (5.3.26).

(i) Combinations of collision matrix elements:

$$\begin{aligned} D_{pq}^{\alpha} &= c_{pp}^{\alpha} c_{qq}^{\alpha} - c_{pq}^{\alpha} c_{pq}^{\alpha}, \\ H_{pqr}^{\alpha} &= c_{qq}^{\alpha} c_{pr}^{\alpha} - c_{pq}^{\alpha} c_{qr}^{\alpha}, \\ F_{135}^{\alpha} &= c_{11}^{\alpha} D_{35}^{\alpha} - c_{13}^{\alpha} H_{153}^{\alpha} - c_{15}^{\alpha} H_{135}^{\alpha}. \end{aligned} \tag{3.15}$$

Table 3 1
Dimensionless parallel transport and quasi-transport coefficients

Coefficient	13M	21M	Numerical value for $Z=1$	
			13M	21M
Electron coefficients				
$\sigma \equiv \tilde{\sigma}_{\parallel}$	c^e_{33}/D^e_{13}	D^e_{35}/F^e_{135}	1 932	1.950
$\alpha \equiv \tilde{\alpha}_{\parallel}$	$-c^e_{13}/D^e_{13}$	$-H^e_{153}/F^e_{135}$	−0.982	−0 877
$\kappa^e \equiv \tilde{\kappa}^e_{\parallel}$	c^e_{11}/D^e_{13}	D^e_{15}/F^e_{135}	1.036	1 659
$\gamma \equiv \tilde{\gamma}_{\parallel}$	0	$-H^e_{135}/F^e_{135}$	0	0 132
$\delta^e \equiv \tilde{\delta}^e_{\parallel}$	0	$-H^e_{315}/F^e_{135}$	0	0 778
$\epsilon^e \equiv \tilde{\epsilon}^e_{\parallel}$	0	D^e_{13}/F^e_{135}	0	0 970
Ion coefficients				
$\kappa^i \equiv \tilde{\kappa}^i_{\parallel}$	$1/c^i_{33}$	c^i_{55}/D^i_{35}	1 768	2.210
$\delta^i \equiv \tilde{\delta}^i_{\parallel}$	0	$-c^i_{35}/D^i_{35}$	0	0.780
$\epsilon^i \equiv \tilde{\epsilon}^i_{\parallel}$	0	c^i_{33}/D^i_{35}	0	1.375

(ii) Combinations of pseudo-viscosity coefficients:

$$\mathscr{D}^{\alpha}_{pq} = \mu^{\alpha}_{pp}\mu^{\alpha}_{qq} - \mu^{\alpha}_{pq}\mu^{\alpha}_{pq},$$

$$\mathscr{H}^{\alpha}_{pqr} = \mu^{\alpha}_{qq}\mu^{\alpha}_{pr} - \mu^{\alpha}_{pq}\mu^{\alpha}_{qr},$$

$$\mathscr{F}^{\alpha}_{135} = \mu^{\alpha}_{11}\mathscr{D}^{\alpha}_{35} - \mu^{\alpha}_{13}\mathscr{H}^{\alpha}_{153} - \mu^{\alpha}_{15}\mathscr{H}^{\alpha}_{135}, \tag{3.16}$$

(iii) Combination of parallel transport coefficients:

$$\Delta^e_{13} = \sigma\kappa^e - \alpha^2, \qquad \Delta^e_{15} = \sigma\epsilon^e - \gamma^2, \qquad \Delta^{\alpha}_{35} = \kappa^{\alpha}\epsilon^{\alpha} - \delta^{\alpha 2},$$
$$\chi^e_{315} = \sigma\delta^e - \alpha\gamma, \qquad \chi^e_{135} = \kappa^e\gamma - \alpha\delta^e, \qquad \chi^e_{153} = \epsilon^e\alpha - \gamma\delta^e,$$
$$\Phi^e_{135} = \sigma\kappa^e\epsilon^e + 2\alpha\gamma\delta^e - \sigma\delta^{e2} - \kappa^e\gamma^2 - \epsilon^e\alpha^2. \tag{3.17}$$

Among these various coefficients there exist relationships which are quite useful in the calculations. We do not list them here, however, because they are not explicitly used in the forthcoming argument. The main relations of this type are derived and listed in appendix 17A.1.

We now define the following functions:

$$D_e(\varphi) = 1 + \varphi(\sigma\mu^e_{11} - 2\alpha\mu^e_{13} + \kappa^e\mu^e_{33}) + \varphi^2\Delta^e_{13}\mathscr{D}^e_{13},$$

$$D_i(\varphi) = \mu^i_{11} + \varphi\kappa^i\mathscr{D}^i_{13},$$

$$d^i(\varphi) = \frac{\mu^i_{13}}{D_i(\varphi)}, \qquad [13M]. \tag{3.18}$$

In these functions, the parallel transport coefficients must be evaluated in the 13M approximation (first column in table 6.1).

$$M_e(\varphi) = 1 + \varphi(\sigma\mu^e_{11} + \kappa^e\mu^e_{33} + \epsilon^e\mu^e_{55} - 2\alpha\mu^e_{13} - 2\gamma\mu^e_{15} + 2\delta^e\mu^e_{35})$$
$$+ \varphi^2(\Delta^e_{13}\mathscr{D}^e_{13} + \Delta^e_{15}\mathscr{D}^e_{15} + \Delta^e_{35}\mathscr{D}^e_{35} + 2\chi^e_{315}\mathscr{H}^e_{315}$$
$$- 2\chi^e_{135}\mathscr{H}^e_{135} - 2\chi^e_{153}\mathscr{H}^e_{153}) + \varphi^3\Phi^e_{135}\mathscr{F}^e_{135},$$

$$M_i(\varphi) = \mu^i_{11} + \varphi(\kappa^i\mathscr{D}^i_{13} + \epsilon^i\mathscr{D}^i_{15} + 2\delta^i\mathscr{H}^i_{315}) + \varphi^2\Delta^i_{35}\mathscr{F}^i_{135},$$

$$m^i_3(\varphi) = M_i^{-1}(\varphi)[\mu^i_{13} + \varphi(\epsilon^i\mathscr{H}^i_{153} - \delta^i\mathscr{H}^i_{135})],$$

$$m^i_5(\varphi) = M_i^{-1}(\varphi)[\mu^i_{15} + \varphi(\kappa^i\mathscr{H}^i_{135} - \delta^i\mathscr{H}^i_{153})], \qquad [21M]. \tag{3.19}$$

In these expressions, the parallel transport coefficients must be evaluated in the 21M approximations (second column in table 3.1).

A non-negligible simplification of the banana transport coefficients occurs in the case, most often met in practice, when

$$A^{-1} \equiv \frac{\Omega_{i0}\,\tau_i}{|\,\Omega_{e0}\,|\,\tau_e} \ll 1. \tag{3.20}$$

This condition is satisfied whenever (see eq. (5.5.37)

$$\frac{T_i}{T_e} \ll Z^{2/3}\left(\frac{m_i}{m_e}\right)^{1/3}. \tag{3.21}$$

We only give here the formulae for the banana transport coefficients (as derived in section 15.2) in which terms of relative order A^{-1} are omitted. This approximation is used by many authors (see, e.g. Galeev 1971). It is, of course, not difficult, though tedious, to calculate the complete expressions whenever (3.20) is not satisfied.

Table 3.2
Dimensionless banana transport coefficients in the 13M approximation ($A \gg 1$).

Pure diffusive coefficients

$$l_{11}^{ee}(\varphi) = D_e^{-1}(\varphi)(\mu_{11}^e + \varphi \kappa^e \mathscr{D}_{13}^e)$$

$$l_{33}^{ee}(\varphi) = D_e^{-1}(\varphi)(\mu_{33}^e + \varphi \sigma \mathscr{D}_{13}^e)$$

$$l_{33}^{ii}(\varphi) = D_i^{-1}(\varphi)\, \mathscr{D}_{13}^i$$

$$l_{13}^{ee}(\varphi) = l_{31}^{ee}(\varphi) = D_e^{-1}(\varphi)(\mu_{13}^e + \varphi \alpha \mathscr{D}_{13}^e)$$

Mixed diffusive coefficients

$$l_{13}^{ei}(\varphi) = l_{31}^{ie}(\varphi) = d^i(\varphi)\, l_{11}^{ee}(\varphi)$$

$$l_{33}^{ei}(\varphi) = l_{33}^{ie}(\varphi) = d^i(\varphi)\, l_{13}^{ee}(\varphi)$$

Electrical coefficients

$$l_{1E}^e(\varphi) = -l_{E1}^e(\varphi) = \sigma l_{11}^{ee}(\varphi) - \alpha l_{13}^{ee}(\varphi)$$

$$l_{3E}^e(\varphi) = -l_{E3}^e(\varphi) = \sigma l_{31}^{ee}(\varphi) - \alpha l_{33}^{ee}(\varphi)$$

$$l_{3E}^i(\varphi) = -l_{E3}^i(\varphi) = \sigma l_{31}^{ie}(\varphi) - \alpha l_{33}^{ie}(\varphi)$$

$$l_{EE}(\varphi) = -\sigma l_{1E}^e(\varphi) + \alpha l_{3E}^e(\varphi)$$

We decided, for easy reference, to list explicit formulae in both the 13M and the 21M approximations. These formulae are collected in tables 3.2 and 3.3, respectively. Actually, the two sets of formulae are very simply related. It suffices to

- put the quasi-transport coefficients γ, δ^α, ϵ^α equal to zero;
- use the 13M values for σ, α, κ^α in the 21M expressions

in order to obtain the 13M formulae.

We note that some of the banana transport coefficients are expressed as combinations of others: these important relations were derived in section 15.2.

The dimensionless parallel transport coefficients as well as the pseudo-viscosity coefficients are purely numerical coefficients: they only depend on the charge number Z. Their values for any desired Z are easily calculated by using the analytical expressions, combined with the values collected in tables 5.3.1 and 5.4.1. Hence, the formulae of tables 3.2 and 3.3 provide an easy and explicit way for calculating the *banana transport coefficients for arbitrary values of Z*.

Table 3.3
Dimensionless banana transport and quasi-transport coefficients in the 21M approximation ($A \gg 1$).

Pure diffusive coefficients

$$l^{ee}_{11}(\varphi) = M_e^{-1}(\varphi)\left[\mu^e_{11} + \varphi(\kappa^e \mathscr{D}^e_{13} + 2\delta^e \mathscr{H}^e_{315} + \epsilon^e \mathscr{D}^e_{15}) + \varphi^2 \Delta^e_{35} \mathscr{F}^e_{135}\right]$$

$$l^{ee}_{33}(\varphi) = M_e^{-1}(\varphi)\left[\mu^e_{33} + \varphi(\epsilon^e \mathscr{D}^e_{35} - 2\gamma \mathscr{H}^e_{135} + \sigma \mathscr{D}^e_{13}) + \varphi^2 \Delta^e_{15} \mathscr{F}^e_{135}\right]$$

$$l^{ee}_{55}(\varphi) = M_e^{-1}(\varphi)\left[\mu^e_{55} + \varphi(\sigma \mathscr{D}^e_{15} - 2\alpha \mathscr{H}^e_{153} + \kappa^e \mathscr{D}^e_{35}) + \varphi^2 \Delta^e_{13} \mathscr{F}^e_{135}\right]$$

$$l^{ee}_{13}(\varphi) = l^{ee}_{31}(\varphi) = M_e^{-1}(\varphi)\left[\mu^e_{13} + \varphi(\alpha \mathscr{D}^e_{13} + \gamma \mathscr{H}^e_{315} - \delta^e \mathscr{H}^e_{135} + \epsilon^e \mathscr{H}^e_{153}) + \varphi^2 \chi^e_{153} \mathscr{F}^e_{135}\right]$$

$$l^{ee}_{15}(\varphi) = l^{ee}_{51}(\varphi) = M_e^{-1}(\varphi)\left[\mu^e_{15} + \varphi(\gamma \mathscr{D}^e_{15} + \alpha \mathscr{H}^e_{315} + \kappa^e \mathscr{H}^e_{135} - \delta^e \mathscr{H}^e_{153}) + \varphi^2 \chi^e_{135} \mathscr{F}^e_{135}\right]$$

$$l^{ee}_{35}(\varphi) = l^{ee}_{53}(\varphi) = M_e^{-1}(\varphi)\left[\mu^e_{35} + \varphi(-\delta^e \mathscr{D}^e_{35} + \sigma \mathscr{H}^e_{315} + \alpha \mathscr{H}^e_{135} + \gamma \mathscr{H}^e_{153}) - \varphi^2 \chi^e_{315} \mathscr{F}^e_{135}\right]$$

$$l^{ii}_{33}(\varphi) = M_i^{-1}(\varphi)\left[\mathscr{D}^i_{13} + \varphi \epsilon^i \mathscr{F}^i_{135}\right]$$

$$l^{ii}_{53}(\varphi) = M_i^{-1}(\varphi)\left[\mathscr{H}^i_{315} - \varphi \delta^i \mathscr{F}^i_{135}\right]$$

Mixed diffusive coefficients

$$l^{ei}_{13}(\varphi) = l^{ie}_{31}(\varphi) = m^i_3(\varphi)\, l^{ee}_{11}(\varphi)$$

$$l^{ei}_{33}(\varphi) = l^{ie}_{33}(\varphi) = m^i_3(\varphi)\, l^{ee}_{13}(\varphi)$$

$$l^{ei}_{53}(\varphi) = m^i_3(\varphi)\, l^{ee}_{51}(\varphi)$$

$$l^{ie}_{51}(\varphi) = m^i_5(\varphi)\, l^{ee}_{11}(\varphi)$$

$$l^{ie}_{53}(\varphi) = m^i_5(\varphi)\, l^{ee}_{13}(\varphi)$$

Electrical coefficients

$$l^e_{1E}(\varphi) = -l^e_{E1}(\varphi) = \sigma l^{ee}_{11}(\varphi) - \alpha l^{ee}_{13}(\varphi) - \gamma l^{ee}_{15}(\varphi)$$

$$l^e_{3E}(\varphi) = -l^e_{E3}(\varphi) = \sigma l^{ee}_{31}(\varphi) - \alpha l^{ee}_{33}(\varphi) - \gamma l^{ee}_{35}(\varphi)$$

$$l^e_{5E}(\varphi) = -l^e_{E5}(\varphi) = \sigma l^{ee}_{51}(\varphi) - \alpha l^{ee}_{53}(\varphi) - \gamma l^{ee}_{55}(\varphi)$$

$$l^i_{3E}(\varphi) = -l^i_{E3}(\varphi) = \sigma l^{ei}_{13}(\varphi) - \alpha l^{ei}_{33}(\varphi) - \gamma l^{ei}_{53}(\varphi)$$

$$l^i_{5E}(\varphi) = m^i_5(\varphi)\, l^e_{1E}(\varphi)$$

$$l_{EE}(\varphi) = -\sigma l^e_{1E}(\varphi) + \alpha l^e_{3E}(\varphi) + \gamma l^e_{5E}(\varphi)$$

Table 3 4
Relations between dimensional and dimensionless transport coefficients

Pure diffusive coefficients

$$L_{11}^{ee} = \frac{n_e \rho_{e0}^2}{\tau_e} \mathscr{J}^2 \tfrac{1}{2} \varphi \, l_{11}^{ee}(\varphi)$$

$$L_{33}^{ee} = \frac{n_e \rho_{e0}^2}{\tau_e} \mathscr{J}^2 \tfrac{5}{4} \varphi \, l_{33}^{ee}(\varphi)$$

$$L_{13}^{ee} = L_{31}^{ee} = \frac{n_e \rho_{e0}^2}{\tau_e} \mathscr{J}^2 \tfrac{1}{2} \sqrt{\tfrac{5}{2}} \varphi \, l_{13}^{ee}(\varphi)$$

$$L_{33}^{ii} = \frac{n_i \rho_{i0}^2}{\tau_i} \mathscr{J}^2 \tfrac{5}{4} \varphi \, l_{33}^{ii}(\varphi)$$

Mixed diffusive coefficients

$$L_{13}^{ei} = \frac{T_i}{T_e} L_{31}^{ie} = \frac{n_e \rho_{e0}^2}{\tau_e} \frac{T_i}{Z T_e} \mathscr{J}^2 \tfrac{1}{2} \sqrt{\tfrac{5}{2}} \varphi \, l_{13}^{ei}(\varphi)$$

$$L_{33}^{ei} = \frac{T_i}{T_e} L_{33}^{ie} = \frac{n_e \rho_{e0}^2}{\tau_e} \frac{T_i}{T_e} \mathscr{J}^2 \tfrac{5}{4} \varphi \, l_{33}^{ei}(\varphi)$$

Electrical coefficients

$$L_{1E}^{e} = -L_{E1}^{e} = -\frac{n_e c}{\mathscr{B}_0} \mathscr{J} \varphi \, l_{1E}^{e}(\varphi)$$

$$L_{3E}^{e} = -L_{E3}^{e} = -\frac{n_e c}{\mathscr{B}_0} \mathscr{J} \sqrt{\tfrac{5}{2}} \varphi \, l_{3E}^{e}(\varphi)$$

$$L_{3E}^{i} = -L_{E3}^{i} = -\frac{n_i c}{\mathscr{B}_0} \mathscr{J} \sqrt{\tfrac{5}{2}} \varphi \, l_{3E}^{i}(\varphi)$$

$$L_{EE} = \frac{n_e e^2}{m_e} \tau_e \varphi \, l_{EE}(\varphi)$$

All these coefficients are functions of the neoclassical factor φ. The formulae given here are valid for *any positive value of* φ.

Finally, table 3.4 gives the relation between the dimensional and the dimensionless banana transport coefficients. These relations are most conveniently expressed in terms of the reference Larmor radius $\rho_{\alpha 0}$, introduced in

(**13**.2.20),

$$\rho_{\alpha 0}^2 = \frac{V_{T\alpha}^2}{\Omega_{\alpha 0}^2} = \left(\frac{m_\alpha c}{e_\alpha \mathscr{B}_0}\right)^2 \cdot 2\frac{T_\alpha}{m_\alpha}. \tag{3.22}$$

We also introduced the specific symbol $\mathscr{J}$ for the following, frequently occurring combination (see eq. **13**.3.12)

$$\mathscr{J} = \frac{\mathscr{I} l_\rho}{R_0 \mathscr{B}_P}. \tag{3.23}$$

The comparison of the banana transport coefficients (table 3.4) with the corresponding classical transport coefficients (table **13**.2.1) and the Pfirsch–Schlüter coefficients (table **13**.3.1) is now very easy.

15.4. Numerical values of the transport coefficients. Limiting values. Convergence of the approximation procedure

The formulae of section 15.3 enable us to calculate very simply the values of all the banana transport and quasi-transport coefficients over the whole range of values of the *neoclassical factor* φ and of the *charge number* Z.

The only limitation in these formulae is condition (3.20) or (3.21). Whenever this condition is violated, the transport coefficients must be calculated more precisely. This means that eqs. (2.3), (2.4) and (2.6) must be solved without any approximation. There is no difficulty at all in performing this operation: it just requires patience and leads to more cumbersome formulae. The main new feature in that case is the following: the complete transport coefficients depend, in addition to φ and Z, on a third dimensionless variable, the temperature ratio (T_e/T_i). For simplicity, we do not discuss the general case here, as condition (3.21) is satisfied in most situations of interest for fusion.

In the present section we only discuss the numerical and graphical aspects of the transport coefficients, a detailed physical discussion will be given in section 15.5.

A first set of data is obtained by plotting the dimensionless transport coefficients $\varphi l_{pq}^{\alpha\beta}(\varphi)$ as functions of φ for a given charge number. The graphs for $Z = 1$ are shown in figs. 4.1–4.4. For each transport coefficient we give a pair of pictures. One of them shows the global behaviour over a wide range of values of the neoclassical factor $(0 \leqslant \varphi \leqslant 20)$; the other one gives a more detailed picture of the behaviour for small values of φ $(0 \leqslant \varphi \leqslant 1)$: numerical values can easily be determined graphically from these figures.

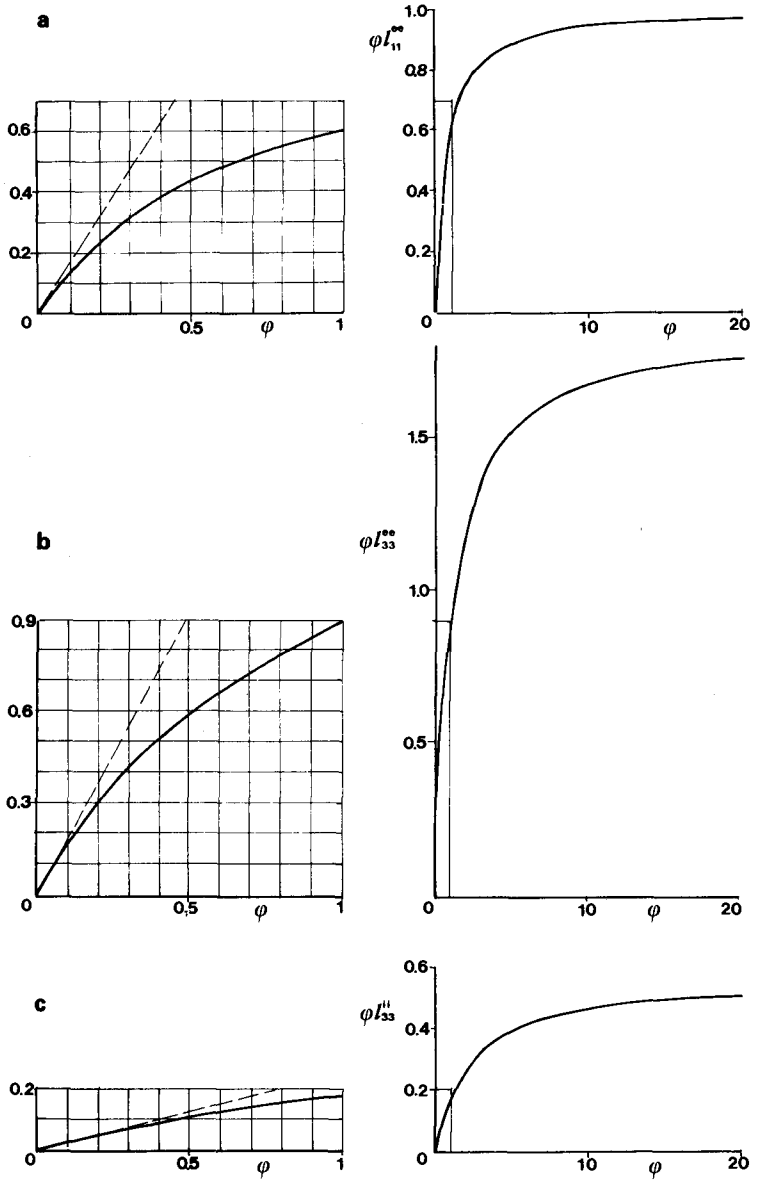

Fig. 4.1. Diagonal diffusive coefficients: (a) electron diffusion coefficient, (b) electron thermal conductivity, (c) ion thermal conductivity

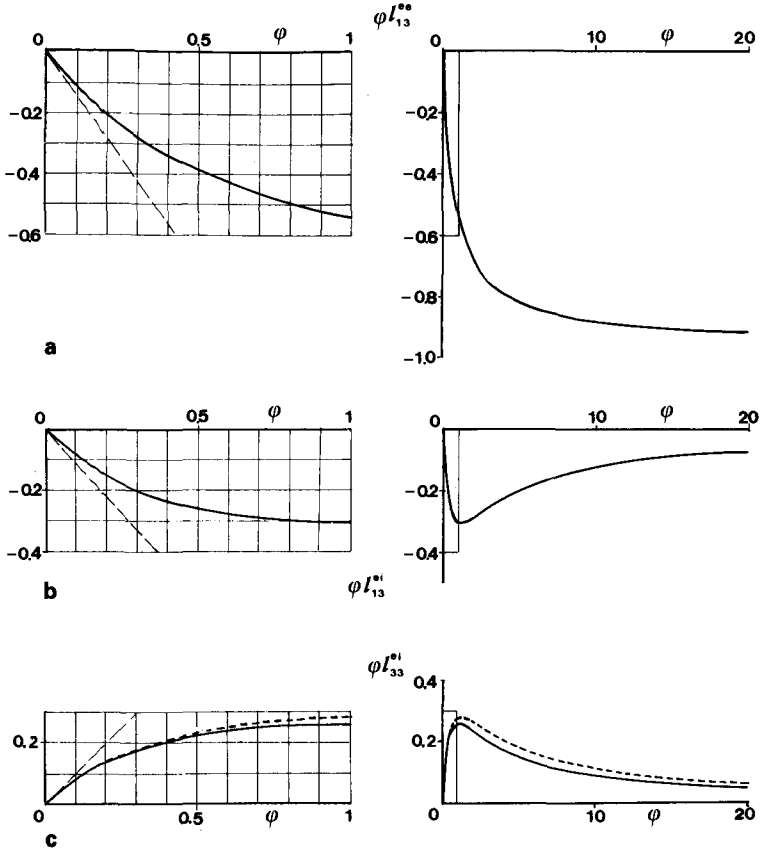

Fig. 4.2. Non-diagonal diffusive transport coefficients· (a) electron thermodiffusion coefficient, (b) mixed thermodiffusion coefficient, (c) mixed thermal conductivity.

In each graph, a solid line represents the 21M result and a dotted line represents the 13M approximation (whenever it is distinguishable from the former).

The *dimensionless transport coefficients* have two common features: *they all start linearly from the origin* and *they all tend toward a finite limiting value for large* φ. In between, two different types of behaviour are found: for most coefficients, the variation between these two limits is monotonic. But for some of them (φl^{ei}_{13}, φl^{ei}_{33}, φl^{i}_{3E}, φl^{e}_{3E}, φl^{e}_{5E}) the asymptotic value is zero. It is very interesting to have explicit formulae for these two limiting values.

The *limit* $\varphi \to 0$ is easily obtained from tables 3.2 or 3.3: the coefficients $l^{\alpha\beta}_{pq}(0)$, which represent the *slope at the origin* of the banana transport coeffi-

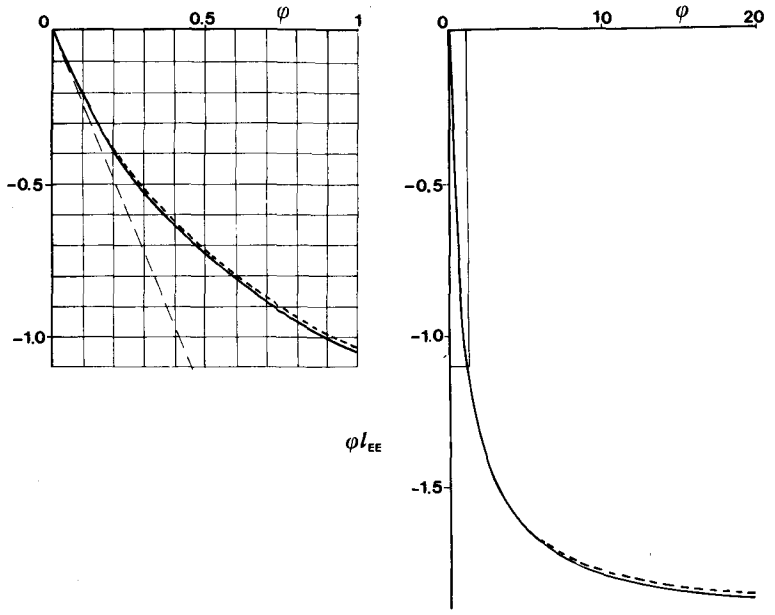

Fig 4.3. Parallel electrical conductivity

cients are simply constants, collected in table 4.1. These numbers are directly comparable with the only explicit numbers available in the literature, i.e. the values given by Hinton and Hazeltine (1976). The comparison is shown in the same table.

The *limit* $\varphi \to \infty$ offers a surprisingly simple form. In order to exhibit this simplicity, the following identities should be noted; they are easily checked by using the definitions of eq. (5.3.26) and the expressions in table 3.1:

- In the 13M approximation:

$$\Delta^{c}_{13} = \left(D^{e}_{13}\right)^{-1}. \tag{4.1}$$

- In the 21M approximation:

$$\Phi^{c}_{135} = \left(F^{c}_{135}\right)^{-1}. \tag{4.2}$$

These relations actually result from the fact that, for each truncation level, the matrix of the parallel quasi-transport coefficients is the inverse of the collision matrix. This point is more fully developed in chapter 17 (see also the

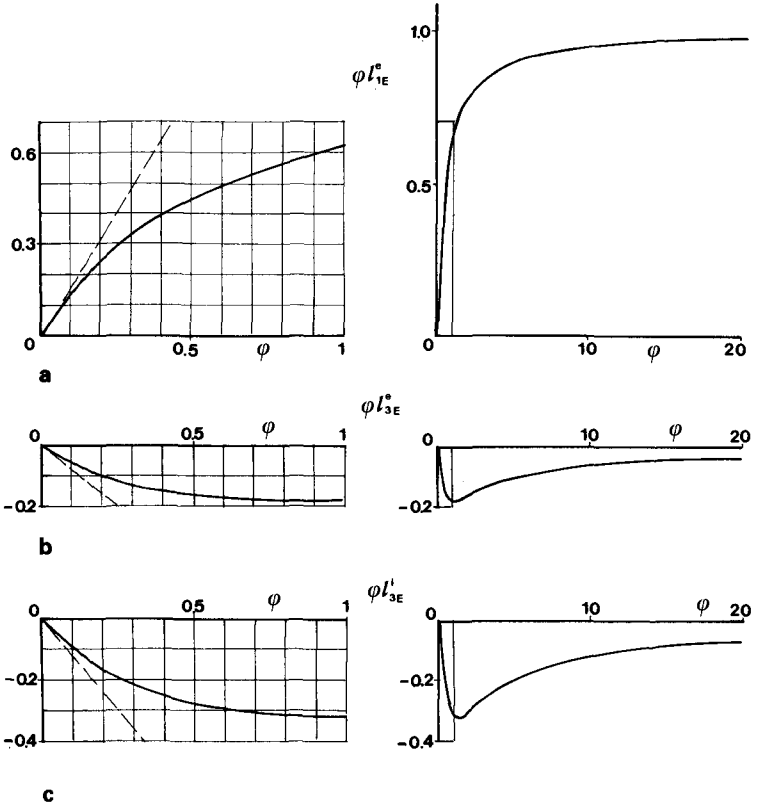

Fig. 4.4. Non-diagonal electrical transport coefficients: (a) Ware–Galeev pinch coefficient, (b, c) transverse thermomagnetic coefficients.

end of section 5.2). The following asymptotic forms then follow from (3.18):

$$D_e(\varphi) \to \varphi^2 \frac{\mathscr{D}^e_{13}}{D^e_{13}},$$

$$D_i(\varphi) \to \varphi \frac{\mathscr{D}^i_{13}}{D^i_{13}}, \quad \varphi \to \infty, \qquad [13M], \tag{4.3}$$

and

$$M_e(\varphi) \to \varphi^3 \frac{\mathscr{F}^e_{135}}{F^e_{135}},$$

$$M_i(\varphi) \to \varphi^2 \frac{\mathscr{F}^i_{135}}{D^i_{35}}, \quad \varphi \to \infty, \qquad [21M]. \tag{4.4}$$

Table 4.1
Initial slopes of the banana transport coefficients.

Coefficient	Analytic expression	Numerical value for $Z=1$		
		13M	21M	[HH76] *
Pure diffusive coefficients				
$l^{ee}_{11}(0)$	μ^e_{11}	1.533	1.533	1.526
$l^{ee}_{33}(0)$	μ^e_{33}	1.854	1.854	1.846
$l^{ii}_{33}(0)$	$\mathscr{D}^i_{13}/\mu^i_{11}$	0.261	0.261	
$l^{ee}_{13}(0)$	μ^e_{13}	−1.344	−1.344	−1.338
Mixed diffusive coefficients				
$l^{ei}_{13}(0)$	$\mu^e_{11}\mu^i_{13}/\mu^i_{11}$	−1.137	−1.137	−1.099
$l^{ei}_{33}(0)$	$\mu^e_{13}\mu^i_{13}/\mu^i_{11}$	0.997	0.997	0.965
Electrical coefficients				
$l^e_{1E}(0)$	$\sigma\mu^e_{11}-\alpha\mu^e_{13}-\gamma\mu^e_{15}$	1.641	1.651	1.673
$l^e_{3E}(0)$	$\sigma\mu^e_{31}-\alpha\mu^e_{33}-\gamma\mu^e_{35}$	−0.775	−0.761	−0.753
$l^i_{3E}(0)$	$(\mu^i_{13}/\mu^i_{11})l^e_{1E}(0)$	−1.217	−1.225	−1.206
$l_{EE}(0)$	$-\sigma l^e_{1E}(0)+\alpha l^e_{3E}(0)+\gamma l^e_{5E}(0)$	−2.409	−2.484	−3.830

* The values in this column were calculated from the results of Hinton and Hazeltine (1976).

For the derivation of the asymptotic forms of the electrical coefficients one also uses the following, easily checked identities (see Appendix **17**A.1, eq. (**17**A1.7)]:

$$\sigma\Delta^e_{35}-\alpha\chi^e_{153}-\gamma\chi^e_{135}=\Phi^e_{135},$$

$$\sigma\chi^e_{153}-\alpha\Delta^e_{15}+\gamma\chi^e_{315}=0,$$

$$\sigma\chi^e_{135}+\alpha\chi^e_{315}-\gamma\Delta^e_{13}=0. \tag{4.5}$$

A short calculation now shows that the limiting values of the banana transport coefficients have the very simple forms collected in table 4.2. We stress that these forms are *universal values, independent of the truncation level.* The physical significance of these results will be discussed in detail in the next section.

Next, we comment on the *convergence of the generalized moment method* used here for the calculation of the banana transport coefficients. To the best

Table 4.2
Asymptotic values of the banana transport coefficients for $\varphi \to \infty$

Coefficient	Value	Coefficient	Value
Diffusive coefficients		**Electrical coefficients**	
$\varphi l_{11}^{ee}(\varphi)$	c_{11}^{e}	$\varphi l_{1F}^{e}(\varphi)$	1
$\varphi l_{33}^{ee}(\varphi)$	c_{33}^{e}	$\varphi l_{3E}^{e}(\varphi)$	0
$\varphi l_{33}^{ii}(\varphi)$	c_{33}^{i}	$\varphi l_{3F}^{i}(\varphi)$	0
$\varphi l_{13}^{ee}(\varphi)$	$-c_{13}^{e}$	$\varphi l_{EE}(\varphi)$	$-\sigma$
$\varphi l_{13}^{ei}(\varphi)$	0		
$\varphi l_{33}^{ei}(\varphi)$	0		

of our knowledge, the results given here are the only ones in the available literature that are capable of addressing this question. The previous results of Hinton & Hazeltine (1976), of Hirshman and Sigmar (1981) and of Shaing and Callen (1983) are essentially equivalent to the 13M approximation.

For almost all the banana transport coefficients, the difference between the 13M and the 21M approximation is so slight that the corresponding graphs are undistinguishable on the scale drawn in fig. 4.1 [only for φl_{33}^{ei} can we note a very slight difference]. To make this point more precise, we collected in table 4.3 the numerical values of all the banana transport coefficients $\varphi l_{pq}^{\alpha\beta}(\varphi)$ for the particular value $\varphi = 1$: the numbers speak for themselves.

We also note that the asymptotic values of all the banana coefficients (table 4.2) are expressed by *universal formulae*, independent of the level of truncation. The same statement holds true for the initial slopes (table 4.1) of the *diffusive* transport coefficients (both pure and mixed). For the electrical coefficients, the expressions of the initial slopes contain contributions of additional quasi-transport coefficients as we increase the level of truncation.

This *fast rate of convergence* of the banana transport coefficients is quite surprising as compared to the relatively large difference between 13M and 21M coefficients in the classical case (chapter 5); it is only beyond the 21M that the rate of convergence becomes very fast in that case. It could be objected that the comparison should be made with the large-field, drift approximation limits of the classical coefficients, which are expressed by a universal formula (section 5.6 or 13.2). This however does not explain the present situation, because the expressions in tables 3.1, 3.2 involve combinations of *parallel* transport coefficients, which are quite sensitive to the truncation level (see fig. 5.4.1). Moreover, we note that the banana coefficients

Table 4.3
Numerical values of the banana transport coefficients

Coefficient	Numerical values for $\varphi = 1$, $Z = 1$	
	13M	21M
Diffusive coefficients		
$l_{11}^{ee}(1)$	0.604	0.604
$l_{33}^{ee}(1)$	0.894	0.890
$l_{33}^{ii}(1)$	0.179	0.174
$l_{13}^{ee}(1)$	−0.548	−0.549
$l_{13}^{ei}(1)$	−0 307	−0 303
$l_{33}^{ei}(1)$	0 278	0 258
Electrical coefficients		
$l_{1E}^{e}(1)$	0.629	0 629
$l_{3E}^{e}(1)$	−0.180	−0.179
$l_{3E}^{i}(1)$	−0.319	−0.316
$l_{EE}(1)$	−1.039	−1.057

are expressed as ratios of polynomials in φ of different degrees in different approximations (for instance, φl_{11}^{ee} is a ratio of two 2-*d* degree polynomials in 13M, and as a ratio of two 3-*d* degree polynomials in 21M). We may only say that the differences in the values of the parallel transport coefficients are compensated by the involvement of the quasi-transport coefficients γ, δ^{α}, ϵ^{α} in the 21M approximation.

We now discuss the *dependence of the banana transport coefficients on the charge number Z*. We collected in fig. 4.5 the graphs of these coefficients as functions of φ, for three particular values of Z: 1, 4 and ∞. We can see that the response of the various coefficients to the variation in Z is very different. The ion thermal conductivity φl_{33}^{ii} is strictly independent of Z (because it is expressed only in terms of c_{pq}^{i} and μ_{pq}^{i} which are independent of Z). The coefficients φl_{11}^{ee}, φl_{13}^{ee}, φl_{13}^{ei} are very slightly influenced by a change in Z. On the contrary, the "*purely thermal*" coefficients φl_{33}^{ee}, φl_{33}^{ei} display a rather strong dependence on Z. The *electrical conductivity* is a rather interesting case. The banana contribution is strongly dependent on Z, but in the *total* conductivity $\sigma^{CL} + \sigma^{B}$, the effect of Z on σ^{B} is compensated by its effect on σ^{CL}. The *electrical cross-coefficients* have a peculiar behaviour. φl_{1E}^{e} and φl_{E3}^{i} are almost independent of Z; the remaining coefficient, φl_{E3}^{e}, is quite different: it decreases monotonically as Z increases and vanishes uniformly as $Z = \infty$. (This peculiarity was also noted by Hinton and Hazeltine 1976.)

15.5. Discussion of the banana transport coefficients

The main result of the present chapter is the set of *transport equations* (3.1)–(3.3), relating a set of four *surface-averaged fluxes* to a set of four *surface-averaged thermodynamic forces*. The very existence of such relations between *delocalized quantities* in a regime where *collisions are very rare* is a novelty from the point of view of "orthodox" non-equilibrium thermodynamics and is worth a deeper study. This will be done in chapter 17.

It should be kept in mind that the transport equations mentioned above define only a part of the total fluxes, namely the *banana fluxes* (see table **12**.9.1). To these must be added the *classical fluxes* and the *Pfirsch–Schlüter fluxes*. (It was shown in section **13**.1 that the electric drift fluxes give negligible contributions.) It is a remarkable fact that all three contributions are related to the forces by linear relations of the same type. Indeed, comparing eqs. (**13**.2.9), (**13**.3.10) and (3.12), we may write for the *total average fluxes* the global expression

$$J_\mu = \sum_{\nu=1}^{4} \left(\mathscr{L}_{\mu\nu}^{\mathrm{CL}} + \mathscr{L}_{\mu\nu}^{\mathrm{PS}} + \mathscr{L}_{\mu\nu}^{\mathrm{B}} \right) X_\nu . \tag{5.1}$$

A first remark concerns the *form* of the three partial transport matrices, which are quite different. Putting together our earlier results (**13**.2.10), (**13**.3.11) and (3.13), and writting explicitly only the non-zero elements, we find

$$\mathscr{L}^{\mathrm{CL}} = \begin{pmatrix} \mathscr{L}_{11}^{\mathrm{CL}} & \mathscr{L}_{12}^{\mathrm{CL}} & 0 & 0 \\ \mathscr{L}_{21}^{\mathrm{CL}} & \mathscr{L}_{22}^{\mathrm{CL}} & 0 & 0 \\ 0 & 0 & \mathscr{L}_{33}^{\mathrm{CL}} & 0 \\ 0 & 0 & 0 & \mathscr{L}_{44}^{\mathrm{CL}} \end{pmatrix}, \tag{5.2}$$

$$\mathscr{L}^{\mathrm{PS}} = \begin{pmatrix} \mathscr{L}_{11}^{\mathrm{PS}} & \mathscr{L}_{12}^{\mathrm{PS}} & 0 & 0 \\ \mathscr{L}_{21}^{\mathrm{PS}} & \mathscr{L}_{22}^{\mathrm{PS}} & 0 & 0 \\ 0 & 0 & \mathscr{L}_{33}^{\mathrm{PS}} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{5.3}$$

$$\mathscr{L}^{\mathrm{B}} = \begin{pmatrix} \mathscr{L}_{11}^{\mathrm{B}} & \mathscr{L}_{12}^{\mathrm{B}} & \mathscr{L}_{13}^{\mathrm{B}} & \mathscr{L}_{14}^{\mathrm{B}} \\ \mathscr{L}_{21}^{\mathrm{B}} & \mathscr{L}_{22}^{\mathrm{B}} & \mathscr{L}_{23}^{\mathrm{B}} & \mathscr{L}_{24}^{\mathrm{B}} \\ \mathscr{L}_{31}^{\mathrm{B}} & \mathscr{L}_{32}^{\mathrm{B}} & \mathscr{L}_{33}^{\mathrm{B}} & \mathscr{L}_{34}^{\mathrm{B}} \\ \mathscr{L}_{41}^{\mathrm{B}} & \mathscr{L}_{42}^{\mathrm{B}} & \mathscr{L}_{43}^{\mathrm{B}} & \mathscr{L}_{44}^{\mathrm{B}} \end{pmatrix}. \tag{5.4}$$

We immediately note that, contrary to the classical and the PS case, *the banana transport matrix is completely full*: all its elements are different from zero. This means that there are *many more cross-effects* in this regime. It will be seen in chapter 17 that this situation introduces some difficulties in the thermodynamic interpretation of the results.

A *general signature of the banana transport coefficients* is the presence of the *neoclassical factor* φ. In the limit $\varphi \ll 1$, all the coefficients become simply

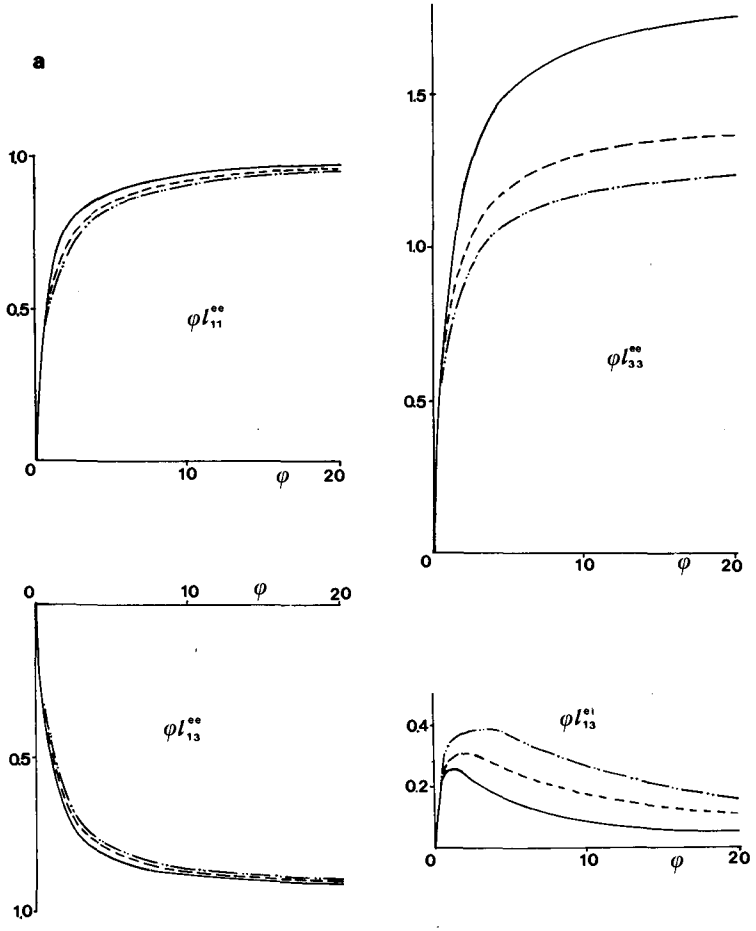

Fig. 4.5. Dependence of the banana transport coefficients on the charge number Z. Solid line: $Z = 1$; dashed line: $Z = 4$; dashed–dotted line: $Z = \infty$. (a) Diffusive coefficients; (b) electrical cross coefficients, (c) banana electrical conductivity and total electrical conductivity: $\sigma^{tot} = \sigma^{B} + \sigma^{CL}$.

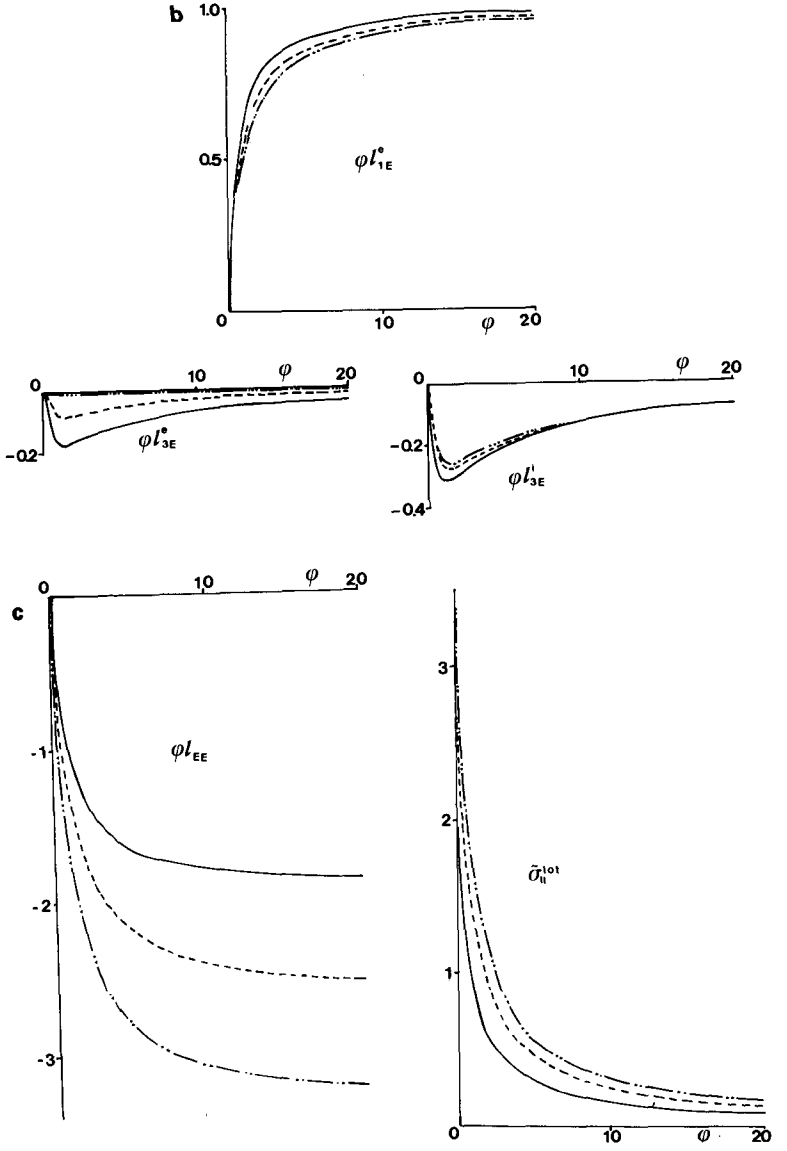

Fig. 4.5. Continued.

proportional to φ. We recall that this factor is roughly interpreted as the ratio of the number of trapped particles to the number of passing particles. This signature results from the predominant role played by the trapped (especially

the "barely trapped") particles in this transport mechanism (as appears clearly in the derivation of chapter 14) *.

Another important common feature of the banana transport coefficients is their expression as *combinations of the pseudo-viscosity coefficients* μ^{α}_{pq} on one hand, and of the *classical parallel transport and quasi-transport coefficients* (σ, α, κ^{α}; γ, δ^{α}, ϵ^{α}) on the other hand. These expressions arise quite naturally in our formulation (but were not noticed by previous investigators). It is precisely from an analysis of these forms that we were able to establish the interconnections between banana transport coefficients found in section 15.2. These demonstrate the deep structural coherence of the banana transport theory.

We now proceed to a more detailed discussion of the transport coefficients, which will be adequately subdivided into two groups, viz. The *diffusive transport coefficients* (pure and mixed) and the *electrical transport coefficients*. In each group, we can further distinguish between the *diagonal* and the *non-diagonal* transport coefficients.

A. Diffusive transport coefficients

A.1. Diagonal diffusive transport coefficients: L^{ee}_{11}, L^{ee}_{33}, L^{ii}_{33}

These describe the "direct" response of the plasma to the presence of a thermodynamic force.

A radial pressure gradient produces a radial particle flux (Fick's law). The corresponding coefficient is the *radial diffusion coefficient*.

A radial electron (ion) temperature gradient produces a radial electron (ion) heat flux (Fourier's law). The corresponding coefficient is the *radial electron (ion) thermal conductivity*.

These three coefficients have a remarkable common form:

$$L^{\alpha\alpha}_{rr} = A_r \frac{n_\alpha \rho^2_{\alpha 0}}{\tau_\alpha} \mathscr{J}^2 \, \varphi l^{\alpha\alpha}_{rr}(\varphi), \quad r = 1, 3, \quad \alpha = \mathrm{e}, \mathrm{i}, \tag{5.5}$$

where A_r is a numerical factor (1/2 or 5/4) and $\mathscr{J}$ was defined in (3.23). We know that $l^{\alpha\alpha}_{rr}(\varphi)$ is a dimensionless quantity which, for all values of φ, remains bounded (see figs. 4.1).

An important property of these coefficients is their dependence of the relaxation time τ_α (or on the collision frequency $\nu_\alpha \equiv \tau_\alpha^{-1}$). We see that *all the*

* In many papers a random-walk type of argument is given in order to provide a rough estimate of the diffusion coefficient from the width of a banana orbit (see, e.g., Hinton and Hazeltine 1976, sec. III.D). I personally do not feel very happy with this kind of argument, which does not explain all the transport coefficients, and always contains some arbitrary assumptions which can only be justified a posteriori.

diagonal diffusive coefficients are proportional to the collision frequency. They share this property with the *classical perpendicular transport coefficients* in the limit of a very strong magnetic field, as can be seen in table **13**.2.1. The same property holds for the *Pfirsch–Schlüter transport coefficients* exhibited in table **13**.3.1. The origin of this important property was discussed in detail in section **5**.6. If, however, the coefficients are plotted as functions of the collision frequency ν_α, the *slopes* of the straight line are found to be very different in the three cases. We now discuss this feature.

In a first step, we consider $\varphi \ll 1$, in which case $A_r l_{rr}^{\alpha\alpha}(\varphi)$ is approximately constant, of order 1. We note, by comparing (5.5) with the expressions of tables **13**.2.1 and **13**.3.1, that the factor $n_\alpha \rho_{\alpha 0}^2/\tau_\alpha$ is a measure of the classical diffusive coefficients. We thus find the following, very illuminating ratios:

$$\mathscr{L}^{\mathrm{CL}} : \mathscr{L}^{\mathrm{PS}} : \mathscr{L}^{\mathrm{B}} \approx 1 : \mathscr{J}^2 \frac{\mathscr{G}-1}{\mathscr{G}} : \mathscr{J}^2 \frac{\varphi}{\mathscr{G}}. \tag{5.6}$$

Thus $\mathscr{G}^{-1}\mathscr{J}^2\varphi$ deserves the name of *banana amplification factor*: it is quite a large number. Indeed, the factor $\mathscr{J}$ is easily identified from (**14**.2.8) as the ratio of the toroidal to the poloidal magnetic field components,

$$\mathscr{J} = \frac{B_\zeta}{B_\theta} \gg 1. \tag{5.7}$$

We know that this ratio is very large in a tokamak configuration. In the standard model of inverse aspect ratio $\bar{\eta}$, we obtain from the results of section **13**.4 [eqs. (**13**.4.3, **13**.4.4 and **13**.4.6)]:

$$\mathscr{J} \simeq \frac{q(r)R_0}{r} \sim q\eta^{-1}. \tag{5.8}$$

If we combine this value with (**13**.4.15) and (**14**.8.11), we find the following estimate for the banana amplification factor:

$$\mathscr{G}^{-1}\mathscr{J}^2\varphi \simeq 1.469\, q^2\eta^{-2}\eta^{1/2} = 1.469\, q^2\eta^{-3/2}. \tag{5.9}$$

Hence, we find in the banana regime, an additional, *very strong enhancement of the radial diffusion coefficient and of both thermal conductivities*, by a factor $\eta^{-3/2}$ over the Pfirsch–Schlüter values, i.e. by a factor $q^2\eta^{-3/2}$ over the classical value.

This is again a *geometrical effect on the transport processes*, which goes to zero when the magnetic field is straight and homogeneous, .because no trapping is possible in such a field (see eq. **14**.8.2).

It can be concluded that, in the long mean free path regime, the banana contribution to the diagonal diffusive transport coefficients strongly dominates

the classical and the Pfirsch–Schlüter contributions, which may be safely neglected.

We note that the three diagonal diffusive coefficients are *definite positive* for all (positive) values of φ: this is in agreement with the requirements of non-equilibrium thermodynamics.

In a tokamak configuration, the total pressure and the two temperatures are always decreasing radially from the magnetic axis outwards; hence

$$-\frac{\partial p}{\partial \rho} \geqslant 0, \qquad -\frac{\partial T_\alpha}{\partial \rho} \geqslant 0.$$

It then follows that *the particle and heat fluxes due to the diagonal transport processes are always directed outwards.*

It is also important to compare the diagonal diffusive transport coefficients among themselves. In particular, we see that the ratio of the ionic to the electronic thermal conductivities is, in order of magnitude,

$$\frac{L_{33}^{\mathrm{iiB}}}{L_{33}^{\mathrm{eeB}}} \approx \frac{n_\mathrm{i}\rho_{\mathrm{i}0}^2}{\tau_\mathrm{i}} \frac{\tau_\mathrm{e}}{n_\mathrm{e}\rho_{\mathrm{e}0}^2} \approx \left(\frac{m_\mathrm{i}}{m_\mathrm{e}}\right)^{1/2} \left(\frac{T_\mathrm{e}}{T_\mathrm{i}}\right)^{1/2}. \tag{5.10}$$

Thus, *the ionic heat flux clearly dominates the electronic heat flux.* This conclusion was also reached in the classical, strong magnetic field regime (see eq. **5.6.12**).

We now consider the dependence of the diffusive transport coefficients on the *neoclassical factor* φ: it is determined by the dimensionless functions $\varphi l_{rr}^{\alpha\alpha}(\varphi)$. It is noteworthy that, by the present method, we obtain explicit analytical expressions for these functions, covering all values of φ from zero to infinity (tables 3.2, 3.3); no limitation on the values of φ was found necessary in the calculation. [The difficult part of the problem is the *actual* calculation of the neoclassical factor from (**14**.8.8) and (**14**.8.1) for a *given geometry of the magnetic field configuration*.] The variation of the diagonal diffusive coefficients with the neoclassical factor is shown in fig. 4.1.

The first calculation of the banana fluxes (Galeev and Sagdeev 1968, Rosenbluth et al. 1972, Hinton and Hazeltine 1976) yielded analytical results only in the limit $\varphi \to 0$; for larger values of φ, they resorted to variational methods. It was the development of the moment-like methods of Hirshman and Sigmar (1981) which opened the possibility of an explicit calculation of the banana fluxes for arbitrary φ. Unfortunately, these authors did not publish any results for the simple electron–ion plasma.

It was shown in sec. 15.4 (table 4.1) that the coefficients $\varphi l_{rr}^{\alpha\alpha}(\varphi)$ become simply proportional to φ in the limit $\varphi \to 0$. It is also interesting to discuss the opposite limiting values, for $\varphi \to \infty$. Although physically unrealizable, this

limit is very interesting from a conceptual point of view. If the standard interpretation of the neoclassical factor is accepted, it represents a situation where *all the particles are trapped* ($f_P = 0$).

This problem was also considered by Hazeltine et al. (1973), where they interpret it as the limit of a "fat torus", $a/R \to 1$. But, as explained above, their method does not allow for a continuous variation of φ from 0 to ∞. They calculated only the "limiting value", ab initio, by setting $\overline{G}_1^{\alpha(0)} = 0$ in our eq. (**14**.4.16). Their "limiting value" does not agree with ours: the reason for the discrepancy is discussed under subsection B.1 below.

In our formalism, these asymptotic values are very easily calculated, as shown in section 15.4; the results are collected in table 4.2. The contrast between the two limiting situations is very illuminating. The banana diffusive coefficients in the small-φ limit are *entirely* determined by the *pseudo-viscosity coefficients* μ_{rr}^{α}; at the other extreme (large-φ limit), they are *entirely* determined by the collision matrix elements c_{rr}^{α} or, what amounts to the same, by the *parallel transport coefficients*. This expresses the fact that, basically, the banana transport coefficients result from a subtle balance between the generalized stresses and the generalized frictions. As a result, the general expressions are complicated combinations of factors μ_{rs}^{α} and c_{rs}^{α} relating, respectively, to these two quantities (tables 3.2, 3.3). What we see in tables 4.1 and 4.2 is that the generalized *stresses* "win" in the small-φ limit, whereas the generalized *frictions* "take over" in the large-φ limit. (This point was not clearly recognized by previous authors.)

It is also important to recall that this conclusion, and the values of the transport coefficients in both limits, are *universally valid*, independently of the truncation level. Actually, the large-φ limits of the diffusion and thermal conductivity coefficients equal exactly the *asymptotic perpendicular conductivity* $\tilde{\sigma}_{\perp}^{\infty}$ *and thermal conductivities* $\tilde{\kappa}_{\perp}^{\alpha\infty}$, as defined by (**5**.6.2), *amplified by the factor* $\mathscr{G}^{-1}\mathscr{J}^2$.

As regards the numerical values of the diffusive coefficients, they can be compared, *in the small-φ limit*, to the "standard" reference, i.e. the review paper by Hinton and Hazeltine (1976). The comparison is given in table 4.1, where we see that these coefficients agree quite well with those calculated by the former authors. * A further discussion of the thermodynamic aspects of the plasma diffusion process will be given in section 17.5.

* It could be mentioned at this point that the calculation of the *ionic* fluxes by the method of Rosenbluth et al. is particularly "tricky"; it involves extra assumptions ("weak coupling") and an additional fitting of a parameter ("y"). In other words their result strongly depends on the details of the modelling of the collision term. Such unpleasant features are absent from the present treatment.

A.2. Non-diagonal transport coefficients: Diffusive cross-effects: L_{13}^{ee}, L_{31}^{ee}; L_{13}^{ei}, L_{31}^{ie}; L_{33}^{ei}, L_{33}^{ie}

It appears from table 3.4 that these coefficients have a common form, similar to the form (5.5) of the diagonal diffusive coefficients,

$$L_{rs}^{e\beta} = A_{rs} \frac{n_e \rho_{e0}^2}{\tau_e} \frac{T_\beta}{|Z_\beta| T_e} \mathscr{J}^2 \varphi l_{rs}^{e\beta}(\varphi). \tag{5.11}$$

This implies that the banana diffusive cross-effects are *strongly enhanced*, compared to their classical and Pfirsch–Schlüter counterparts; the amplification factor is again $\mathscr{G}^{-1}\mathscr{J}^2\varphi$, thus eq. (5.6) holds for these coefficients as well.

The physical interpretation of these cross-coefficients is easily found (see, e.g., de Groot and Mazur 1984). The presence of a coefficient L_{13}^{ee} implies that a radial electron temperature gradient produces a radial electron flux: this is the well-known *thermodiffusion effect* (or *Soret effect*). Note that $L_{13}^{ee} < 0$, hence the radial thermodiffusion flux is directed *inwards*. The thermodiffusion effect thus partly compensates the strong outward diffusion flux [curiously, this point was not stressed by previous authors, except in a recent paper by Becker (1987)].

The reciprocal effect is described by the coefficient L_{31}^{ee}: a radial pressure gradient produces a radial electron heat flux (*electronic Dufour effect*). Here again, $L_{31}^{ee} < 0$, hence the Dufour heat flux is directed *inwards*. The Soret and Dufour effects are interrelated by a simple Onsager symmetry relation

$$L_{13}^{ee} = L_{31}^{ee}.$$

The remaining cross-coefficients represent new effects, not met in our previous studies: the corresponding transport coefficients vanish for the classical and the PS fluxes (see eqs. 5.2, 5.3).

We first recall that these mixed diffusive coefficients are closely connected to the pure diffusive coefficients, as results from eqs. (2.11) and (2.16) (see also tables 3.2 and 3.3). Thus, L_{13}^{ei} is proportional to the diffusion coefficient L_{11}^{ee}, and L_{33}^{ei} is proportional to the thermodiffusion coefficient L_{31}^{ee} (with the *same* constant of proportionality).

The physical effects produced by the mixed diffusive coefficients can be described as follows:

– A radial electron flux is produced by a radial ion temperature gradient (*mixed thermodiffusion*); it is measured by the coefficient L_{13}^{ei} and is directed inwards. It therefore reinforces the inward flux produced by the electron temperature gradient.

– A radial *electron* heat flux is produced by a radial *ion* temperature gradient (*mixed heat conduction*); it is measured by L_{33}^{ei} and is directed outwards.

– A radial ion heat flux is produced by a radial pressure gradient (*ion Dufour effect*); it is measured by L_{31}^{ie} and is directed inwards.
– A radial *ion* heat flux is produced by a radial *electron* temperature gradient (*mixed heat conduction*); it is measured by L_{33}^{ie} and is directed outwards.

The relations between reciprocal mixed diffusive coefficients require a generalization of the Onsager relations. Indeed, the reference state being a two-temperature state (hence not a true equilibrium state), the generalized Onsager relations that come out of the calculation reflect this temperature difference between electrons and ions:

$$T_e L_{13}^{ei} = T_i L_{31}^{ie}, \qquad T_e L_{33}^{ei} = T_i L_{33}^{ie}.$$

It is again interesting to study the limits of these coefficients for $\varphi \to 0$ and for $\varphi \to \infty$. These are collected in tables 4.1 and 4.2.

We find here the same characteristics as for the diagonal diffusive coefficients. In the small-φ limit (few trapped particles), they are determined solely by the pseudo-viscosity coefficients μ_{rs}^{α}, whereas in the large-φ limit, they are determined by the collision matrix elements c_{rs}^{e}.

The large-φ limit of $\varphi l_{13}^{ee}(\varphi)$ agrees exactly *in size* with the asymptotic classical perpendicular thermoelectric coefficient $\tilde{\alpha}_{\perp}^{\infty}$ (eq. **5**.6.2), *amplified by the factor* $\mathscr{G}^{-1}\mathscr{J}^{2}$, but the *sign* is opposite. (This difference merely stems from the classical definition of α, in which the temperature gradient is related to an *electric current*, rather than an *electron flux*: in the electric current, the electron flux is counted negatively.)

The mixed diffusive coefficients vanish in both limits, $\varphi = 0$ and $\varphi = \infty$. Their effects are relatively small. For instance, the ion heat flux due to the electron temperature gradient is always smaller by a factor $(m_e/m_i)^{1/2}$ compared to the contribution of the ion temperature gradient.

B. Electrical transport coefficients

We now go over to the class of transport phenomena involving the electric field and the electric current. More important, they involve thermodynamic forces and fluxes in the *parallel direction*, as opposed to the diffusive processes, which all take place in the radial direction. We first discuss the diagonal transport coefficient.

B.1. The parallel electrical conductivity: L_{EE}

Here we have a purely parallel transport process: a parallel average electric field produces a parallel average electric current. Thus, if the cross-effects were neglected, eq. (3.11), combined with the classical equation (**13**.2.8) yields the following expression for the *total* average parallel electric current:

$$\mathscr{B}_0^{-1}\langle Bj_{\parallel}\rangle^{(\mathrm{diag})} = \left(\sigma_{\parallel}^{\mathrm{CL}} + \sigma_{\parallel}^{\mathrm{B}}\right)\mathscr{B}_0^{-1}\left\langle BE_{\parallel}^{(A)}\right\rangle, \tag{5.12}$$

with *

$$\sigma_{\parallel}^{CL} = \frac{e^2 n_e}{m_e} \tau_e \sigma, \tag{5.13}$$

and (see table 3.4)

$$\sigma_{\parallel}^{B} = \frac{e^2 n_e}{m_e} \tau_e \varphi I_{EE}(\varphi). \tag{5.14}$$

Note that there is no Pfirsch–Schlüter contribution to the parallel electrical conductivity (see eq. 5.3). Three facts are striking.

(a) *The banana electrical conductivity is inversely proportional to the collision frequency.* This behaviour is in sharp contrast with the diffusive coefficients; on the other hand, it is shared by the classical parallel conductivity, as can be seen by comparing the two previous formulae. The origin of this property lies in the different role played by the collisions in promoting transport in the parallel and in the perpendicular directions. This problem has been discussed in detail in section 5.5.

(b) *There is no amplification factor in the banana electrical conductivity.* Thus, the classical and the banana contributions are of the same order of magnitude. This is in contrast to relations (5.6) for the diffusive radial coefficients.

(c) *The banana contribution to the electrical conductivity is negative.* This fact was first discovered by Hinton and Oberman (1969). It clearly shows that the banana flux cannot have an independent thermodynamical status, because it would lead to a negative entropy production. ** Only the *total electric current* (classical + banana) has a physical meaning. [Note that in the case of the diffusive processes, the banana fluxes – being so much larger than the classical and the PS fluxes – are by themselves a good approximation of the total fluxes].

The fact that, in the long mean free path regime, the parallel electric current is *smaller* than its classical value can be easily understood. Indeed, the *trapped particles* are prevented from going around the torus, and therefore they cannot participate in the parallel electric current.

This interpretation is fully confirmed by considering the asymptotic limit described above. It was shown in table 4.2 that, in the limit $\varphi \to \infty$, the

* Recall the abbreviated notation for the dimensionless parallel electrical conductivity: $\sigma \equiv \tilde{\sigma}_{\parallel}$.

** The relation between the banana transport coefficients and the entropy production is actually much more "exotic" than in the classical case, as will be seen in detail in chapter 17

banana (dimensionless) electrical conductivity tends exactly towards $-\sigma$. The consistency of this remarkable result can be checked by the following argument. It was shown in eq. (2.21) that the banana electrical conductivity is related to the other banana transport and quasi-transport coefficients as

$$l_{\mathrm{EE}} = -\sigma l^{\mathrm{e}}_{1\mathrm{E}} + \alpha l^{\mathrm{e}}_{3\mathrm{E}} + \gamma l^{\mathrm{e}}_{5\mathrm{E}}.$$

In the limit $\varphi \to \infty$, we note the simple limiting values of table 4.2: $\varphi l^{\mathrm{e}}_{1\mathrm{F}} \to 1$, $\varphi l^{\mathrm{e}}_{3\mathrm{F}} \to 0$, $\varphi l^{\mathrm{e}}_{5\mathrm{E}} \to 0$, and find immediately $\varphi l_{\mathrm{EE}} \to -\sigma$. As a result, the *dimensional banana electrical conductivity tends exactly towards the Spitzer conductivity, with opposite sign*:

$$\lim_{\varphi\to\infty} \sigma^{\mathrm{B}}_{\parallel}(\varphi) = -\frac{e^2 n_{\mathrm{e}}}{m_{\mathrm{e}}}\tau_{\mathrm{e}}\sigma = -\sigma^{\mathrm{CL}}_{\parallel}. \tag{5.15}$$

Hence, from (5.11),

$$\lim_{\varphi\to\infty} \mathscr{B}_0^{-1}\langle Bj_{\parallel}\rangle^{(\mathrm{diag})} = 0. \tag{5.16}$$

The total parallel electric current vanishes in this limit. This agrees with the picture in which all the particles are trapped, hence no electric current can flow in the parallel direction. We recall, however, that this conclusion holds only if the cross-effects are neglected (see below).

Hazeltine et al. (1973) proposed to investigate the transport coefficients in the limit of a "small aspect ratio", in which all the particles are trapped. Their limiting values of the diffusive coefficients disagree with ours (as stated above), but the limiting electrical conductivity is the same as ours.

This curious disagreement is understandable when their procedure is closely analyzed. Their theory does not provide a formula for the transport coefficients over the whole range of φ. Hence, they calculate the "limit" by arguing that, if all the particles are trapped, the function $\bar{G}_1^{\alpha(0)}$ in (**14**.2.13) must vanish (see eq. **14**.5.16). They then calculate directly the fluxes, by using $\bar{f}_1^{\alpha(0)} = \bar{F}_1^{\alpha(0)}$. However, in our perspective, it is clear from (**14**.11.3) that with $\bar{G}_1^{\alpha(0)} = 0$, *the generalized stresses vanish identically*, and so do all the banana fluxes (1.4). On the other hand, eq. (**12**.10.5) shows that the parallel banana electric current is a linear combination of the generalized stresses, and is therefore trivially zero when $\bar{G}_1^{\alpha(0)} = 0$; this is inconsistent with the result of Hazeltine et al. Thus, *the radial particle and heat fluxes calculated in this "limit" by Hazeltine et al. are not banana fluxes*. We conclude that their method is inadequate for the study of the limit $\varphi \to \infty$.

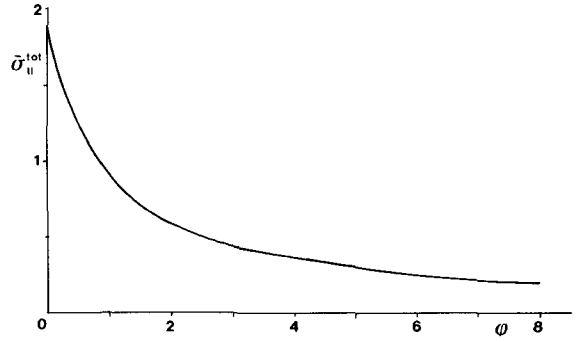

Fig. 5.1. Total parallel electrical conductivity (classical and banana contributions).

It may also be noted from Table 4.1 that the electrical conductivity is the only transport coefficient for which an important discrepancy appears between our results and those of Hinton and Hazeltine (1976) in the opposite limit $\varphi \to 0$. This coefficient is thus particularly sensitive to the extra assumptions used by these authors (use of a "Spitzer function", use of a model collision operator, etc.).

In our method, we obtain formulae for the banana transport coefficients expressed as continuous functions of φ over the whole range of this parameter. The limiting values can therefore be calculated by a true mathematical limiting process, rather than by a brutal, discontinuous procedure. In this way, we find that the banana fluxes tend towards well-defined, simple and universal limiting values.

We also note that, in between the two limits, *the total electrical conductivity is always positive*: $\sigma_{\parallel}^{TOT} \geqslant 0$ (fig. 5.1). The shape of this curve is quite illuminating. It shows that the total electrical conductivity has significant values only for $\varphi \lesssim 1$. For $\varphi > 1$, when trapped particles are more numerous than passing ones, the conductivity quickly becomes very small and tends steadily to zero.

B.2. Electrical cross-effects: L_{1E}^{e}, L_{3E}^{e}, L_{3F}^{i}; L_{F1}^{e}, L_{F3}^{e}, L_{E3}^{i}
These coefficients have again a common structure, shown in table 3.4,

$$L_{rE}^{\alpha} = -A_r n_\alpha c \mathcal{J} \varphi l_{rE}^{\alpha}(\varphi). \tag{5.17}$$

This form is quite different from (5.5) or (5.14). We note, in particular, that these coefficients are *independent of the collision frequency*. In this sense, they are intermediate between the radial diffusive coefficients ($\sim \nu_\alpha$) and the parallel electrical conductivity ($\sim \nu_\alpha^{-1}$). This does not mean, however, that the

collisions play no role in the electrical cross-effects: on the contrary, we know that the functions $l^{\alpha}_{r\mathrm{E}}(\varphi)$ are combinations of the parallel transport coefficients and of the pseudo-viscosities, which are both determined by the collision term.

Next, we see here a new *amplification factor*: $\mathscr{J}\varphi$ (relative to the diagonal electrical conductivity L_{FE}). This factor is much smaller than the amplification factor for the diffusive coefficients; in the standard model we find

$$\mathscr{J}\varphi \simeq 1.469\, q\eta^{-1/2}, \tag{5.18}$$

which is to be compared to eq. (5.9).

All these electrical cross-effects are new; they have no classical or PS counterpart (see eqs. 5.2, 5.3). Their peculiarity results from the fact that *a thermodynamic force directed in the parallel direction produces an effect in the radial direction (and vice-versa).* Why are such effects so peculiar? Thermodynamics does not impose the vanishing of any element in the transport matrix of an anisotropic system. But, in the case of the classical as well as the PS fluxes we have always found that there is a *strict decoupling* of the phenomena occurring in the parallel and in the perpendicular directions. This is no longer so for the banana fluxes. We now describe these cross-effects more precisely.

– *A parallel electric field produces a radial electron flux* (through $L^{\mathrm{e}}_{1\mathrm{E}}$), *a radial electron heat flux* (through $L^{\mathrm{e}}_{3\mathrm{E}}$) *and a radial ion heat flux* (through $L^{\mathrm{i}}_{3\mathrm{E}}$).

– *A parallel electric current is produced by a radial pressure gradient* (through $L^{\mathrm{e}}_{\mathrm{E}1}$), *by a radial electron temperature gradient* (through $L^{\mathrm{e}}_{\mathrm{F}3}$), *and by a radial ion temperature gradient* (through $L^{\mathrm{i}}_{\mathrm{F}3}$).

In discussing these effects, an important question is the *orientation* of these fluxes. From eqs. (3.9–3.11) and (5.17), we see that the sign of the radial fluxes is determined by the sign of the factor:

$$\mathscr{J}X_{\mathrm{E}} = \frac{B_\zeta}{B_\theta}\,\mathscr{B}_0^{-1}\langle \boldsymbol{B}\cdot\boldsymbol{E}^{(A)}\rangle.$$

A simple argument, based on the corkscrew rule, shows that if we choose $B_\zeta > 0$, and if $E^{(A)}_\zeta > 0$ (thus $\mathscr{B}_0^{-1}\langle \boldsymbol{B}\cdot\boldsymbol{E}^{(A)}\rangle > 0$), the corresponding toroidal current j_ζ (>0) produces a poloidal magnetic field oriented in the positive direction $B_\theta > 0$; conversely, if $E^{(A)}_\zeta < 0$, then $B_\theta < 0$ as well, and for all cases we have

$$\mathscr{J}X_{\mathrm{F}} > 0. \tag{5.19}$$

It then follows that the radial electron flux due to the electric cross-effect in (3.11) is *negative*, because $l^{\mathrm{e}}_{1\mathrm{E}} > 0$ and $L^{\mathrm{e}}_{1\mathrm{F}} = -n_{\mathrm{e}}c\mathscr{J}\varphi l^{\mathrm{e}}_{1\mathrm{F}}(\varphi) < 0$ (see table 3.4 and fig. 4.4a). In other words, *a parallel electric field produces an inward*

electron flux. As the diffusion is ambipolar, this flux is accompanied by an equivalent ion flux and the overall effect is a *pinching of the plasma torus*: this is the celebrated *Ware–Galeev pinch effect*. It was discovered independently by Ware (1970a) and by Galeev (1971); its calculation was refined by Rutherford et al. (1970). It was explained (Ware 1970a) by the existence, in the tokamak configuration, of an enhanced drift velocity for the *trapped particles*.

As a result, the global radial particle flux, as given by (3.11) contains a positive (outward) component (L_{11}^{ee}) and three negative (inward) components (L_{13}^{ee}, L_{13}^{ei}, L_{1E}^{e}). The diffusive effects (L_{11}^{ee}, $L_{13}^{e\alpha}$) usually dominate, because of the large amplification factor $\mathscr{J}^2\varphi$, but near the magnetic axis, where the pressure and temperature profiles are very flat, the Ware–Galeev effect may be quite important [see Ware (1970b) for an analysis of the relaxation oscillations in a tokamak on this basis].

Consider now the additional effects of this type: a parallel electric field produces an electron heat flux and an ion heat flux in the radial direction: this is called the *Ettingshausen effect* in non-equilibrium thermodynamics (de Groot and Mazur 1984) *. The same argument as above, combined with the fact that $l_{3E}^{e}(\varphi) < 0$ and $l_{3E}^{i}(\varphi) < 0$ shows that *these heat fluxes are directed outward*. This remarkable and, at first sight, surprising result shows that there is a correlation between particle flux and heat flux when cross-effects set in. By analogy, it may be pointed out that the thermodiffusive radial heat flux produced by a radial pressure gradient is inward ($L_{13}^{ee} < 0$), whereas the radial heat flux produced by a radial temperature gradient is outward ($L_{33}^{ee} > 0$).

The existence of the Ettingshausen effect was briefly mentioned in Ware's (1970a) paper, without a specific calculation. His intuition led him, however, to predict an inward flux, i.e. the wrong direction. Galeev (1971) also found the wrong sign for this coefficient. Hinton and Hazeltine (1976) found the correct value (and the correct sign!) for the coefficient L_{3E}^{e} (but did not point out the discrepancy with the former results!).

We believe that the discrepancy can be understood as follows, If we consider not only the heat flux q_ρ^e but also the *total energy flux*, i.e. the sum of

* It is no longer well-known that Albert von Ettingshausen was Ludwig Boltzmann's colleague at the University of Graz around 1885. He was professor of theoretical physics (whereas Boltzmann taught experimental physics, a rather strange inversion!) and was an admirer of Boltzmann. He was the director of Walter Nernst, who came to Graz as a "postdoc" Ettingshausen proposed to Nernst a measurement of the Hall phenomena in the various directions: this work resulted in the discovery of the whole group of thermomagnetic and galvanomagnetic phenomena (in metals) and attracted great interest in its time Boltzmann followed this work with increasing interest and discussed these problems very frequently with the authors. A very interesting quotation of Nernst's souvenirs is to be found in the article by Urban (1982).

the heat flux and of the *convective* energy flux,

$$\langle q_\rho^e \rangle + \tfrac{5}{2} T_e \langle \Gamma_\rho^e \rangle,$$

then this total energy flux is related to the parallel electric field by the coefficient

$$L_{3E}^e + \tfrac{5}{2} L_{1E}^e = -n_e c \mathcal{J} \varphi \sqrt{\tfrac{5}{2}} \left[l_{3E}^e(\varphi) + \sqrt{\tfrac{5}{2}}\, l_{1E}^e(\varphi) \right].$$

In the small-φ limit, the values taken from table 4.1 yield

$$l_{3E}^e(0) + \sqrt{\tfrac{5}{2}}\, l_{1E}^e(0) = -0.761 + \sqrt{2.5} \times 1.651 = 1.849 > 0.$$

Thus, the convective term dominates the conductive one and reverses the sign of the *total* energy flux.

The relation between a description in terms of the true heat flux and in terms of the total energy flux is not trivial. It will be discussed in detail in section **16**.7.

Hinton and Hazeltine (1976) also found that, if only electron–ion collisions are used in the theory, the coefficient L_{3E}^e vanishes. This curious property can be verified in the present formulation as well. Retaining only contributions from the electron-ion collisions amounts to taking the limit $Z \to \infty$ both in the classical transport coefficients (see eq. **4**.6.31) and in the pseudo-viscosity coefficients (see eqs. **14**.12.2, **14**.1.10, **11**.4.14). It can be seen from fig. 4.5b that the coefficient $\varphi l_{3E}^e(\varphi)$ is indeed identically zero for $Z = \infty$.

We now discuss the converse effects, related to the parallel electric current produced by radial pressure and temperature gradients. In the terminology of non-equilibrium thermodynamics, these effects are classified as *transverse thermomagnetic effects* (de Groot and Mazur 1984). The corresponding transport coefficients are related to the "pinch" coefficients by the appropriate Onsager relations:

$$L_{pE}^\alpha(\boldsymbol{B}) = L_{Ep}^\alpha(-\boldsymbol{B}). \tag{5.20a}$$

As these coefficients are odd functions of the magnetic field (see the discussion of Sec. **5**.5), the result for a given magnetic field is:

$$L_{1E}^e = -L_{E1}^e, \quad L_{3E}^\alpha = -L_{E3}^\alpha. \tag{5.20b}$$

A dissident result appeared in the paper by Molvig and Hizanidis (1984). These authors find, in the framework of the so-called "Lagrangian transport

theory" (Bernstein and Molvig 1983, Cohen et al. 1984), that the Onsager symmetry principle does not hold for the electrical cross-coefficients.

Molvig and Hizanidis' results are, however, open to objections. They present a decomposition of the transport coefficients into an "implicit part" and an "explicit part": $T_{kl} = T_{kl}^{\text{im}} + T_{kl}^{\text{ex}}$. According to their definition, the implicit part "requires the calculation of the first-order distribution function", whereas the "explicit part" is "given explicitly in terms of the Maxwellian distribution function and the Fokker–Planck coefficients". But, upon looking at their formula (86) for T_{13}^{ex} (corresponding to our $l_{1\text{E}}^{\text{e}}$), we find $T_{13}^{\text{ex}} = (T\alpha_3, f_0)$, with $\alpha_3 = C_l(g_3)$, [their eq. (77)], and with g_3 defined by their eq. (76): $\hat{f}_1 = \Sigma g_i A_i$ ($A_3 = V_{\text{T}}/T$, V_{T} is the loop voltage). Thus, the calculation of T_{13}^{ex} *also* requires the knowledge of $\hat{f}_1$. Furthermore, Molvig and Hizanidis find the symmetry relations $T_{13}^{\text{im}} = T_{31}^{\text{im}}$, $T_{23}^{\text{im}} = T_{32}^{\text{im}}$, which should actually be *anti*-symmetry relations: $T_{n3}^{\text{im}} = -T_{3n}^{\text{im}}$. As for the ionic contributions (our $l_{3\text{E}}^{\text{i}}$) they are completely lacking in their theory.

Finally, their interpretation of the non-symmetric fluxes as "kinematic fluxes" appears quite arbitrary. Indeed, no clear definition has been given of a kinematic flux and no structural features of the equations can be related to this concept.

The fact that a gradient in the *radial* direction affects a current in the *parallel* direction suggests a mechanism involving a drift motion. Indeed, the presence of a radial pressure gradient, combined with the poloidal magnetic field, produces a drift of the banana orbits in the toroidal direction, an effect analogous to the *diamagnetic drift* of the ordinary Larmor orbits (remember that the diamagnetic drift is in opposite directions for the electrons and for the ions, and thus produces an electric current). The effect of the temperature gradients produces a current in the opposite direction (remember that L_{E1}^{c} and L_{E3}^{α} have opposite signs). The coefficients $\varphi l_{\text{E3}}^{\text{e}}(\varphi)$ and $\varphi l_{\text{E3}}^{\text{i}}(\varphi)$ are quantitatively smaller than $\varphi l_{\text{E1}}^{\text{e}}(\varphi)$: they have a maximum near $\varphi = 1$ and go to zero as $\varphi \to \infty$. On the contrary, $\varphi l_{\text{E1}}^{\text{e}}(\varphi) \to -1$ in the limit $\varphi \to \infty$; this confirms the interpretation in terms of a "generalized diamagnetic drift" of the trapped particles. Recalling that the total electrical conductivity vanishes exactly as $\varphi \to \infty$, we see that, in this limit, the electrical current is driven *exclusively* by the radial pressure gradient and not at all by the parallel electric field,

$$\mathscr{B}_0^{-1}\langle Bj_{\parallel}\rangle = -L_{\text{E1}}^{\text{e}}\, n_e^{-1}\, \nabla_\rho P, \qquad \varphi \to \infty. \tag{5.21}$$

These properties are very interesting for the tokamak physics, and were pointed out by Bickerton et al. (1971) and by Kadomtsev and Shafranov (1972). Contribution (5.21) (for finite φ) was called the *bootstrap current*. It leads to the attractive idea of operating a tokamak in a steady state, with

$E^{(A)} = 0$. Indeed, after a transient period, in which the radial pressure profile builds up, the electric field (which is an expensive item!) can be switched off. The parallel current necessary for producing the poloidal field, hence the indispensable rotational transform of the configuration, would be driven by the pressure gradient alone, as in (5.21). The great advantage of a steady-state operation is that the materials of a potential reactor suffer much less fatigue than in a pulsed-mode regime.

In a very recent work, Zarnstorff and Prager (1984, 1986) found a clear confirmation of the bootstrap current (as well as of the Pfirsch–Schlüter current) in experiments done on the levitated octupole device of the University of Wisconsin. The results appear to be in excellent agreement with the neoclassical theory. Later on, Zarnstorff found a clear indication of the presence of a bootstrap current in the large TFTR tokamak in Princeton (Schwarzschild 1986). These experimental findings can be interpreted as a "happy prospect of a steady-state tokamak".

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16

The intermediate mean free path regime. Interpolation formulae for the transport coefficients

16.1. Introduction

In chapters 12–15 we have been able to derive quite precise expressions for the radial particle and heat fluxes in two extreme cases.

When the *mean free path is very short*, i.e. when *the collisions are very frequent*, the particles have no time to perform the characteristic banana orbits (or perform a full poloidal circuit) between two collisions. As a result, no stress anisotropy can develop, and the banana contribution to the radial fluxes is negligible. Among the remaining contributions to the radial fluxes, given in table **12**.9.1, it was shown in chapter 13 that the dominant ones, in a tokamak configuration, are the *Pfirsch–Schlüter fluxes*.

At the other extreme of a *very long mean free path* i.e. of *very rare collisions*, the form of the (order ε) distribution function is mainly determined by the collisionless motion of the particles. The main contribution to the radial fluxes is then due to the generalized stresses. These can be evaluated by solving the drift kinetic equation in the regime of very small collision frequencies. This leads to the *banana fluxes*, which strongly dominate the Pfirsch–Schlüter fluxes in this regime.

A natural question arises now: *What happens in between these two extreme regimes*? Clearly, there is no exact answer to this question, because it is impossible to solve the drift kinetic equation, analytically or numerically, over the whole range of collisionality regimes.

It appears, however, that *in the case of a large aspect ratio*, i.e. when $\bar{\eta} \equiv (a/R_0) \ll 1$, *it is possible to find a "window" in between the two extremes*, in which the drift kinetic equation can again be solved analytically. This window was called the *plateau regime*, for reasons to be presently clarified. In this range, the collision frequency is large enough to destroy the banana orbits of the particles, but not large enough to eliminate the anisotropy of the plasma. As a result, the generalized stresses are still producing a set of banana

fluxes, but their form and value is quite different from the long mean free path regime.

This remarkable window was already suggested (but not really worked out) by the "fathers" of the neoclassical theory, Galeev and Sagdeev (1968). It was later introduced by Stringer (1970), who used a quite different argument. Rosenbluth et al. (1972) then took up the problem on a more rigorous basis: their work is reviewed by Hinton and Hazeltine (1976). Hirshman and Sigmar (1977) treated the problem for a many-component plasma, but using a method which was the continuation of the Hinton–Hazeltine theory. It was only in 1983 that Shaing and Callen (1983a) provided a clear application to this problem of the methods developed by Hirshman and Sigmar (1981) for the study of the banana fluxes. These modern methods were also applied to non-axisymmetric geometries by Shaing & Callen (1983b), by Shaing et al. (1986) and by Coronado and Wobig (1986) (see section 16.8). (The method developed in this chapter is inspired from the work by Shaing and Callen 1983a). This short discussion shows that, in spite of its long history, the problem of the plateau regime was clearly and rigorously formulated only in very recent years.

The plateau regime is a quite interesting problem from the point of view of theoretical physics; it should be stressed, however, that it cannot always be realized in practice. In particular, if the inverse aspect ratio $\bar{\eta}$ is not small enough, the constraints defining the plateau regime cannot be satisfied.

Whenever this regime exists, it provides a possibility of interpolating the values of the transport coefficients from their extreme banana value to their other extreme Pfirsch–Schlüter value. One thus obtains *approximate formulae for the transport coefficients which may be thought to apply over the whole range of parameters*. Such interpolation formulae, together with their limitations, will be discussed in sections 16.6 and 16.7.

16.2. The NGC variables x, ξ, ϕ

The decomposition of the average radial fluxes and of the parallel current obtained in sections **12**.9 and **12**.10 provides a set of identities valid independently of the "degree of collisionality". On the other hand, as shown in sections **13**.2, **13**.3, the expressions for the classical and the Pfirsch–Schlüter fluxes are also universally valid. The only component of these fluxes for which a specific assumption had to be made is the *banana component*. Its evaluation required a solution of the drift kinetic equation by an algorithm specifically designed for the very long mean free path regime.

It will now be shown that, under some conditions, it is possible to define a regime in which the magnetic terms of the drift kinetic equation can be

partially balanced by the collisions in the leading order, and in which the equation can still be solved analytically. The solution of this equation defines a non-equilibrium state in which the collisions and the magnetic effects stand on equal footing, and which is therefore precisely the intermediate regime between the two extreme cases we were looking for.

Our starting point is the *drift kinetic equation* (**14**.1.3), which we rewrite by taking into account relations (**14**.1.10), (**14**.1.11), (**14**.2.2), (**14**.2.6) and (**14**.2.9), and multiplying both sides by (B/U_α),

$$\boldsymbol{B}\cdot\nabla\tilde{f}_1^\alpha + \frac{\mathscr{J}}{R_0\mathscr{B}_P}\left(\boldsymbol{B}\cdot\nabla\frac{U_\alpha}{\Omega_\alpha}\right)\frac{\partial f_0^\alpha}{\partial\rho} = \frac{B}{U_\alpha}\nu_\alpha\mathscr{L}_\alpha\tilde{f}_1^\alpha + B\mathscr{M}_\alpha f_0^\alpha, \tag{2.1}$$

where the *pitch angle scattering operator* $\mathscr{L}_\alpha$ is defined as

$$\mathscr{L}_\alpha = \frac{m_\alpha}{B}U_\alpha\frac{\partial}{\partial M}U_\alpha M\frac{\partial}{\partial M}. \tag{2.2}$$

We recall that $\tilde{f}_1^\alpha$ is the deviation of the gyro-averaged distribution function from the reference local equilibrium state f_0^α. It is a quantity of first order in the drift parameter ε (see eq. **14**.1.1) and is expressed as a function of the natural guiding centre (NGC) variables $\mathscr{E}$, M; θ, ρ. There is no dependence on either the gyrophase ϕ (because of the gyro-average) or on the toroidal angle ζ (because of the assumed *axisymmetry*). All the remaining symbols have been defined in chapter 14. Let us only recall that the factor $\mathscr{M}_\alpha$ contains both the effect of the external electric field $\boldsymbol{E}^{(A)}$ and of the "modelled part" of the collision term.

In order to identify the particular regime we are looking for, we shall use a different set of NGC variables, related to the variables x, λ, ϕ of section **14**.5, but replacing λ by a different variable, ξ,

$$x = T_\alpha^{-1}\left[\mathscr{E} - e_\alpha\Phi(\rho)\right],$$

$$\xi = \frac{U_\alpha}{(2/m_\alpha)^{1/2}\sqrt{\mathscr{E} - e_\alpha\Phi(\rho)}}. \tag{2.3}$$

The inverse transformation is

$$\mathscr{E} = T_\alpha x + e_\alpha\Phi(\rho),$$

$$M = \frac{T_\alpha x}{B(\theta,\rho)}(1-\xi^2). \tag{2.4}$$

The variable ξ has a simple physical interpretation. To leading order in ε it is just the ratio of the parallel velocity $v_{\parallel} = \boldsymbol{b} \cdot \boldsymbol{v}$ to the absolute value of the total velocity,

$$v\,\xi = v_{\parallel} = \boldsymbol{b} \cdot \boldsymbol{v} = v \cos \vartheta.$$

Thus, ξ is the cosine of the angle ϑ between the velocity and the magnetic field or, as often called, the *cosine of the pitch-angle*. We note the (important) expression of the parallel velocity,

$$U_\alpha = \left(\frac{2T_\alpha}{m_\alpha}\right)^{1/2} \sqrt{x}\,\xi. \tag{2.5}$$

In terms of these variables, the reference distribution function has the same form as in (**14**.6.11),

$$f_0^\alpha(x;\, \rho) = \frac{1}{2\pi}\left(\frac{m_\alpha}{2T_\alpha}\right)^{3/2} n_\alpha(\rho)\; \phi^0(x), \tag{2.6}$$

with

$$\phi^0(x) = \frac{2}{\sqrt{\pi}} \mathrm{e}^{-x}. \tag{2.7}$$

The transformation formula for any integral over the whole velocity space (corresponding to eq. **14**.5.6) is, for an arbitrary function F,

$$\int \mathrm{d}\boldsymbol{c}\; \phi^0(c)\; F(\boldsymbol{c}) = \frac{1}{4\pi}\,\frac{2}{\sqrt{\pi}} \int_0^\infty \mathrm{d}x \sqrt{x}\, \mathrm{e}^{-x} \int_{-1}^{1} \mathrm{d}\xi \int_0^{2\pi} \mathrm{d}\phi\; F(x,\, \xi,\, \phi). \tag{2.8}$$

The differential operators are readily shown to transform as

$$\frac{\partial}{\partial M} \to -\frac{B}{2T_\alpha x \xi}\,\frac{\partial}{\partial \xi},$$

$$\frac{\partial}{\partial \theta} \to -\frac{1}{2B}\,\frac{1-\xi^2}{\xi}\,\frac{\partial B}{\partial \theta}\,\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \theta}. \tag{2.9}$$

In order to facilitate the calculations, we give here some partial results. The pitch-angle scattering operator $\mathscr{L}_\alpha$ (2.2) transforms into the extremely simple form

$$\mathscr{L} = \frac{1}{2}\frac{\partial}{\partial \xi}(1-\xi^2)\frac{\partial}{\partial \xi} \tag{2.10}$$

(in these variables, $\mathscr{L}_\alpha$ is independent of α). The operator describing the parallel motion becomes

$$U_\alpha \boldsymbol{B}\cdot\nabla = \left(\frac{2T_\alpha}{m_\alpha}\right)^{1/2}\sqrt{x}\,\frac{B_\theta}{l_\theta}\left(\xi\frac{\partial}{\partial\theta} - \frac{1}{2B}\frac{\partial B}{\partial\theta}(1-\xi^2)\frac{\partial}{\partial\xi}\right). \tag{2.11}$$

Hence, the drift kinetic equation (2.1) is expressed as (using the quantity $\mathscr{J}$ defined in eq. **15**.3.23)

$$\left(\frac{2T_\alpha}{m_\alpha}\right)^{1/2}\sqrt{x}\,\frac{B_\theta}{Bl_\theta}\left(\xi\frac{\partial}{\partial\theta} - \frac{1}{2B}\frac{\partial B}{\partial\theta}(1-\xi^2)\frac{\partial}{\partial\xi}\right)\tilde{f}_1^\alpha(x,\,\xi;\,\theta,\,\rho)$$

$$-\nu_\alpha(x)\frac{1}{2}\frac{\partial}{\partial\xi}(1-\xi^2)\frac{\partial}{\partial\xi}\tilde{f}_1^\alpha(x,\,\xi;\,\theta,\,\rho)$$

$$= \left[\left(\frac{2T_\alpha}{m_\alpha}\right)^{1/2}\sqrt{x}\,\xi\mathscr{M}_\alpha - \frac{\mathscr{J}}{B}U_\alpha\left(\boldsymbol{B}\cdot\nabla\frac{U_\alpha}{\Omega_\alpha}\right)s_0^\alpha\right]f_0^\alpha, \tag{2.12}$$

where we introduced the following symbol [see (**14**.2.6)]:

$$s_0^\alpha = -\frac{1}{\tau_\alpha}\left(\frac{m_\alpha}{T_\alpha}\right)^{1/2}\left[g_\rho^{\alpha(1)} - g_\rho^{\alpha(1)A} + \sqrt{\tfrac{2}{5}}\left(x-\tfrac{5}{2}\right)g_\rho^{\alpha(3)}\right]. \tag{2.13}$$

In order to make progress, we must exhibit the orders of magnitude of the various terms. We must therefore introduce a specific model for the magnetic field geometry: we adopt the *standard model* of section **8**.9, which describes a large aspect ratio tokamak. In that model we have a natural small parameter

$$\eta \equiv \eta(r) = \frac{r}{R_0} \ll 1. \tag{2.14}$$

By using the formulae of section **8.9**, we easily derive the following results, to leading order in η,

$$\frac{B_\theta}{Bl_\theta} = (1 + \eta \cos\theta)\frac{1}{r}\frac{r}{q(r)\,R_0} \approx \frac{1}{q(r)\,R_0}, \tag{2.15}$$

$$\frac{1}{B}\frac{\partial B}{\partial \theta} = (1 + \eta \cos\theta)\frac{\partial}{\partial\theta}(1 + \eta\cos\theta)^{-1} \approx \eta \sin\theta, \tag{2.16}$$

$$\frac{U_\alpha}{B}\boldsymbol{B}\cdot\nabla\frac{U_\alpha}{\Omega_\alpha} = \frac{2T_\alpha}{m_\alpha}x\frac{B_\theta}{l_\theta}\frac{1}{\Omega_\alpha}\left(\xi\frac{\partial}{\partial\theta} - \tfrac{1}{2}\eta\sin\theta(1-\xi^2)\frac{\partial}{\partial\xi}\right)\frac{\xi}{B(\theta,\rho)}$$

$$= -\eta\frac{1}{qR_0}\frac{T_\alpha}{m_\alpha\Omega_\alpha}x\sin\theta\,(1+\xi^2). \tag{2.17}$$

We also introduce the following dimensionless collision frequency

$$\bar{\nu}_\alpha(x) = \frac{Bl_\theta}{B_\theta}\left(\frac{m_\alpha}{2T_\alpha}\right)^{1/2}\frac{\nu_\alpha(x)}{\sqrt{x}} \approx qR_0\left(\frac{m_\alpha}{2T_\alpha}\right)^{1/2}\frac{\nu_\alpha(x)}{\sqrt{x}}. \tag{2.18}$$

With these results, the drift kinetic equation (2.12) reduces to

$$\left(\xi\frac{\partial}{\partial\theta} - \tfrac{1}{2}\eta\sin\theta(1-\xi^2)\frac{\partial}{\partial\xi}\right)\tilde{f}_1^\alpha - \tfrac{1}{2}\bar{\nu}_\alpha\frac{\partial}{\partial\xi}(1-\xi^2)\frac{\partial}{\partial\xi}\tilde{f}_1^\alpha$$

$$= \eta\mathscr{J}\left(\frac{2T_\alpha}{m_\alpha}\right)^{1/2}\sqrt{x}\,\frac{1}{2\Omega_\alpha}\sin\theta(1+\xi^2)s_0^\alpha f_0^\alpha + qR_0\,\xi\,\mathscr{M}_\alpha f_0^\alpha. \tag{2.19}$$

The equation is further transformed by choosing the following specific Ansatz for the distribution function:

$$\tilde{f}_1^\alpha = -\left(\frac{2T_\alpha}{m_\alpha}\right)^{1/2}\frac{\mathscr{J}\sqrt{x}}{\Omega_\alpha(\theta,\rho)}\xi\,s_0^\alpha(x,\rho)\,f_0^\alpha + \frac{B(\theta,\rho)}{\mathscr{B}_0}\xi\sqrt{x}\,S_\alpha(x,\rho)\,f_0^\alpha$$

$$+\bar{h}_1^\alpha(x,\xi;\theta,\rho), \tag{2.20}$$

where $\mathscr{B}_0$ is defined, as usual, by

$$\mathscr{B}_0 = \langle B^2\rangle^{1/2}.$$

Recalling result (**14**.2.13), we note that the first term in (2.20) is a solution of the collisionless drift kinetic equation, i.e. the equation obtained by neglecting the last term in both sides of eq. (2.19). The quantity $S_\alpha(x, \rho)$ will be defined below in such a way as to "optimize" the equation in a certain sense. $\bar{h}_1^\alpha$ then becomes the new unknown function. The drift kinetic equation (2.19) thus becomes

$$\left(\xi\frac{\partial}{\partial\theta} - \tfrac{1}{2}\eta\,\sin\theta(1-\xi^2)\frac{\partial}{\partial\xi}\right)\bar{h}_1^\alpha - \bar{\nu}_\alpha \mathscr{L}\bar{h}_1^\alpha$$

$$= \tfrac{1}{2}\eta\frac{B}{\mathscr{B}_0}\sin\theta\,(1-3\xi^2)\sqrt{x}\,S_\alpha$$

$$-\bar{\nu}_\alpha\xi\left(\frac{B(\theta)}{\mathscr{B}_0}\sqrt{x}\,S_\alpha(x,\rho) - G_\alpha(x,\theta,\rho)\right)f_0^\alpha, \tag{2.21}$$

where we introduced the abbreviation

$$G_\alpha(x,\theta,\rho) = \frac{qR_0}{\bar{\nu}_\alpha(x)}\mathscr{M}_\alpha(\theta,\rho) + \left(\frac{2T_\alpha}{m_\alpha}\right)^{1/2}\frac{\mathscr{J}\sqrt{x}}{\Omega_\alpha(\theta,\rho)}s_0^\alpha(x,\rho). \tag{2.22}$$

This quantity contains, in particular, the collision-model dependent term $\mathscr{M}_\alpha$. We now decide to *determine* S_α *in such a way as to "minimize" the contribution of the last term in* (2.21). More precisely, we require the *vanishing of its surface average*,

$$\left\langle\frac{B}{\mathscr{B}_0}\left(\frac{B}{\mathscr{B}_0}\sqrt{x}\,S_\alpha - G_\alpha\right)\right\rangle = 0.$$

This leads to the definition

$$\sqrt{x}\,S_\alpha = \frac{1}{\mathscr{B}_0}\langle BG_\alpha\rangle. \tag{2.23}$$

A reasonable argument (first used by Hirshman and Sigmar 1977) suggests that, for reasonably smooth functions $G_\alpha(\theta)$, the following relation holds in order of magnitude:

$$\frac{B(\theta)}{\mathscr{B}_0}\frac{\langle B(\theta)\,G_\alpha(\theta)\rangle}{\mathscr{B}_0} - G_\alpha(\theta) \approx \eta\frac{\langle B(\theta)\,G_\alpha(\theta)\rangle}{\mathscr{B}_0} \equiv \eta\sqrt{x}\,S_\alpha, \tag{2.24}$$

where the factor η appears as a measure of the magnetic field modulation Ξ (eq. **14**.8.5) in the standard model. Upon accepting this argument, we approximate the drift kinetic equation (2.21) by

$$\left(\xi\frac{\partial}{\partial\theta} - \tfrac{1}{2}\eta\sin\theta(1-\xi^2)\frac{\partial}{\partial\xi}\right)\bar{h}_1^\alpha - \tfrac{1}{2}\bar{\nu}_\alpha\frac{\partial}{\partial\xi}(1-\xi^2)\frac{\partial}{\partial\xi}\bar{h}_1^\alpha$$

$$= \left(\tfrac{1}{2}\eta\frac{B}{\mathscr{B}_0}\sin\theta(1-3\xi^2) - \eta\,\bar{\nu}_\alpha\,\xi\right)\sqrt{x}\,S_\alpha f_0^\alpha. \tag{2.25}$$

16.3. The plateau regime

In section **14**.1, the drift kinetic equation for the (order ε) distribution function was expanded in a manner appropriate for the long mean free path regime. This is clearly recognizable in the decomposition of the complete equation (**14**.1.3) into the two equations (**14**.1.8) and (**14**.1.9). The leading order distribution function $\bar{f}_1^{\alpha(0)}$ is determined by a *collisionless* kinetic equation.

We now look for a regime in which *the collisions may balance the free streaming term already in the leading order*. This implies that the dimensionless collision frequency $\bar{\nu}_\alpha$ must be much larger than in the long mean free path regime, though remaining small compared to 1. As a result, most particles in the plasma have no time for completing a banana orbit (or a revolution in the poloidal direction) between two collisions. Therefore, there exists a class of particles (*resonant particles*) which cannot explore the upper and the lower parts of the torus; hence their radial drifts cannot be compensated, and this leads to an enhanced radial transport. This mechanism was first discovered by Stringer (1970). It turns out that these particles are localized in a narrow range of pitch-angles (i.e. of ξ).

In order to identify this regime, we use a technique which is well-known in perturbation theory (Nayfeh 1973): it consists of *scaling* the relevant variable. We thus study the narrow neighbourhood of the interesting domain "*with a microscope*".

We now formulate the problem quantitatively. For simplicity, we abbreviate the dimensionless collision frequency $\bar{\nu}_\alpha$ (Eq. 2.18) as $\bar{\nu}_\alpha \equiv \nu$. The starting assumptions are

$$\eta \ll 1, \tag{3.1}$$

$$\nu \ll 1, \tag{3.2}$$

$$\eta \ll \nu. \tag{3.3}$$

The last inequality is not yet sufficiently precise: the exact relation between the two small parameters η and ν must be determined by the nature of the problem.

We now introduce the *scaled pitch-angle cosine* p,

$$\xi = \nu^a p, \quad a > 0. \tag{3.4}$$

(The "microscope effect" is clear: when p is of order 1, ξ is of order ν^a.) Substituting this relation into (2.25), we find that the first (parallel streaming term becomes

$$\xi \frac{\partial}{\partial \theta} \bar{h}_1^\alpha \to \nu^a p \frac{\partial}{\partial \theta} \bar{h}_1^\alpha,$$

and the pitch-angle scattering term is

$$\nu \mathscr{L} \bar{h}_1^\alpha \to \tfrac{1}{2} \nu^{1-2a} \frac{\partial}{\partial p} (1 - \nu^{2a} p^2) \frac{\partial}{\partial p} \bar{h}_1^\alpha.$$

We want to determine a regime in which there is a real competition between these two terms. In order that they be of the same order of magnitude, we require

$$\nu^a = \nu^{1-2a},$$

which fixes the value of the exponent, $a = 1/3$. Thus, (3.4) is now replaced by

$$\xi = \nu^{1/3} p. \tag{3.5}$$

We now substitute this value into the kinetic equation (2.25) and neglect all terms proportional to positive powers of η and ν, on the basis of (3.1) and (3.2),

$$p \frac{\partial}{\partial \theta} \bar{h}_1^\alpha - \frac{1}{2} \frac{\partial^2}{\partial p^2} \bar{h}_1^\alpha - \tfrac{1}{2} (\eta \nu^{-2/3}) \sin \theta \frac{\partial}{\partial p} \bar{h}_1^\alpha$$

$$= \tfrac{1}{2} (\eta \nu^{-1/3}) \sin \theta \, (B/\mathscr{B}_0) \sqrt{x} \, S_\alpha f_0^\alpha. \tag{3.6}$$

We now take advantage of the specific form of the source term and set

$$\bar{h}_1^\alpha = \tfrac{1}{2} (\eta \nu^{-1/3}) \frac{B(\theta)}{\mathscr{B}_0} \sqrt{x} \, S_\alpha f_0^\alpha \, k_1^\alpha(p, x, \theta, \rho). \tag{3.7}$$

Noting that in the standard model $\mathscr{B}_0^{-1}[\partial B(\theta)/\partial\theta] \sim \eta$ (see eq. 2.16), we reduce (3.6) to

$$p\frac{\partial}{\partial\theta}k_1^\alpha - \frac{1}{2}\frac{\partial^2}{\partial p^2}k_1^\alpha - \tfrac{1}{2}(\eta\nu^{-2/3})\sin\theta\frac{\partial}{\partial p}k_1^\alpha = \sin\theta. \tag{3.8}$$

At this stage, we are left with a single parameter $(\eta\nu^{-2/3})$ in the equation. We are now in a position to make condition (3.3) precise by requiring that

$$\eta\nu^{-2/3} \ll 1. \tag{3.9}$$

As a result, eq. (3.8) can be solved by perturbation theory:

$$k_1^\alpha = k_1^{\alpha 0} + (\eta\nu^{-2/3})k_1^{\alpha 1} + \cdots. \tag{3.10}$$

The leading term in this expansion is determined by the following equation (Hinton and Hazeltine 1976)

$$p\frac{\partial}{\partial\theta}k_1^{\alpha 0} - \frac{1}{2}\frac{\partial^2}{\partial p^2}k_1^{\alpha 0} = \sin\theta. \tag{3.11}$$

It is not difficult to solve this equation. Its form suggests the Ansatz

$$k_1^{\alpha 0}(p,\theta) = \mathrm{Im}[g(p)\,\mathrm{e}^{\mathrm{i}\theta}]. \tag{3.12}$$

It follows that the function $g(p)$ satisfies the equation

$$\frac{1}{2}\frac{\mathrm{d}^2 g(p)}{\mathrm{d}p^2} - \mathrm{i}pg(p) = -1. \tag{3.13}$$

The solution of this equation, tending to zero as $p \to \mp\infty$, was first given (in a different context) by Su and Oberman (1968) in the form of the *resonance function*

$$g(p) = \int_0^\infty \mathrm{d}\tau\,\mathrm{e}^{-\mathrm{i}p\tau}\,\mathrm{e}^{-(\tau^3/6)}. \tag{3.14}$$

Hence, from (3.12) follows

$$k_1^{\alpha 0}(p,\theta) = \int_0^\infty \mathrm{d}\tau\,\sin(\theta - p\tau)\,\mathrm{e}^{-(\tau^3/6)}. \tag{3.15}$$

This function can also be written as a sum of an odd and an even function of θ:

$$k_1^{\alpha 0}(p, \theta) = \sin\theta \int_0^\infty d\tau \cos p\tau \, e^{-(\tau^3/6)} - \cos\theta \int_0^\infty d\tau \sin p\tau \, e^{-(\tau^3/6)}. \tag{3.16}$$

This function has a finite value in $p = 0$,

$$k_1^{\alpha 0}(0, \theta) = 6^{1/3}\Gamma(4/3) \sin\theta \approx 1.623 \sin\theta. \tag{3.17}$$

It decays monotonically to zero for large p.

Collecting now all these partial results, we return to eqs. (**14**.1.1) and (2.20) and use (2.6), (3.7), (3.10) and (3.15). We thus obtain the following representation of the full gyro-averaged distribution function in the *plateau regime*:

$$\bar{f}^\alpha(\xi, x; \theta, \rho) = \frac{1}{2\pi}\left(\frac{m_\alpha}{2T_\alpha}\right)^{1/2} n_\alpha(\rho)\, \phi^0(x)\left[1 + \bar{\chi}_1^\alpha(\xi, x; \theta, \rho)\right], \tag{3.18}$$

with

$$\bar{\chi}_1^\alpha = \bar{\chi}_1^{\alpha[0]} + \left(\eta\bar{\nu}_\alpha^{-1/3}\right)\bar{\chi}_1^{\alpha[1]} + \left(\eta\bar{\nu}_\alpha^{-1/3}\right)\left(\eta\bar{\nu}_\alpha^{-2/3}\right)\bar{\chi}_1^{\alpha[2]} + \cdots, \tag{3.19}$$

where

$$\bar{\chi}_1^{\alpha[0]} = -\left(\frac{2T_\alpha}{m_\alpha}\right)^{1/2} \frac{\mathscr{J}\sqrt{x}}{\Omega_\alpha(\theta, \rho)}\, \xi s_0^\alpha(x, \rho) + \frac{B(\theta, \rho)}{\mathscr{B}_0}\sqrt{x}\, \xi S_\alpha(x, \rho)$$

$$\equiv \bar{\chi}_1^{\alpha[0]} + \bar{\chi}_{11}^{\alpha[0]}, \tag{3.20}$$

and

$$\bar{\chi}_1^{\alpha[1]} = \frac{1}{2}\frac{B(\theta, \rho)}{\mathscr{B}_0}\sqrt{x}\, S_\alpha(x, \rho) \int_0^\infty d\tau \, \sin\left(\theta - \bar{\nu}_\alpha^{-1/3}\xi\tau\right) e^{-(\tau^3/6)}. \tag{3.21}$$

The function $\bar{\chi}_1^{\alpha[2]}$ will not be needed in the forthcoming treatment. Expansion (3.19) replaces representation (**14**.1.7) in the plateau regime. It is worth noting that this expansion is *not analytic* in the collision frequency $\bar{\nu}_\alpha$.

16.4. The pseudo-viscosity coefficients in the plateau regime

At this point, the form of our results is remarkably close to the results obtained for the long mean free path regime in chapter 14. By properly exploiting this analogy, the derivation of the plateau transport equations will be straightforward. We note that eqs. (3.18) and (3.19) correspond to (**14**.6.14) and (**14**.6.15). The analogy becomes evident upon comparing eqs. (3.20) and (**14**.6.16). In both cases, the zeroth-order deviation $\bar{\chi}_1^{\alpha[0]}$ appears as a sum of two terms. The first one is a linear combination of the thermodynamic source terms. Written explicitly by using (2.13) and the definition of the Laguerre–Sonine polynomials (table **G1**.3.2), this term has practically the same form as (**14**.6.17):

$$\bar{\chi}_1^{\alpha[0]} = \sqrt{3}\,\frac{\mathscr{J}}{\Omega_\alpha \tau_\alpha}\xi\sqrt{x}\left[\left(g_\rho^{\alpha(1)} - g_\rho^{\alpha(1)A}\right)L_0^{3/2}(x) + g_\rho^{\alpha(3)}L_1^{3/2}(x)\right]. \tag{4.1}$$

The second term of (3.20) has a form which is also closely related to (**14**.6.21). In both cases we find a function [$S_\alpha(x, \rho)$ and $\mathscr{G}_\alpha(x, \rho)$, respectively] which depends on the external field $\boldsymbol{E}^{(A)}$ and on the modelling of the collision term, through the moments $\mathscr{M}_\alpha$; compare eqs. (2.22), (2.23) and (**14**.4.16). The alternative described at the end of section **14**.5 is fully applicable here. The way chosen by the earlier authors was to assume a specific collision model and to calculate the fluxes in this framework (Hinton and Hazeltine 1976; but also Hirshman and Sigmar 1977!) *.

The alternative, much more elegant way, consists of expanding the unknown function into a series of Laguerre-Sonine polynomials and deriving the pseudo-viscosity coefficients and the transport equations from an analysis of the moment equations. This method was first applied to the plateau regime by Shaing and Callen (1983a).

Adopting the latter philosophy, we represent the function $S_\alpha(x)$ in the following form, closely analogous to (**14**.5.17),

$$S_\alpha(x) = \sum_m c_m^\alpha L_m^{3/2}(x). \tag{4.2}$$

In practice, we truncate the infinite series at $m = 2$, corresponding to the *21M approximation*, just as for the long mean free path theory (We note here again that the previously published treatments correspond to a 13M approxi-

* It is interesting to note here that in the review paper by Hirshman and Sigmar (1981), in which these authors develop their "moment method", the pseudo-viscosity coefficients in the plateau regime were not calculated by a direct method, but were inferred from their "old style calculation" of 1977

mation.) The second term of the zeroth-order deviation is then similar to (**14**.6.22),

$$\bar{\chi}_{\mathrm{II}}^{\alpha[0]} = \frac{B}{\mathscr{B}_0}\xi\sqrt{x}\left[c_0^\alpha L_0^{3/2}(x) + c_1^\alpha L_1^{3/2}(x) + c_2^\alpha L_2^{3/2}(x)\right]. \tag{4.3}$$

Next, we note that the analysis in Sec. **14**.7 is perfectly applicable here too, with the result that the Laguerre–Sonine coefficients of $\bar{\chi}_{\mathrm{II}}^{\alpha[0]}$ are directly related to the "poloidal fluxes" ω_n^α; eq. (4.2) is therefore rewritten as

$$S_\alpha(x) = \sqrt{3}\left[\omega_1^\alpha L_0^{3/2}(x) + \omega_3^\alpha L_1^{3/2}(x) + \omega_5^\alpha L_2^{3/2}(x)\right]. \tag{4.4}$$

We now proceed to the evaluation of the generalized stresses. The calculation is more straightforward here than in chapter 15, because the deviation $\bar{\chi}_1^{\alpha[1]}$ is explicitly known. From (**12**.11.7), (2.8) and (2.11) we obtain

$$\sqrt{2}\,\langle \boldsymbol{B}\cdot(\nabla\cdot\mathbf{h}^{\alpha(2)})\rangle = \frac{m_\alpha}{n_\alpha T_\alpha}\left\langle B\int \mathrm{d}\boldsymbol{v}\, U_\alpha^2\, \boldsymbol{B}\cdot\nabla \tilde{f}_1^\alpha\right\rangle$$

$$= \frac{2}{\sqrt{\pi}}\left\langle B\int_0^\infty \mathrm{d}x\, \mathrm{e}^{-x}\, x^{3/2}\int_{-1}^{1}\mathrm{d}\xi\,\xi^2\frac{B_\theta}{l_\theta}\left(\frac{\partial}{\partial\theta} - \tfrac{1}{2}\eta\,\sin\theta\frac{1}{\xi}(1-\xi^2)\frac{\partial}{\partial\xi}\right)\right.$$

$$\left.\times\left[\bar{\chi}_1^{\alpha[0]} + \left(\eta\bar{\nu}_\alpha^{-1/3}\right)\bar{\chi}_1^{\alpha[1]}\right]\right\rangle. \tag{4.5}$$

It is easily checked that the contribution of $\bar{\chi}_1^{\alpha[0]}$ vanishes (either because of the surface average or because of an integral over an odd function of ξ); in the remaining terms, the contribution of $(\partial\bar{\chi}_1^{\alpha[1]}/\partial\theta)$ vanishes for similar reasons. Using (3.21) (or 3.16) and noting that $B/\mathscr{B}_0 = 1 + \mathrm{O}(\eta)$, we find

$$\langle \boldsymbol{B}\cdot(\nabla\cdot\mathbf{h}^{\alpha(2)})\rangle$$

$$= \frac{2}{\sqrt{\pi}}\left\langle \int \mathrm{d}x\, x^{3/2}\,\mathrm{e}^{-x}\int \mathrm{d}\xi\,\xi^2\frac{B_\theta}{l_\theta}\left(-\tfrac{1}{2}\eta\,\sin\theta\frac{1}{\xi}(1-\xi^2)\frac{\partial}{\partial\xi}\right)\right.$$

$$\times\frac{\eta}{2\bar{\nu}_\alpha^{1/3}}\frac{B}{\mathscr{B}_0}\sqrt{x}\,S_\alpha(x)\left[\sin\theta\int \mathrm{d}\tau\,\cos\left(\nu_\alpha^{-1/3}\xi\tau\right)\mathrm{e}^{-\tau^3/6}\right.$$

$$\left.\left.-\cos\theta\int \mathrm{d}\tau\,\sin\left(\bar{\nu}_\alpha^{-1/3}\xi\tau\right)\mathrm{e}^{-\tau^3/6}\right]\right\rangle. \tag{4.6}$$

We note that, in the framework of the standard model, B_θ/l_θ is, to leading order, independent of θ and $B/\mathscr{B}_0 \approx 1$. The formula thus reduces to

$$\begin{aligned}
&\sqrt{2}\,\langle \boldsymbol{B}\cdot(\nabla\cdot\mathbf{h}^{\alpha(2)})\rangle \\
&\quad = -\frac{1}{4}\frac{B_\theta}{l_\theta}\langle\sin^2\theta\rangle\frac{2}{\sqrt{\pi}}\int_0^\infty \mathrm{d}x\, x^{3/2}\,\mathrm{e}^{-x}\,S_\alpha(x)\big(\eta^2\bar{\nu}_\alpha^{1/3}\big) \\
&\qquad \times\int_{-1}^{1}\mathrm{d}\xi\,\xi\frac{\partial}{\partial\xi}\int_0^\infty \mathrm{d}\tau\cos\big(\bar{\nu}_\alpha^{-1/3}\xi\tau\big)\,\mathrm{e}^{-\tau^3/6} \\
&\quad = \frac{2}{8\sqrt{\pi}}\frac{B_\theta}{l_\theta}\int_0^\infty \mathrm{d}x\, x^2\,\mathrm{e}^{-x}\,S_\alpha(x)\cdot\frac{\eta^2}{\bar{\nu}_\alpha^{1/3}}\cdot\pi\bar{\nu}_\alpha^{1/3} \\
&\quad = \frac{\sqrt{\pi}}{4}\eta^2\frac{B_\theta}{l_\theta}\int_0^\infty \mathrm{d}x\, x^2\,\mathrm{e}^{-x\sqrt{\frac{3}{2}}}\,L_0^{3/2}(x) \\
&\qquad \times\sqrt{3}\left[\omega_1^\alpha L_0^{3/2}(x)+\omega_3^\alpha L_1^{3/2}(x)+\omega_5^\alpha L_2^{3/2}(x)\right] \\
&\quad = \frac{\sqrt{2\pi}}{4}\frac{B_\theta}{l_\theta}\eta^2\left(2\omega_1^\alpha+\sqrt{\tfrac{2}{5}}\,\omega_3^\alpha-\frac{1}{\sqrt{70}}\omega_5^\alpha\right).
\end{aligned} \tag{4.7}$$

All the integrals necessary for this calculation are explained in Appendix 16A.1. Going over to the dimensionless generalized source terms defined in (**12**.6.4), we find

$$\left\langle\frac{B}{\mathscr{B}_0}\bar{g}_\parallel^{\alpha(1)}\right\rangle = -\frac{\sqrt{\pi}}{4}\tau_\alpha\left(\frac{2T_\alpha}{m_\alpha}\right)^{1/2}\eta^2\frac{B_\theta}{l_\theta\mathscr{B}_0}\left(2\omega_1^\alpha+\sqrt{\tfrac{2}{5}}\,\omega_3^\alpha-\frac{1}{\sqrt{70}}\omega_5^\alpha\right). \tag{4.8}$$

An exactly analogous calculation provides the values of the 4th and 6th order generalized stresses. We find a set of equations which can be cast in the same form as relations (**15**.1.1):

$$\begin{aligned}
\left\langle\frac{B}{\mathscr{B}_0}\bar{g}_\parallel^{\alpha(1)}\right\rangle &= -\varphi_{\alpha*}(\lambda_{11}\omega_1^\alpha+\lambda_{13}\omega_3^\alpha+\lambda_{15}\omega_5^\alpha), \\
\left\langle\frac{B}{\mathscr{B}_0}\bar{g}_\parallel^{\alpha(3)}\right\rangle &= -\varphi_{\alpha*}(\lambda_{31}\omega_1^\alpha+\lambda_{33}\omega_3^\alpha+\lambda_{35}\omega_5^\alpha), \\
\left\langle\frac{B}{\mathscr{B}_0}\bar{g}_\parallel^{\alpha(5)}\right\rangle &= -\varphi_{\alpha*}(\lambda_{51}\omega_1^\alpha+\lambda_{53}\omega_3^\alpha+\lambda_{55}\omega_5^\alpha).
\end{aligned} \tag{4.9}$$

Table 4.1
The plateau pseudo-viscosity coefficients.

Coefficient	Value
λ_{11}	2
$\lambda_{13}=\lambda_{31}$	$\sqrt{2/5}=$ 0.6325
$\lambda_{15}=\lambda_{51}$	$-1/\sqrt{70}=-0$ 1195
λ_{33}	$13/5=$ 2.6000
$\lambda_{35}=\lambda_{53}$	$23/(10\sqrt{7})=$ 0.8693
λ_{55}	$433/140=$ 3.0929

The coefficients λ_{pq}, which are *independent of the species index* α, are the *plateau pseudo-viscosity coefficients*; their values are collected in table 4.1. The matrix of these coefficients possesses the characteristic "Onsager-like" symmetry,

$$\lambda_{pq}=\lambda_{qp}. \tag{4.10}$$

On the other hand, the *plateau factor* $\varphi_{\alpha *}$ is defined as

$$\varphi_{\alpha *}=\frac{\sqrt{\pi}}{4}\eta^2\left(\frac{2T_\alpha}{m_\alpha}\right)^{1/2}\frac{B_\theta}{l_\theta \mathscr{B}_0}\tau_\alpha. \tag{4.11}$$

In order to understand its meaning, we note that in the standard model,

$$\frac{\mathscr{B}_0 l_\theta}{B_\theta}\simeq qR_0. \tag{4.12}$$

This characteristic length is called the *connection length* in the fusion literature. We may construct with it a characteristic frequency, $V_{T\alpha}/qR_0$ (where $V_{T\alpha}$ is the thermal speed of species α). This is called the *untrapped particle transit frequency* (Hinton and Hazeltine 1976); its inverse measures the time it takes on the average for an untrapped particle to make a poloidal circuit. We thus define the characteristic transit time $\mathscr{T}_{T\alpha}$ as

$$\frac{1}{\mathscr{T}_{T\alpha}}\equiv\frac{V_{T\alpha}}{qR_0}=\left(\frac{2T_\alpha}{m_\alpha}\right)^{1/2}\frac{B_\theta}{l_\theta \mathscr{B}_0}. \tag{4.13}$$

Finally, we construct a natural *dimensionless collision frequency* $\nu_{\alpha *}$ as the ratio of the collision frequency to the transit frequency,

$$\nu_{\alpha *}\equiv\frac{\mathscr{T}_{T\alpha}}{\tau_\alpha}=\mathscr{T}_{T\alpha}\nu_\alpha=qR_0\left(\frac{m_\alpha}{2T_\alpha}\right)^{1/2}\nu_\alpha. \tag{4.14}$$

[This quantity is not to be confused with the quantity $\bar{\nu}_\alpha(x)$, which is a function of the energy.]

The *plateau factor* $\varphi_{\alpha *}$ is thus proportional to (but much smaller than) the inverse dimensionless collision frequency,

$$\varphi_{\alpha *} = \tfrac{1}{4}\sqrt{\pi}\,\eta^2 \nu_{\alpha *}^{-1}. \tag{4.15}$$

The result (4.9) is fundamental. It shows that in the plateau regime *there exists again a linear relationship between the generalized stresses and the "poloidal fluxes"*. From here on, the transport equations are derived exactly as in section **15**.2 from an analysis of the moment equations. Hence, in one stroke, we obtain the final result:

The plateau transport equations are of the same form as the equations (**15**.3.1)–(**15**.3.3); *they are obtained by the mere substitution*

$$\varphi \to \varphi_{\alpha *}.$$

The expressions of the plateau transport coefficients (both dimensionless and dimensional) are obtained from those of tables **15**.3.2–**15**.3.4 *by the substitutions*

$$\varphi \to \varphi_{\alpha *}, \qquad \mu_{pq}^{\alpha} \to \lambda_{pq}. \tag{4.16}$$

16.5 The plateau transport equations

In spite of the identity *in form* of the banana and the plateau transport coefficients, the *content* of the formulae is very different. Indeed, *the neoclassical factor* φ *is a purely geometrical quantity*, determined entirely by the shape of the confining magnetic field, as appears clearly from its definition (**14**.8.1), (**14**.8.8). On the contrary, *the plateau factor* $\varphi_{\alpha *}$ *depends specifically on the dynamics of the particles* through the collision frequency. As a result, the main difference between the plateau transport coefficients on one hand, and the banana and Pfirsch–Schlüter coefficients on the other hand, lies in their dependence on the collision frequency. In discussing this problem, we must again make a distinction between the diffusive and the electrical transport coefficients, as defined in section **15**.3.

A. Diffusion coefficient

As all the diffusive coefficients have the same structure (table **15**.3.4), we discuss the diffusion coefficient L_{11}^{ee} as an illustration of their common properties. Their specific differences are introduced in subsection B.

The *diffusion coefficient* in the plateau domain is obtained from table **15**.3.4 with the substitutions (4.16),

$$L_{11}^{ee} \equiv \mathscr{L}_{11}^{P} = \tfrac{1}{2} n_e \rho_{e0}^2 \mathscr{J}^2 \frac{1}{\tau_e} \varphi_{e*} \hat{l}_{11}^{ee}(\varphi_{e*}), \tag{5.1}$$

where $\hat{l}_{11}^{ee}(\varphi_{e*})$ denotes the dimensionless diffusion coefficient $l_{11}^{ee}(\varphi)$ (table **15**.3.3), evaluated with the plateau pseudo-viscosities λ_{pq} instead of the banana pseudo-viscosities μ_{pq}^{α}. This coefficient is to be compared with the banana diffusion coefficient on one hand (table **15**.3.4),

$$\mathscr{L}_{11}^{B} = \tfrac{1}{2} n_e \rho_{e0}^2 \mathscr{J}^2 \frac{1}{\tau_e} \varphi l_{11}^{ee}(\varphi), \tag{5.2}$$

and with the Pfirsch–Schlüter diffusion coefficient on the other hand [table **13**.3.1],

$$\mathscr{L}_{11}^{PS} = \tfrac{1}{2} n_e \rho_{e0}^2 \mathscr{J}^2 \frac{1}{\tau_e} (\mathscr{G} - 1) c_{11}^{e}. \tag{5.3}$$

Both the banana and the Pfirsch–Schlüter coefficients are *simply proportional to the collision frequency* $\nu_e \equiv \tau_e^{-1}$. Their difference lies in the *slope* of these linear functions: *this slope is much larger for the banana coefficient than for the PS coefficient*, because of the large banana amplification factor (section **15**.5). On the contrary, *the plateau coefficient has a non-trivial dependence on the collision frequency*, determined by the function $\varphi_{e*} \hat{l}_{11}^{ee}(\varphi_{e*})$.

In order to make the comparison more explicit, we express these coefficients in the standard model, in which the (local) inverse aspect ratio η is considered very small (this was a necessary condition for the existence of the plateau regime). In this case (see eqs. **13**.4.15, **14**.8.11)

$$\mathscr{J}^2 = q^2 \eta^{-2}, \qquad \mathscr{G} - 1 = 2\eta^2, \qquad \phi = 1.469 \sqrt{\eta}\,. \tag{5.4}$$

Next, we define, for convenience, dimensionless diffusion coefficients $\tilde{\mathscr{L}}_{11}$ scaled with the transit time $\mathscr{T}_{T\alpha}$ rather than the relaxation time τ_α,

$$\tilde{\mathscr{L}}_{11} = \frac{\mathscr{L}_{11}}{\tfrac{1}{2}\left(n_e \rho_{e0}^2 / \mathscr{T}_{Te}\right) q^2}. \tag{5.5}$$

Expressing τ_e in terms of the "natural" collision frequency ν_{e*} (eq. 4.14), our previous formulae become

$$\tilde{\mathscr{D}}_{11}^{B} = 1.469\sqrt{\eta}\; l_{11}^{ee}(1.469\sqrt{\eta})\,\eta^{-2}\nu_{e*}, \tag{5.6}$$

$$\tilde{\mathscr{D}}_{11}^{P} = \varphi_{e*}\,\hat{l}_{11}^{ee}(\varphi_{e*})\,\eta^{-2}\nu_{e*}, \tag{5.7}$$

$$\tilde{\mathscr{D}}_{11}^{PS} = 2\,\hat{l}_{11}^{ee}\eta^{2}\,\eta^{-2}\nu_{e*}, \tag{5.8}$$

where $\hat{l}_{11}^{ee}$ is the PS coefficient,

$$\hat{l}_{11}^{ee} = c_{11}^{e}.$$

We now note that, for small η, we may approximate

$$l_{11}^{ee}(1.469\sqrt{\eta}) \approx \mu_{11}^{ee} + O(\sqrt{\eta}). \tag{5.9}$$

Similarly, for small φ_{e*} (see eq. 3.9) we take

$$\hat{l}_{11}^{ee}(\varphi_{e*}) \approx \lambda_{11} + O(\varphi_{e*}). \tag{5.10}$$

Using also (4.15) and inserting the numerical values, we find (for $Z = 1$)

$$\tilde{\mathscr{D}}_{11}^{B} \simeq 2.252\,\eta^{-3/2}\nu_{e*},$$

$$\tilde{\mathscr{D}}_{11}^{P} \simeq \frac{\sqrt{\pi}}{2},$$

$$\tilde{\mathscr{D}}_{11}^{PS} \simeq 2\,\nu_{e*}. \tag{5.11}$$

The remarkable fact which appears here is that *in the plateau regime the diffusion coefficient is (approximately) independent of the collision frequency.* This property justifies the name given to this regime by Galeev and Sagdeev (1968).

Imagine now that, starting at a very small value, the collision frequency ν_{e*} is progressively increased. The plasma then crosses the three regimes in succession, from the banana through the plateau to the PS regime. On the basis of the approximate relations (5.11), the dimensionless diffusion coefficient varies *schematically* as shown in fig. 5.1. In the banana regime it is represented by a steep straight line; in the plateau regime by a horizontal "plateau", and in the PS regime by a line with a small slope.

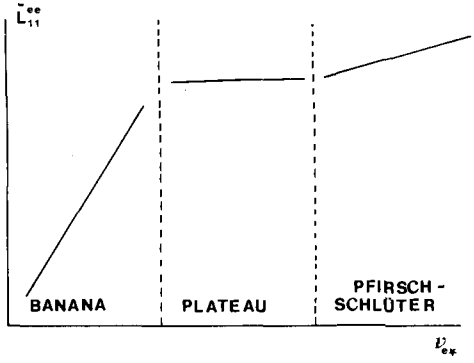

Fig. 5.1 "Traditional" view of the diffusion coefficient $\tilde{L}_{11}^{ee}$ as a function of the collision frequency ν_{e*} (schematic)

An interesting question is to determine the approximate boundaries of the three regimes. This can be done by the following rough argument. We keep in mind that $\varphi_{\alpha*}$ is inversely proportional to $\nu_{\alpha*}$. Looking at eqs. (5.8) and (5.7), we see that when

$$\varphi_{\alpha*} \approx 1.469\sqrt{\eta} \equiv \varphi_B \qquad [P \leftrightarrow B] \tag{5.12}$$

the plateau coefficient goes over into the banana coefficient. Similarly, when

$$\varphi_{\alpha*} \approx 2\eta^2 \equiv \varphi_{PS} \qquad [P \leftrightarrow PS] \tag{5.13}$$

the plateau coefficient goes over into the PS coefficient. We thus bound the three regimes as

$$\varphi_{PS} \ll \varphi_B \ll \varphi_{\alpha*}: \quad \text{Banana,} \tag{5.14}$$

$$\varphi_{PS} \ll \varphi_{\alpha*} \ll \varphi_B: \quad \text{Plateau,} \tag{5.15}$$

$$\varphi_{\alpha*} \ll \varphi_{PS} \ll \varphi_B: \quad \text{Pfirsch–Schlüter.} \tag{5.16}$$

This criterion can also be expressed as a set of conditions on the natural collision frequency *,

$$\begin{aligned} \nu_{\alpha*} &\ll 0.302\,\eta^{3/2} &&: \text{Banana,} \\ 0.302\,\eta^{3/2} \ll \nu_{\alpha*} &\ll 0.222 &&: \text{Plateau,} \\ 0.222 \ll \nu_{\alpha*} & &&: \text{Pfirsch-Schlüter.} \end{aligned} \tag{5.17}$$

This criterion is in agreement with condition (3.9) used in the solution of the drift kinetic equation. Clearly, in order to have a well-defined plateau regime, we need a very small value of η. We also see that the picture in fig. 5.1

* The numbers in these formulae are $(\sqrt{\pi}/8) \simeq 0.222$, $(1.469\sqrt{\pi}/4) \simeq 0.302$.

is strongly exaggerated: when expressed as a function of $\nu_{\alpha *}$, the banana regime is an extremely narrow strip (of width $\sim \eta^{3/2}$) near the origin [see also fig. 6.1].

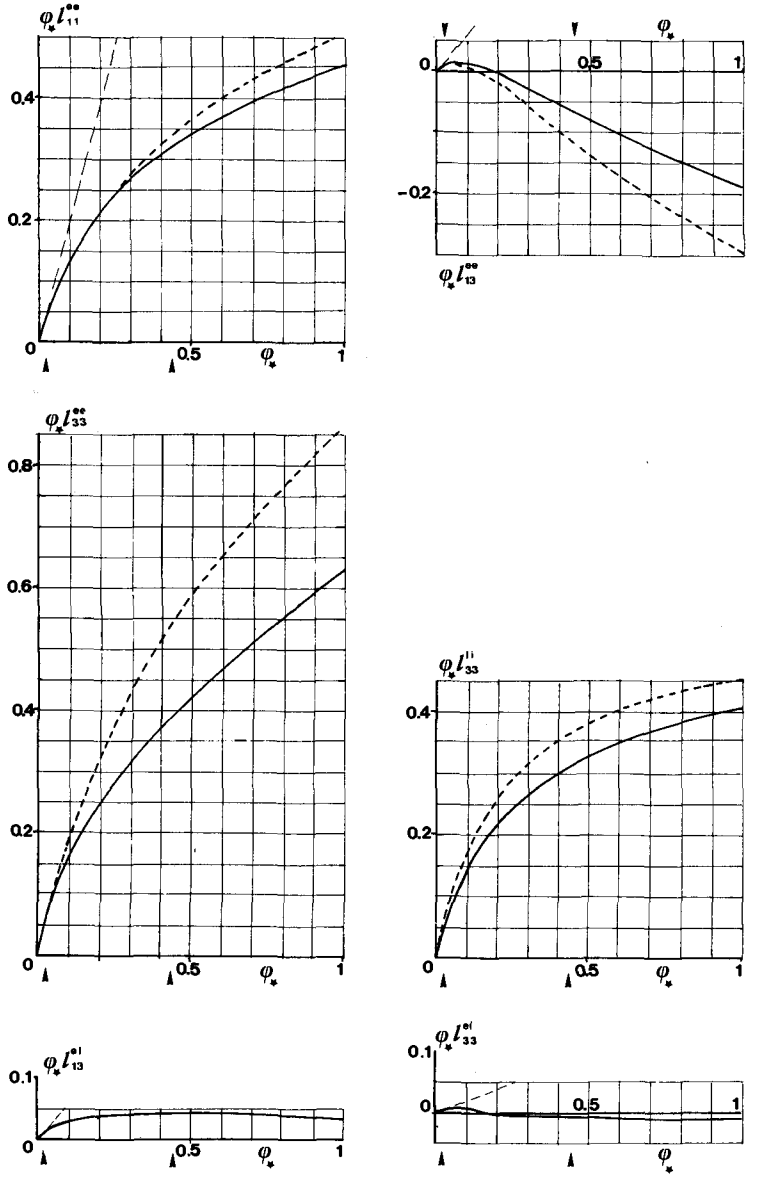

Fig. 5.2. Dimensionless transport coefficients in the plateau regime, as functions of the plateau factor φ_* ($Z = 1$). Solid line: 21M approximation; dashed line: 13M approximation.

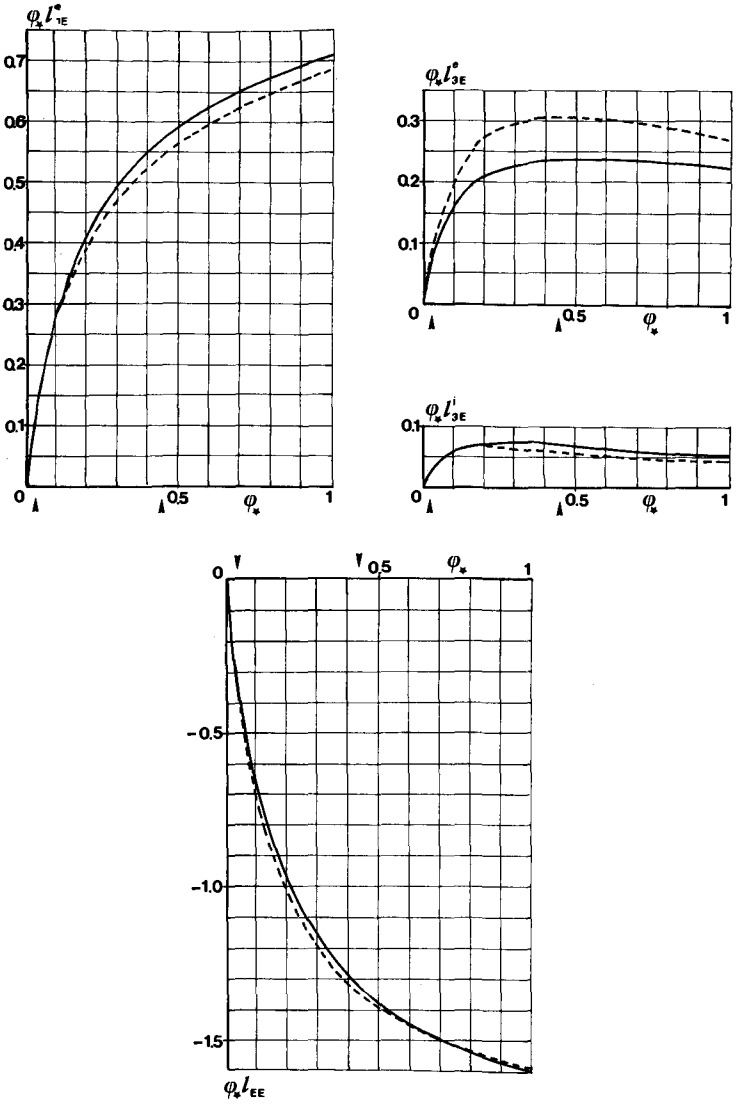

Fig. 5.2. Continued.

Summing up this discussion, we see that the main interest of the plateau regime is to provide a better understanding of the transition from the long mean free path regime to the short mean free path regime. This question will be further discussed in the next section. We now wish to investigate the

specific features of the various transport coefficients in the plateau regime. These are determined by the function $\hat{l}^{\alpha\beta}_{pq}(\varphi_{\alpha *})$.

B. Diffusive coefficients

The dimensionless diffusive coefficients $\varphi_{\alpha *}\hat{l}^{\alpha\beta}_{pq}(\varphi_{\alpha *})$ are represented in figs. 5.2 as functions of $\varphi_{\alpha *}$. Our first remark concerns the use to be made of these figures. There is a serious difference with the graphs of figs. **15**.4.1–**15**.4.4, because of the different meaning of the factors φ and $\varphi_{\alpha *}$. In the latter figures, the purely geometrical *neoclassical factor* φ is not limited by any condition: in principle, the whole range of φ between 0 and ∞ is physically accessible. On the contrary, the *plateau factor* $\varphi_{\alpha *}$ is limited to a range defined by eq. (5.15). The location and the width of this range depends on the value of η. As a result, the value of $\varphi_{\alpha *}$ exceeding 1 are certainly unphysical (they are in the banana regime). As an illustration, the (maximal) boundaries of the plateau regime are shown in the figures for the particular case $\eta = 0.1$. *The evolution from the banana regime to the PS regime corresponds to a variation from right to left between these two boundaries.*

From the graphs of fig. 5.2 it appears that the *convergence of the moment method* is, globally viewed, not as fast as in the banana regime. The difference between the 13M and 21M values is more like in the classical regime [see

Table 5.1
Initial slopes of the plateau transport coefficients

Coefficient	Analytic expression	Numerical value *	
		13M	21M
Diffusive coefficients			
$\hat{l}^{ee}_{11}$	2	2.000	2.000
$\hat{l}^{ee}_{33}$	13/5	2 600	2 600
$\hat{l}^{ii}_{33}$	12/5	2.400	2.400
$\hat{l}^{ee}_{13}$	$\sqrt{2/5}$	0.633	0.633
$\hat{l}^{ei}_{13}$	$\sqrt{2/5}$	0 633	0 633
$\hat{l}^{ei}_{33}$	1/5	0.200	0.200
Electrical coefficients			
$\hat{l}^{e}_{1E}$	$2\sigma - \sqrt{2/5}\,\alpha + \sqrt{1/70}\,\gamma$	4.485	4 470
$\hat{l}^{e}_{3E}$	$\sqrt{2/5}\,\sigma - (13/5)\alpha - (23/10\sqrt{7})\gamma$	3 776	3.399
$\hat{l}^{i}_{3E}$	$\sqrt{1/10}\,\hat{l}^{e}_{1E}$	1.419	1.414
$\hat{l}_{EE}$	$-\sigma\hat{l}^{e}_{1E} + \alpha\hat{l}^{e}_{3E} + \gamma\hat{l}^{e}_{5F}$	−12.376	−11 681

* All values for $Z = 1$.

section 5.4]. However, *within the range of the plateau regime*, the difference between the two values is, in general, rather small.

In order to make a comparison with the results of other authors, we listed in table 5.1 the values of the initial slopes $\hat{I}^{\alpha\beta}_{pq}(0)$. They are obtained from table 15.4.1 with the substitutions (4.16) and the numerical values of table 4.1. These initial slopes correspond to the numbers calculated by Hinton and Hazeltine (1976). *

Our results *in the 13M approximation* agree exactly with theirs (note, however, that Hinton and Hazeltine do not give any value for the mixed electron–ion coefficients L^{ei}_{13}, L^{ei}_{33}, L^{i}_{3E}). It is actually clear that for the *diffusive coefficients*, these initial slopes are independent of the level of truncation (just as in the banana case), and are determined solely by the pseudo-viscosity coefficients.

The *diagonal transport coefficients*, i.e. the diffusion coefficient $\hat{I}^{ee}_{11}$ and the thermal conductivities $\hat{I}^{ee}_{33}$, $\hat{I}^{ii}_{33}$ behave roughly like the banana coefficient. They are *definite positive* over the whole range of φ_*,

$$\hat{I}^{\alpha\alpha}_{rr}(\varphi_{\alpha *}) > 0, \qquad \alpha = e, i, \quad r = 1, 3. \tag{5.18}$$

They are uniformly increasing functions of φ_*.

A surprise awaits us, however, when we consider the non-diagonal transport coefficients. Consider the most common of these, the *electron thermodiffusion coefficient* $\varphi_{e*}\hat{I}^{ee}_{13}(\varphi_{e*})$. This coefficient starts from zero with a *positive initial slope*, reaches a maximum and then decreases and changes sign as it goes on toward more and more negative values. ** We note that the change of sign occurs within the plateau domain (for reasonable values of η). Hence, *for small φ_{e*}, the sign of the plateau thermodiffusion coefficient differs from both the corresponding banana and PS coefficients.*

The variation of the thermodiffusion coefficient as the collision frequency increases can be described as follows: it first becomes more and more negative [banana regime (5.6)] until it reaches the right-hand side boundary of the plateau curve; at this stage it starts increasing and eventually becomes positive. The transition to the PS regime cannot be described by the present results, because within this regime the thermodiffusion is again negative and goes on decreasing [eq. (5.8) with $c_{11} \to -c_{13}$]. This type of collision-frequency

* In order to make the comparison, one must substitute the numbers $\hat{I}^{\alpha\beta}_{pq}(0)$ from table 5.1 into the expressions of table 15 3 4, and then collect all the numerical factors $\sqrt{\pi}/4$ (from φ_*), $(1/2)$, $\sqrt{5/2}$, etc., appearing in each transport coefficient, in order to obtain the numerical results quoted by these authors

** Hinton and Hazeltine (1976), who find the same *positive* value ($= +\sqrt{2/5}$) for the initial slope, do not comment at all on this unusual property

dependence is completely different from the scheme of fig. 5.1. It also implies that an approximation of type (5.9) (as given by Hinton and Hazeltine 1976) is completely meaningless for the thermodiffusion coefficient.

Exactly the same comments apply to the mixed thermal conductivity coefficient L_{33}^{ei}; however, this coefficient is uniformly very small, as can be seen in fig. 5.2. The mixed coefficient L_{13}^{ei} is also very small: this ensures a smooth transition to the PS values where, as we know, both these mixed coefficients are zero.

C. Electrical coefficients

The behaviour of these coefficients is pretty easily understandable. Consider, for instance, the plateau contribution to the parallel electrical conductivity. It has the main characteristics of the corresponding long mean free path contribution (section **15**.5), namely it is of the same order as the classical conductivity (no amplification factor), but is uniformly negative. Nevertheless, the total conductivity, which in the intermediate mean free path regime is given by

$$\sigma_{\parallel}^{tot} = \sigma_{\parallel}^{CL} + \sigma_{\parallel}^{P},$$

is positive. The interpretation of this feature is, of course different from the long mean free path regime. Here the number of trapped particles is very small. The curves of fig. 5.2 can be easily understood as follows. As the collision frequency increases, the regime changes from banana to plateau at the right boundary. The negative plateau contribution steadily decreases, and the total electrical conductivity practically reaches its classical value at the left boundary.

The electrical cross-coefficients have qualitatively the same properties in the plateau regime as in the long mean free path regime [see section **15**.5]. All these coefficients tend steadily to zero as the PS boundary is approached.

16.6. Interpolation formulae for the diffusion coefficient

After the lengthy discussion of the previous paragraph, we turn our attention to the following question. *Is it possible to derive approximate formulae for the transport coefficients, giving a global view of these quantities over the whole range of collision frequencies*? The term "derivation" is actually not quite appropriate here. The problem can be formulated more precisely as follows. Given our knowledge of the long mean free path, intermediate mean free path and short mean free path regimes, how could we interconnect these partial results in a single formula? Such a formula is just an *interpolation* between the three

regimes. It rests on an "act of faith" that nothing spectacular occurs in between the known regimes. The structure of the drift kinetic equation does not suggest any such spectacular phenomenon, but one should always keep in mind the semi-empirical nature of such formulae.

Many authors proposed interpolation formulae (Hinton and Rosenbluth 1973, Hinton and Moore 1974, Rawls et al. 1975, Tsang and Callen 1976, Hinton and Hazeltine 1976) in which the transport coefficients are fitted (by a least-square method) to rather complicated functional forms. Hirshman and Sigmar (1977) proposed a much simpler interpolation procedure, which they showed to be quite accurate in practice. It is based on the idea that, comparing eqs. (5.6) and (5.7), one (roughly) obtains the plateau coefficient from the banana coefficient by replacing φ_B (eq. (5.12)) by $\varphi_{\alpha *}$. They therefore propose to express all transport coefficients in the plateau form, in terms of a factor φ_t,

$$\varphi_t = \frac{\varphi_B \varphi_{\alpha *}}{\varphi_B + \varphi_{\alpha *}}.$$

This factor reduces to φ_B ($\varphi_{\alpha *}$) in the limit of large (small) $\varphi_{\alpha *}$. One could similarly interpolate between the plateau regime and the PS regime (characterized by $\varphi_{PS} = 2\eta^2$) by using

$$\varphi_t' = \varphi_{\alpha *} + \varphi_t.$$

These substitutions are, however, too rough: they ignore the specificity of the functions $l_{pq}^{\alpha\beta}$, $\hat{l}_{pq}^{\alpha\beta}$. These are different, because the pseudo-viscosities have different values in the banana regime and in the plateau regime. Also, these functions, as well as the corresponding PS functions, are different for the different transport coefficients.

We propose here a set of interpolation formulae that have the simplicity of the Hirshman–Sigmar idea, but are adapted specifically to each transport coefficient. In the present section the interpolation formula is introduced and discussed by considering the diffusion coefficient $\tilde{L}_{11}^{ee}$ (reduced as in eq. 5.5). The procedure will be extended to the other transport coefficients in section 16.7.

We propose the following formula for the diffusion coefficient:

$$\tilde{L}_{11}^{ee} = \frac{\varphi_B l_{11}^{ee}(\varphi_B) \cdot \left(\varphi_{e*} \hat{l}_{11}^{ee}(\varphi_{e*}) + \varphi_{PS} \tilde{l}_{11}^{ee}\right)}{\varphi_B l_{11}^{ee}(\varphi_B) + \varphi_{e*} \hat{l}_{11}^{ee}(\varphi_{e*}) + \varphi_{PS} \tilde{l}_{11}^{ee}} \eta^{-2} \nu_{e*}. \tag{6.1}$$

Clearly, this coefficient reduces to the banana coefficient when relations (5.14) are realized; it reduces to the plateau coefficient in the regime (5.15) and

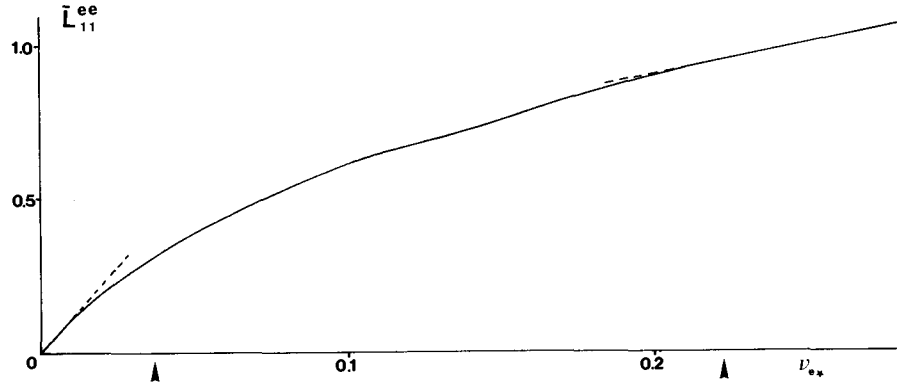

Fig 6.1. Diffusion coefficient $\tilde{L}_{11}^{ee}$ as a function of the collision frequency ν_{e*}. True values for $\eta = 0.25$.

to the PS coefficient when (5.16) prevails. This simple formula provides a good global view of the diffusion coefficient, *provided* η *is small.* If this condition is not well satisfied, the plateau domain is too narrow, i.e. φ_{PS} is too close to φ_B and therefore the limiting value of the diffusion coefficient is not accurate.

We now give a graphical view of the diffusion coefficient over the whole range of collisionalities. Several presentations can be chosen, each one pointing out some characteristic feature.

In fig. 6.1 we show the graph of the reduced diffusion coefficient $\tilde{L}_{11}^{ee}$, as defined by the interpolation formula (6.1), versus the "natural" collision frequency ν_{e*}. We purposely chose a rather large value of $\eta(= 0.25)$ in order to make the three regions clearly visible. Indeed, the banana regime is confined to a very narrow strip near the origin ($\nu_{e*} < 0.04$) in which a linear dependence is recognized. The plateau transition appears very gradually towards a PS linear region for $\nu_{e*} > 0.22$. The same, very gradual shape of the curve appears in the paper by Hinton & Hazeltine (1976). We thus see that the schematic picture of fig. 5.1, which appears in many elementary texts, is grossly exaggerated.

In order to clearly visualize the three regimes, one has to selectively expand or contract the scales of the graphs. This is achieved by drawing the graphs in a log–log representation. In fig. 6.2 we draw the graph of the dimensionless diffusion coefficient $\tilde{L}_{11}^{ee}(\nu_{e*})$, calculated by eq. (6.1), for a rather small value of the inverse aspect ratio ($\eta = 0.01$). In this picture, the three regimes appear quite clearly. At both ends of the graph we find the signature of a linear dependence in the form of straight segments of slope 1. In the middle we have a quite clear plateau region.

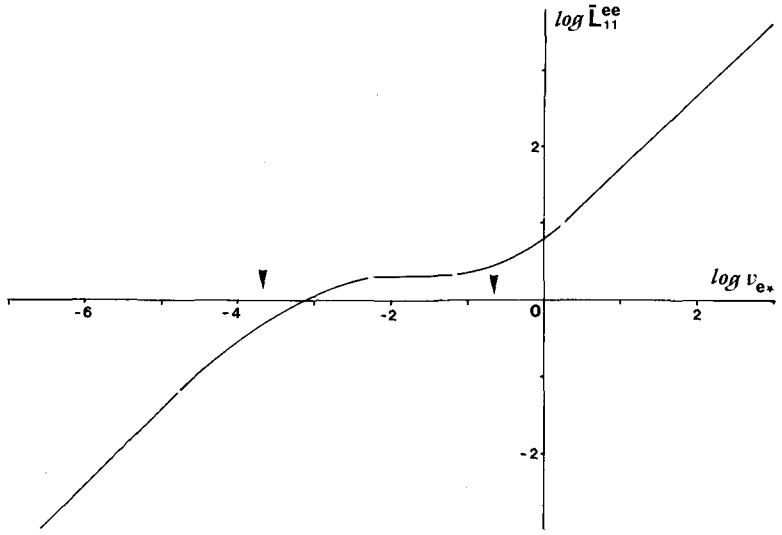

Fig 6.2 Diffusion coefficient as function of the collision frequency, in log–log representation ($\eta = 0.01$).

Finally a rather interesting representation consists of a graph (on a log–log scale) of the quantity $\tilde{L}_{11}^{ee}(\nu_{e*})/\nu_{e*}$ (fig. 6.3). Here, the two extreme regimes show up as constants, corresponding to the limiting values of eq. (5.11). In

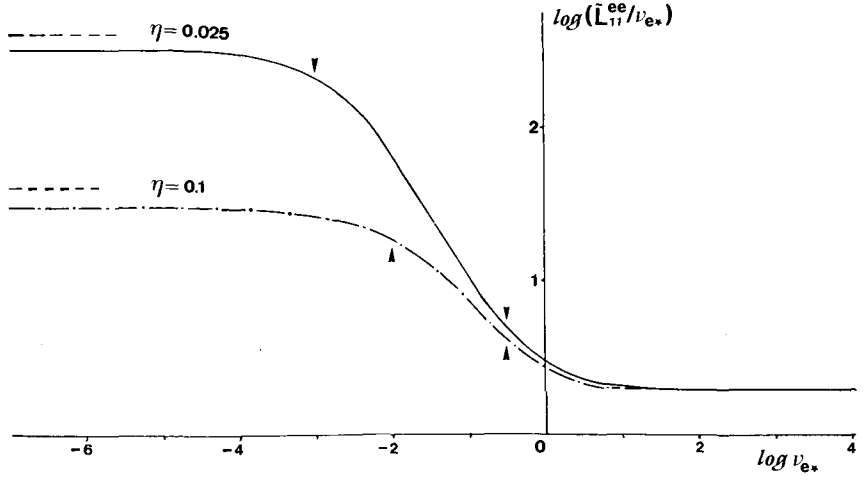

Fig 6 3 The "local slope" $\tilde{L}_{11}^{ee}/\nu_{e*}$ of the diffusion coefficient as a function of the collision frequency, in log–log representation The dashed lines represent the exact limiting values calculated from eq (5.11).

between, there is a region of slope (-1), corresponding to the plateau regime. We clearly see the transition from a steep slope $(2.252\eta^{-3/2})$ in the banana regime to the small slope $(=2)$, independent of η, in the PS regime.

We also have a convenient check on the accuracy of the interpolation formulae in the two extreme regimes. It can be seen that the PS regime is reached almost without any error, whereas in the banana regime there is a relatively small error (which increases with η).

16.7. Approximate transport equations for the entire collision frequency range

Quite naturally, we are tempted to extend the interpolation procedure described in section 16.6 for the diffusion coefficient to all the other transport coefficients. Whereas the method works satisfactorily for some of these (in particular, the diagonal transport coefficients), it badly fails for some of the non-diagonal ones, in particular for the thermodiffusion coefficient $\tilde{L}_{13}^{ee}$ (and for the mixed thermal conductivity $\tilde{L}_{33}^{ei}$). The reason is the change of sign which occurs in the plateau regime. This unusual property was discussed in the previous section. The interpolation formula (6.1) correctly describes the change of sign in going from the banana regime to the plateau regime, but is unable to restore a negative sign as one goes over into the PS regime.

There is a way out of this difficulty: it was suggested by Hinton and Hazeltine (1976) but, strangely, was not consistently pursued to its completion (see the discussion at the end of this section). It starts from the remark that the *total average radial energy flux* Q_ρ^α in the plasma has two components: the heat flux q_ρ^α and the flux of enthalpy carried by the particle flux,

$$\left\langle Q_\rho^\alpha \right\rangle = \left\langle q_\rho^\alpha \right\rangle + \tfrac{5}{2} T_\alpha \left\langle \Gamma_\rho^\alpha \right\rangle. \tag{7.1}$$

Anticipating our forthcoming results, it will be shown in chapter 18 (tables **18**.3.1 and **18**.4.1) that precisely this combination enters the internal energy balance equations for the magnetically confined plasma.

The discussion at the end of section **12**.4 is quite relevant at this point. In "ordinary" (one-fluid) hydrodynamics, the second term in (7.1) is called a convective contribution to the energy flux; it is determined by the differential equation of motion for $n_\alpha \boldsymbol{u}$ (the Navier–Stokes equation). In the present, two-fluid theory of a confined plasma, the average radial particle flux appears as a *diffusive flux*, determined by an algebraic transport equation in terms of the thermodynamic forces, just like the heat flux. Hence, it is quite possible to

combine the transport equations into a new set of four equations for the following fluxes (written in the second set of notations of eq. **15**.3.10):

$$J_1' = J_1 = \langle \Gamma_\rho^e \rangle,$$

$$J_2' = J_2 + \tfrac{5}{2} J_1 = T_e^{-1} \Big(\langle q_\rho^e \rangle + \tfrac{5}{2} T_e \langle \Gamma_\rho^e \rangle \Big),$$

$$J_3' = J_3 + \tfrac{5}{2} Z^{-1} J_1 = T_i^{-1} \Big(\langle q_\rho^i \rangle + \tfrac{5}{2} Z^{-1} T_i \langle \Gamma_\rho^e \rangle \Big),$$

$$J_4' = J_4 = \mathscr{B}_0^{-1} \langle B j_\parallel \rangle, \tag{7.2}$$

where we used the ambipolarity condition $\langle \Gamma_\rho^i \rangle = Z^{-1} \langle \Gamma_\rho^e \rangle$. This linear transformation can be expressed in terms of a transformation matrix,

$$J_\mu' = P_{\mu\nu} J_\nu, \tag{7.3}$$

According to the general transformation theory of non-equilibrium thermodynamics (de Groot and Mazur 1984, Ch. VI, § 5) it is convenient to modify also the definition of the thermodynamic forces, in such a way as to leave invariant the bilinear form in the fluxes and the forces,

$$\mathscr{Q} = \sum_{\mu=1}^{4} J_\mu X_\mu = \sum_{\mu=1}^{4} J_\mu' X_\mu'. \tag{7.4}$$

This condition requires a transformation of the thermodynamic forces,

$$X_\mu' = R_{\mu\nu} X_\nu, \tag{7.5}$$

such that

$$P_{\mu\rho} R_{\mu\sigma} = \delta_{\rho\sigma}. \tag{7.6}$$

It is easily checked that the new forces are

$$X_1' = X_1 - \tfrac{5}{2} X_2 - \tfrac{5}{2} Z^{-1} X_3$$

$$= -\Big\{ (n_e T_e)^{-1} \nabla_\rho P - \tfrac{5}{2} \Big[T_e^{-1} \nabla_\rho T_e + (Z T_i)^{-1} \nabla_\rho T_i \Big] \Big\},$$

$$X_2' = X_2 = -T_e^{-1} \nabla_\rho T_e,$$

$$X_3' = X_3 = -T_i^{-1} \nabla_\rho T_i,$$

$$X_4' = X_4 = \mathscr{B}_0^{-1} \Big\langle B E_\parallel^{(A)} \Big\rangle. \tag{7.7}$$

In non-equilibrium thermodynamics, the bilinear form $\mathcal{Q}$ is interpreted as the entropy production (see sec. **6**.3, and de Groot and Mazur 1984). It will be shown in chapter 17 (Sections **17**.7, **17**.8) that this identification is no longer valid in the banana–plateau regime. Nevertheless, whereas the form-invariance of the entropy production is no longer a good justification for the choice (7.7), it remains true that the latter ensures the *Onsager symmetry of the transformed transport matrix*, a very desirable property.

We also defined a set of *dimensionless fluxes* by the following relations (we only consider the radial fluxes and forces, as the parallel ones are unchanged under the transformation):

$$J_1' \equiv n_e \left(\frac{T_e}{m_e}\right)^{1/2} \left\langle h_\rho'^{e(1)} \right\rangle,$$

$$J_2' \equiv \sqrt{\tfrac{5}{2}}\, n_e \left(\frac{T_e}{m_e}\right)^{1/2} \left\langle h_\rho'^{e(3)} \right\rangle,$$

$$J_3' \equiv \sqrt{\tfrac{5}{2}}\, n_i \left(\frac{T_i}{m_i}\right)^{1/2} \left\langle h_\rho'^{i(3)} \right\rangle. \tag{7.8}$$

The corresponding forces are

$$X_1' \equiv \left(\frac{m_e}{T_e}\right)^{1/2} \frac{1}{\tau_e}\, g_\rho'^{(1)P},$$

$$X_2' \equiv \sqrt{\frac{2}{5}} \left(\frac{m_e}{T_e}\right)^{1/2} \frac{1}{\tau_e}\, g_\rho'^{e(3)},$$

$$X_3' \equiv \sqrt{\frac{2}{5}} \left(\frac{m_i}{T_i}\right)^{1/2} \frac{1}{\tau_i}\, g_\rho'^{i(3)}. \tag{7.9}$$

The transformation relations are now expressed as follows.
Dimensionless fluxes:

$$\left\langle h_\rho'^{e(1)} \right\rangle = \left\langle h_\rho^{e(1)} \right\rangle,$$

$$\left\langle h_\rho'^{e(3)} \right\rangle = \left\langle h_\rho^{e(3)} \right\rangle + \sqrt{\tfrac{5}{2}} \left\langle h_\rho^{e(1)} \right\rangle,$$

$$\left\langle h_\rho'^{i(3)} \right\rangle = \left\langle h_\rho^{i(3)} \right\rangle + \sqrt{\tfrac{5}{2}}\, a^{-1} \left\langle h_\rho^{e(1)} \right\rangle. \tag{7.10}$$

Dimensionless forces:

$$g_\rho'^{(1)P} = g_\rho^{(1)P} - \sqrt{\tfrac{5}{2}}\, g_\rho^{e(3)} - \sqrt{\tfrac{5}{2}}\, aA\, g_\rho^{i(3)},$$

$$g_\rho'^{e(3)} = g_\rho^{e(3)},$$

$$g_\rho'^{i(3)} = g_\rho^{i(3)}. \tag{7.11}$$

The factors a and A were defined in (**15**.3.4).

The *dimensionless transport equations* are now expressed in a form similar to eqs. (**15**.3.1)–(**15**.3.3) (for the privileged fluxes),

$$\left\langle h_\rho'^{e(r)} \right\rangle_B = \mathscr{K}_e^2 \varphi \left[l_{r1}'^{ee}(\varphi)\, g_\rho'^{(1)P} + l_{r3}'^{ee}(\varphi)\, g_\rho'^{e(3)} + l_{r3}'^{ei}(\varphi)\, aA\, g_\rho'^{i(3)} \right]$$

$$+ \mathscr{K}_e\, \varphi l_{rE}'^{e}(\varphi)\, \hat{g}_\parallel^{(1)A}, \quad r = 1, 3, \tag{7.12}$$

$$\left\langle h_\rho'^{i(3)} \right\rangle_B = \mathscr{K}_e^2 a^{-1} \varphi \left[l_{31}'^{ie}(\varphi)\, g_\rho'^{(1)P} + l_{33}'^{ie}(\varphi)\, g_\rho'^{e(3)} \right]$$

$$+ \mathscr{K}_i^2\, \varphi l_{33}'^{ii}(\varphi)\, g_\rho'^{i(3)} + \mathscr{K}_e a^{-1} \varphi l_{3E}'^{i}(\varphi)\, \hat{g}_\parallel^{(1)A}, \tag{7.13}$$

$$\mathscr{B}_0^{-1} \left\langle B h_\parallel^{(1)} \right\rangle_B = \mathscr{K}_e\, \varphi \left[l_{E1}'^{e}(\varphi)\, g_\rho'^{(1)P} + l_{E3}'^{e}(\varphi)\, g_\rho'^{e(3)} + l_{E3}'^{i}(\varphi)\, aA\, g_\rho'^{i(3)} \right]$$

$$+ \varphi l_{EE}'(\varphi)\, \hat{g}_\parallel^{(1)A}. \tag{7.14}$$

The *transformed transport coefficients* $l_{rs}'^{\alpha\beta}$ are collected in table 7.1. The main advantage of these transformed coefficients is that *they keep the same sign in the banana, the plateau and the Pfirsch–Schlüter regimes*. Hence, the difficulty mentioned at the beginning of this section is resolved and we can construct a complete set of interpolated transport coefficients by extending the method of section 16.6. The various classes of transport coefficients must be discussed separately, because of the different dependence on the collision frequency.

A, B. Diffusive coefficients

The common feature here is the form of the relation between dimensional and dimensionless coefficients (see table **15**.3.4). Moreover, all these coefficients have a non-vanishing limit in the PS regime. Note that the latter property was not true for the mixed diffusive coefficients l_{13}^{ei}, l_{33}^{ei}, which vanish in that limit;

Table 7.1
Transformed dimensionless transport coefficients

Pure diffusive coefficients

$$l'^{ee}_{11}(\varphi) = l^{ee}_{11}(\varphi)$$

$$l'^{ee}_{33}(\varphi) = l^{ee}_{33}(\varphi) + 2\sqrt{\tfrac{5}{2}}\, l^{ee}_{13}(\varphi) + \tfrac{5}{2} l^{ee}_{11}(\varphi)$$

$$l'^{ii}_{33}(\varphi) = l^{ii}_{33}(\varphi)$$

$$l'^{ee}_{13}(\varphi) = l'^{ee}_{31}(\varphi) = l^{ee}_{13}(\varphi) + \sqrt{\tfrac{5}{2}}\, l^{ee}_{11}(\varphi)$$

Mixed diffusive coefficients

$$l'^{ei}_{13}(\varphi) = l'^{ie}_{31}(\varphi) = l^{ei}_{13}(\varphi) + \sqrt{\tfrac{5}{2}}\, l^{ee}_{11}(\varphi)$$

$$l'^{ei}_{33}(\varphi) = l'^{ie}_{33}(\varphi) = l^{ei}_{33}(\varphi) + 2\sqrt{\tfrac{5}{2}}\, l^{ei}_{13}(\varphi) + \tfrac{5}{2} l^{ee}_{11}(\varphi)$$

Electrical coefficients

$$l'^{e}_{1F}(\varphi) = -\, l'^{e}_{F1}(\varphi) = l^{e}_{1E}(\varphi)$$

$$l'^{e}_{3E}(\varphi) = -\, l'^{e}_{E3}(\varphi) = l^{e}_{3F}(\varphi) + \sqrt{\tfrac{5}{2}}\, l^{e}_{1E}(\varphi)$$

$$l'^{i}_{3F}(\varphi) = -\, l'^{i}_{F3}(\varphi) = l^{i}_{3E}(\varphi) + \sqrt{\tfrac{5}{2}}\, l^{e}_{1F}(\varphi)$$

$$l'_{FE}(\varphi) = l_{EE}(\varphi)$$

the corresponding transformed coefficients tend, however, to a limiting value determined by l^{ee}_{11}.

All these features are shared with the ordinary diffusion coefficient l^{ee}_{11}; therefore, the procedure described in section 16.6 can be straightforwardly extended. We define the *reduced diffusive transport coefficients* as (see eq. 5.5)

$$\tilde{L}'^{\alpha\beta}_{rs} = \frac{L'^{\alpha\beta}_{rs}}{\tfrac{1}{2}\left(n_\alpha \rho^2_{\alpha 0}/\mathscr{T}_{T\alpha}\right) q^2}, \qquad \alpha, \beta = \mathrm{e}, \mathrm{i}; \quad r, s = 1, 3. \tag{7.15}$$

(For the mixed coefficients, $\alpha \neq \beta$, the scaling factor is evaluated, by definition, with $\alpha = \mathrm{e}$.) The interpolation formula (6.1) is now generalized as

$$\tilde{L}'^{\alpha\beta}_{rs} = \frac{\varphi_B l'^{\alpha\beta}_{rs}(\varphi_B) \cdot \left(\varphi_{\alpha *} \hat{l}'^{\alpha\beta}_{rs}(\varphi_{\alpha *}) + \varphi_{PS} \bar{l}'^{\alpha\beta}_{rs}\right)}{\varphi_B l'^{\alpha\beta}_{rs}(\varphi_B) + \varphi_{\alpha *} \hat{l}'^{\alpha\beta}_{rs}(\varphi_{\alpha *}) + \varphi_{PS} \bar{l}'^{\alpha\beta}_{rs}} \eta^{-2} \nu_{\alpha *}. \tag{7.16}$$

We recall that the dimensionless collision frequency $\nu_{\alpha *}$ was defined in (4.14) and that the plateau factor $\varphi_{\alpha *}$ is related to $\nu_{\alpha *}$ by (4.15); all the other quantities were defined in section 16.5. Expressions (7.16) are proposed as a representation of the transformed diffusive coefficients over the whole range of collision frequencies, *provided the inverse aspect ratio* η *is sufficiently small*, so that the standard model is a good approximation. As we possess complete analytic expressions for all the dimensionless coefficients in all three regimes, the quantities (7.16) are easily evaluated on a personal computer. Their values are plotted as functions of the collision frequency in fig. 7.1(a). These curves all have the same general shape as the diffusion coefficient discussed in section 16.6: they start with a steep linear section in the banana regime, then flatten to a plateau-like shape in the intermediate range and end up by another, less steep linear growth in the high collision frequency regime.

C. *Electrical coefficients*

For these coefficients, the relation between dimensional and dimensionless forms, given in table **15**.3.4 is different from the diffusive case. For the *electrical cross-coefficients*, this relation does not involve the collision frequency. We therefore define the corresponding reduced transport coefficients as

$$\tilde{L}'^{\alpha}_{r\mathrm{E}} = \frac{L'^{\alpha}_{r\mathrm{E}}}{(n_\alpha c/\mathscr{B}_0)\mathscr{J}}, \qquad \alpha = \mathrm{e}, \mathrm{i}, \quad r = 1, 3. \tag{7.17}$$

On the other hand, we note that all these coefficients have a vanishing Pfirsch–Schlüter contribution. Therefore, the interpolation formula (7.16) reduces to

$$\tilde{L}'^{\alpha}_{r\mathrm{E}} = \frac{\varphi_{\mathrm{B}} l'^{\alpha}_{r\mathrm{E}}(\varphi_{\mathrm{B}}) \cdot \varphi_{\alpha *} \hat{l}'^{\alpha}_{r\mathrm{E}}(\varphi_{\alpha *})}{\varphi_{\mathrm{B}} l'^{\alpha}_{r\mathrm{E}}(\varphi_{\mathrm{B}}) + \varphi_{\alpha *} \hat{l}'^{\alpha}_{r\mathrm{E}}(\varphi_{\alpha *})}. \tag{7.18}$$

These coefficients are also plotted in fig. 7.1. They are all decreasing (in absolute value) from their banana value and vanish in the high-collision frequency regime. This behaviour is to be expected for these coefficients, which are typical long mean free path effects.

We now turn to the *parallel electrical conductivity*, for which the relation in table **15**.3.4 is of still another type. For this coefficient, the PS contribution is also zero. Thus, its general behaviour as a function of $\nu_{\alpha *}$ is similar to the electrical cross-coefficients. However, as discussed in section **15**.5, the *classical* parallel electrical conductivity must be added to the neoclassical coefficient in

order to obtain the *total parallel electrical conductivity*. Recalling eqs. (**15**.5.12)–(**15**.5.14), we define a reduced coefficient as

$$\tilde{L}'_{EE} = \frac{L'_{EE}}{\left(e^2 n_e / m_e\right)\tau_e}, \tag{7.19}$$

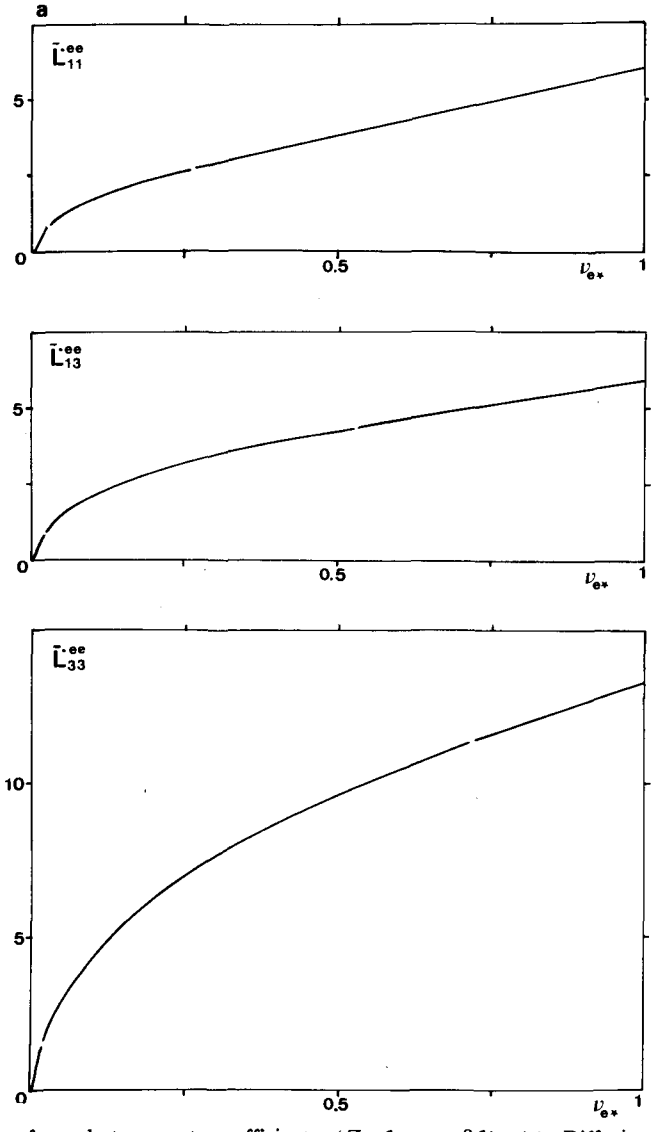

Fig. 7.1. Transformed transport coefficients ($Z = 1$; $\eta = 0.1$). (a) Diffusive coefficients, (b) Electrical coefficients.

and introduce the interpolation as

$$\tilde{L}'_{EE} = \tilde{\sigma}_{\parallel} + \frac{\varphi_B l'_{EE}(\varphi_B) \cdot \varphi_{e*} \hat{l}'_{EE}(\varphi_{e*})}{\varphi_B l'_{EE}(\varphi_B) + \varphi_{e*} \hat{l}'_{EE}(\varphi_{e*})}, \tag{7.20}$$

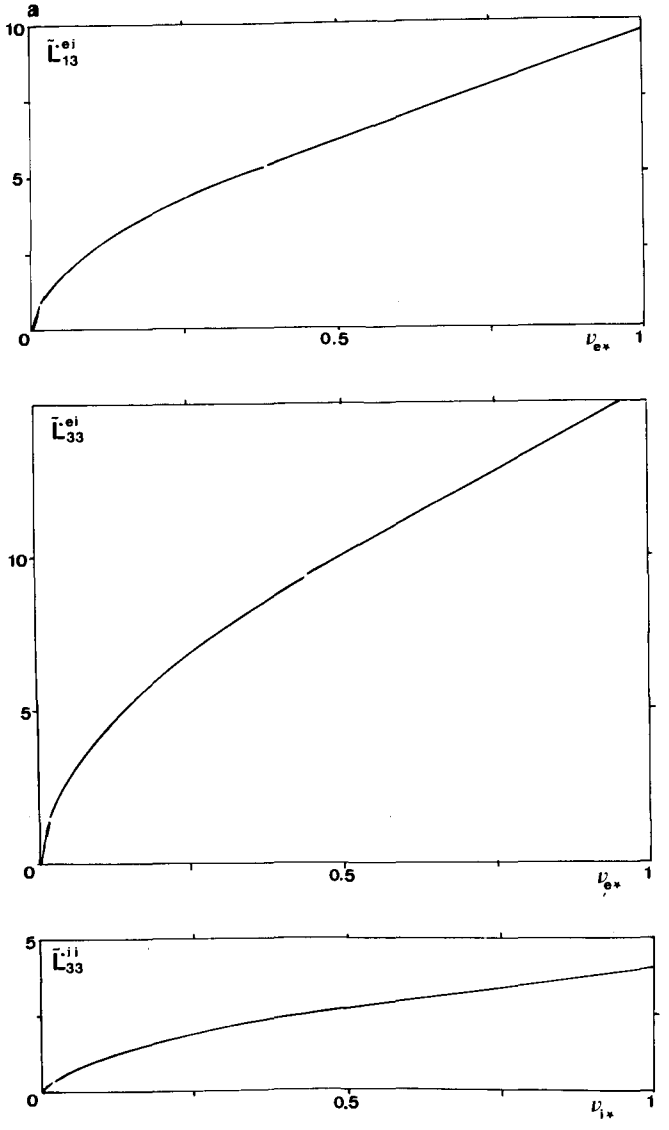

Fig. 7.1. Continued.

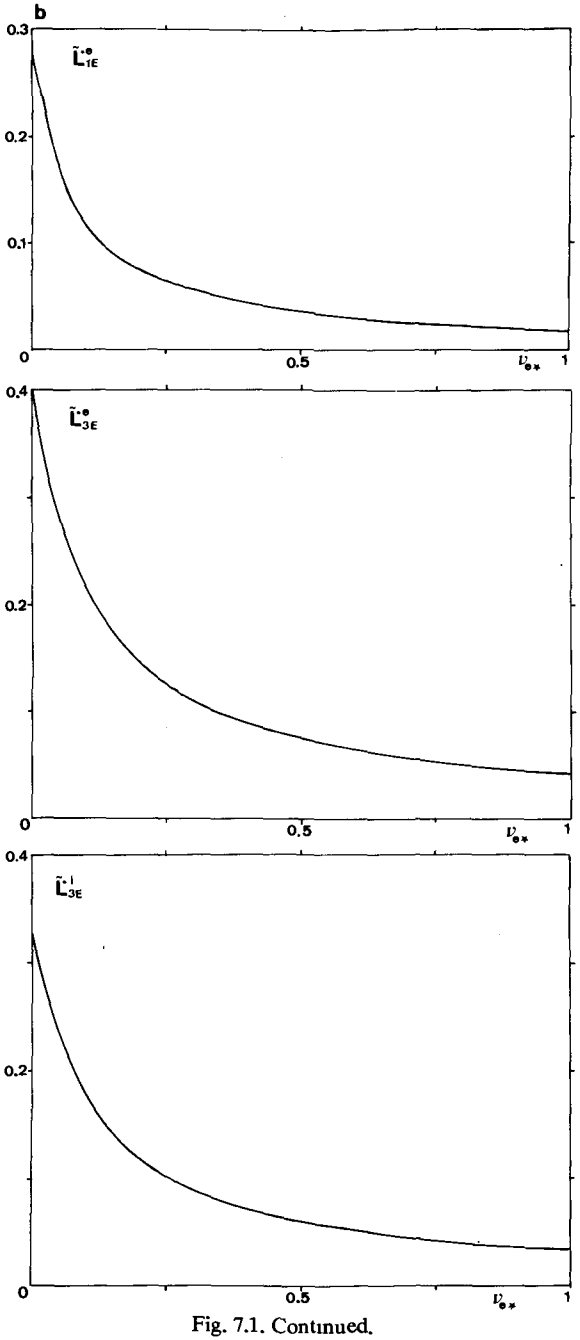

Fig. 7.1. Continued.

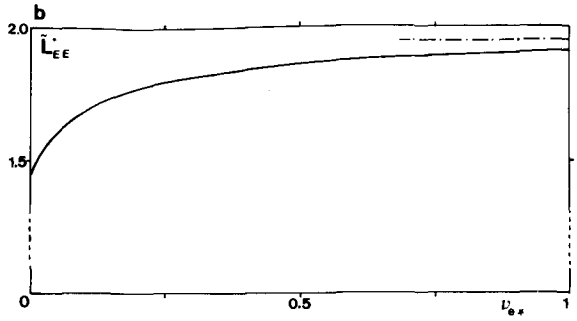

Fig. 7.1. Continued.

where $\tilde{\sigma}_{\|}$ is, as usual, the classical (Spitzer) parallel electrical conductivity. The graph of fig. 7.1(b) shows a monotonous increase of the parallel electrical conductivity from a low value to its classical limit, attained in the high-frequency regime. This provides us with a clear illustration of the trapped particle effect, discussed in section **15**.5.

We have thus obtained a *complete set of interpolated transport coefficients* which provide us with a good approximation, presumably valid over the whole range of collisionalities, in the limit of small inverse aspect ratio $\bar{\eta} \ll 1$. In table 7.2, we collected the transformed transport equations in explicit dimensional form, together with the complete definition of the transport coefficients. The more transparent "first system of notations" was used here, as defined in (**15**.3.11). These equations are in a form suitable for use in computer codes.

We insist on the fact that we constructed a *complete* set of *ten* independent transport coefficients. The transport equations of table 7.2 may be compared to those of Hinton and Hazeltine (1976, sec. VI. F, eqs. 6.114–6.123, 6.130–6.132). These authors use a transformed electron heat flux $J_3'^e$, but leave the ion heat flux untransformed. Their electronic transport equations involve no coupling to the ion temperature gradient (i.e. no mixed diffusive coefficients, in our terminology). Their transformed force A'_{1e} (which should be compared to our X'_1) contains, however, a strange term, proportional to $\nabla_\rho T_i$, with a coefficient whose motivation is rather unconvincing. Similarly, in their treatment of the ion heat flux, they use a transport equation of the form $J_3 = L_{33}X_3 + \alpha J_1$ (in our notation), which can hardly be justified.

Let us be clear about this critique. We do not contend that the results of Hinton and Hazeltine are wrong! Actually, we have seen in table **15**.4.1 (and in table 5.1) that there is a good agreement between their results and ours, wherever such results are comparable. Our point is that their final *transformed* transport equations appear to be quite unnatural and incomplete. In contrast,

Table 7.2
(*Left*) Approximate transport equations for the entire collision frequency range ($\eta \ll 1$).
(*Right*) Transformed transport coefficients.

Dimensional form

$$J_1'^{e} = L_{11}'^{ee} X_1'^{e} + L_{13}'^{ee} X_3'^{e} + L_{13}'^{ei} X_3'^{i} + L_{1E}'^{e} X_E'$$

$$J_3'^{e} = L_{31}'^{ee} X_1'^{e} + L_{33}'^{ee} X_3'^{e} + L_{33}'^{ei} X_3'^{i} + L_{3E}'^{e} X_E'$$

$$J_3'^{i} = L_{31}'^{ie} X_1'^{e} + L_{33}'^{ie} X_3'^{e} + L_{33}'^{ii} X_3'^{i} + L_{3E}'^{i} X_E'$$

$$J_E' = L_{E1}'^{e} X_1'^{e} + L_{E3}'^{e} X_3'^{e} + L_{E3}'^{i} X_3'^{i} + L_{EE}' X_E'$$

Transformed fluxes

$$J_1'^{e} = J_1^{e}$$

$$J_3'^{e} = J_3^{e} + \tfrac{5}{2} J_1^{e}$$

$$J_3'^{i} = J_3^{i} + \tfrac{5}{2} Z^{-1} J_1^{e}$$

$$J_E' = J_E$$

Transformed forces

$$X_1'^{e} = X_1^{e} - \tfrac{5}{2} X_3^{e} - \tfrac{5}{2} Z^{-1} X_3^{i}$$

$$X_3'^{e} = X_3^{e}$$

$$X_3'^{i} = X_3^{i}$$

$$X_E' = X_E$$

Pure diffusive coefficients

$$L_{11}'^{ee} = \frac{n_e \rho_{e0}^2}{\mathscr{T}_{Te}} q^2 \tfrac{1}{2} \tilde{L}_{11}'^{ee}$$

$$L_{33}'^{ee} = \frac{n_e \rho_{e0}^2}{\mathscr{T}_{Te}} q^2 \tfrac{5}{4} \tilde{L}_{33}'^{ee}$$

$$L_{13}'^{ee} = L_{31}'^{ee} = \frac{n_e \rho_{e0}^2}{\mathscr{T}_{Te}} q^2 \tfrac{1}{2} \sqrt{\tfrac{5}{2}}\, \tilde{L}_{13}'^{ee}$$

$$L_{33}'^{ii} = \frac{n_i \rho_{i0}^2}{\mathscr{T}_{Ti}} q^2 \tfrac{5}{4} \tilde{L}_{33}'^{ii}$$

Mixed diffusive coefficients

$$L_{13}'^{ei} = \frac{T_i}{T_e} L_{31}'^{ie} = \frac{n_e \rho_{e0}^2}{\mathscr{T}_{Te}} \frac{T_i}{Z T_e} q^2 \tfrac{1}{2} \sqrt{\tfrac{5}{2}}\, \tilde{L}_{13}'^{ei}$$

$$L_{33}'^{ei} = \frac{T_i}{T_e} L_{33}'^{ie} = \frac{n_e \rho_{e0}^2}{\mathscr{T}_{Te}} \frac{T_i}{Z T_e} q^2 \tfrac{5}{4} \tilde{L}_{33}'^{ei}$$

Electrical coefficients

$$L_{1E}'^{e} = -L_{E1}'^{e} = -\frac{n_e c}{\mathscr{B}_0} q \eta^{-1} \tilde{L}_{1E}'^{e}$$

$$L_{3E}'^{e} = -L_{E3}'^{e} = -\frac{n_e c}{\mathscr{B}_0} q \eta^{-1} \sqrt{\tfrac{5}{2}}\, \tilde{L}_{3E}'^{e}$$

$$L_{3E}'^{i} = -L_{E3}'^{i} = -\frac{n_i c}{\mathscr{B}_0} q \eta^{-1} \sqrt{\tfrac{5}{2}}\, \tilde{L}_{3E}'^{i}$$

$$L_{EE}' = \frac{n_e e^2}{m_e} \tau_e \tilde{L}_{EE}'$$

our transformation law proceeds from a simple and intuitive redefinition of the energy fluxes (7.2), combined with an "orthodox" thermodynamic transformation of the forces (7.7).

Before closing this section, it may be mentioned that the interpolation between the three regimes could be further improved. This improvement is achieved by a more detailed consideration of additional intermediate layers. As an example, Hinton and Rosenbluth (1973) investigated the boundary layer which is immediately adjacent to the banana regime by an elegant, but rather complicated procedure. The plateau–PS transition was also considered in more detail. These refinements are reviewed by Hinton and Hazeltine (1976) and by Hirshman and Sigmar (1981). It appears, however, that these improvements are not really spectacular and result in an interpolation which is not very different from the simple type described in the present section. Given that these refinements must appeal to specific modellings of the collision operator, the hidden uncertainties appear just as uncontrollable as the straight interpolation. We shall not discuss these studies any further.

16.8. Miscellaneous additional topics

In this section we present some topics which were not included in our previous discussion. It is, clearly, impossible to cover all the problems in a single book. We therefore chose to present these subjects very briefly, mainly by indicating the connections with the previously presented material, and to refer the reader to the original literature for the details. We chose three problems for this discussion.

A. Non-axisymmetric confinement configurations

Axisymmetric systems constitute an important class of magnetic confinement devices (such as the tokamak) but are not the only systems of interest. The *stellarator* is a celebrated example of a non-axisymmetric device. The axisymmetry assumption gave rise to many simplifications in our treatment of the neoclassical theory. In the general case, the motion of the particles is more complicated (see section **9**.8); as a result, the solution of the drift kinetic equation is more involved, because of the existence of several distinct classes of trapped particles. Although the problem was considered already in the early days of the neoclassical theory (Galeev and Sagdeev 1968, 1979, Kovrizhnykh 1969, Frieman 1970), this work is still continuing in present days (Pytte and Boozer 1981, Kovrizhnykh 1984, Ho and Kulsrud 1987). Among the recent papers, we wish to point out more especially the following ones, which are a

direct generalization of the Hirshman–Sigmar methods to non-axisymmetric systems and which provide the clearest view into the structure of the theory: Shaing and Callen (1983b), Shaing et al. (1986), Coronado and Wobig (1986), Rodriguez-Solano Ribeiro and Shaing (1987).

A general confinement configuration is best described by using Hamada coordinates (section **8**.4) in which the "radial coordinate" labeling the magnetic surfaces is taken to be the volume enclosed by the surface (see section **8**.3). The three coordinates are denoted by (V, θ, ζ); they are not, in general, orthogonal. The representation of the magnetic field is of the Clebsch type (section **8**.7),

$$\boldsymbol{B} = \phi_{\mathrm{P}}(\nabla\zeta \wedge \nabla V) + \phi_{\mathrm{T}}(\nabla V \wedge \nabla\theta). \tag{8.1}$$

Noting also that, by definition of the Hamada coordinates, the Jacobian of the coordinate system equals one,

$$J \equiv (\nabla V \wedge \nabla\theta)\cdot\nabla\zeta = \mathbf{1}, \tag{8.2}$$

we find [General Appendix G2, eqs. (**G2**.1.15), (**G2**.1.16)] that the contravariant and the covariant basis vectors are defined, respectively, as

$$\begin{aligned} \hat{\boldsymbol{e}}^{V} &= \nabla V, \qquad & \hat{\boldsymbol{e}}_{V} &= \nabla\theta \wedge \nabla\zeta, \\ \hat{\boldsymbol{e}}^{\theta} &= \nabla\theta, \qquad & \hat{\boldsymbol{e}}_{\theta} &= \nabla\zeta \wedge \nabla V, \\ \hat{\boldsymbol{e}}^{\zeta} &= \nabla\zeta, \qquad & \hat{\boldsymbol{e}}_{\zeta} &= \nabla V \wedge \nabla\theta. \end{aligned} \tag{8.3}$$

Comparing (8.1) with (**G2**.1.20) we find that the coefficients $\phi_{\mathrm{P}}(V)$, $\phi_{\mathrm{T}}(V)$ are simply the poloidal and the toroidal contravariant components of the magnetic field, respectively,

$$\hat{B}^{\theta} = \phi_{\mathrm{P}}, \qquad \hat{B}^{\zeta} = \phi_{\mathrm{T}}. \tag{8.4}$$

We know (Section **G2**.1) that these contravariant components do not have the physical dimensions of a magnetic field intensity. Rather, they are identified (up to a factor 2π) with the poloidal and toroidal magnetic flux densities, respectively (see eqs. **8**.3.2, **8**.3.5),

$$\phi_{\mathrm{P}} = \boldsymbol{B}\cdot\nabla\theta, \qquad \phi_{\mathrm{T}} = \boldsymbol{B}\cdot\nabla\zeta. \tag{8.5}$$

The *drift kinetic equation* written in these variables takes the form, generalizing (**14**.1.3),

$$\frac{U_\alpha}{B}\left(\phi_P\frac{\partial \bar{f}_1^\alpha}{\partial\theta}+\phi_T\frac{\partial \bar{f}_1^\alpha}{\partial\zeta}\right)$$

$$+\frac{U_\alpha}{B}\left[\frac{\partial}{\partial\theta}\left(\frac{U_\alpha}{\Omega_\alpha}(\boldsymbol{B}\wedge\nabla V)\cdot\nabla\theta\right)+\frac{\partial}{\partial\zeta}\left(\frac{U_\alpha}{\Omega_\alpha}(\boldsymbol{B}\wedge\nabla V)\cdot\nabla\zeta\right)\right]\frac{\partial f_0^\alpha}{\partial V}$$

$$+\frac{e_\alpha}{m_\alpha}E_\parallel^{(A)}U_\alpha\frac{\partial f_0^\alpha}{\partial\mathscr{E}}=\mathscr{K}^\alpha\{\bar{f}_1^\alpha\}. \tag{8.6}$$

We used here the fact that the identity (**14**.A1.9) is still valid. The construction of the transport theory follows very closely the developments of chaps. 12–16. The reader who has mastered these chapters will have no difficulty going through the papers by Shaing and Callen (1983b) and by Shaing et al. (1986); only the explicit solution of the drift kinetic equation is considerably more complicated. Here, we go over to the final results and, for simplicity, discuss only the case when $E_\parallel^{(A)}=0$ (i.e. we only discuss the diffusive coefficients).

The first step is a decomposition of the radial fluxes that is quite analogous to (**12**.9.9) and (**12**.9.14). One may again distinguish a classical flux, a Pfirsch–Schlüter flux and a banana flux (an additional term, corresponding to a non-ambipolar diffusion also exists in the non-axisymmetric geometry). The general expression of these contributions is very similar to the corresponding axisymmetric ones (Shaing and Callen 1983b).

Here we limit ourselves to the discussion of the banana (or plateau) fluxes. In order to make the comparison more transparent, we go over to the physical components (used throughout this book), by making the following changes in the formulae of Shaing et al. (1986):

$$\Gamma_{bp}^c\equiv\langle\Gamma^c\cdot\nabla V\rangle\to V'\langle\Gamma_\rho^c\rangle_B,$$

$$\left(\frac{\partial P}{\partial V},\frac{\partial T_\alpha}{\partial V}\right)\to\frac{1}{V'}\left(\frac{\partial P}{\partial\rho},\frac{\partial T_\alpha}{\partial\rho}\right),$$

where $V'\equiv \mathrm{d}V(\rho)/\mathrm{d}\rho$. Next, we go over to the usual dimensionless quantities and write the transport equations for the electron fluxes in the form

$$\langle h_\rho^{e(p)}\rangle_B=\tilde{\mathscr{M}}_e\,\varphi\left[l_{p1}^{ee}(\varphi)\;g_\rho^{(1)}+l_{p3}^{ee}(\varphi)\;g_\rho^{e(3)}+l_{p3}^{ei}(\varphi)\;g_\rho^{i(3)}\right]. \tag{8.7}$$

The remarkable fact is the *formal identity of this equation with the corresponding axisymmetric expression* (**15**.3.1). The coefficients $l_{pr}^{\alpha\beta}(\varphi)$ were calculated by Shaing et al. (1986) in the limit of small φ (= large aspect ratio) and in the 13M approximation. Their expression is the same as in table **15**.4.1: they are entirely determined by the *pseudo-viscosity coefficients*, which have the same value as in the axisymmetric case. We thus reach the remarkable conclusion that *the whole effect of the geometry is concentrated in the factor* $\tilde{\mathscr{M}}_e$. The quantity $\mathscr{K}_e^2$ of eq. (**15**.3.4) must be replaced by the factor

$$\tilde{\mathscr{M}}_e = \frac{1}{(\Omega_{e0}\tau_e)^2} \frac{I\tilde{G}_b}{(d\phi_P/dV)V'^2}, \tag{8.8}$$

where I is a generalization of the quantity $\mathscr{I}$ of the axisymmetric case,

$$I = \langle (\boldsymbol{B} \wedge \nabla V) \cdot \nabla \theta \rangle, \tag{8.9}$$

$\tilde{G}_b$ is a rather complicated expression, of which we only display two terms for the purpose of illustration (the complete expression is given in the work by Shaing et al. 1986):

$$\tilde{G}_b = G_1 + G_1 + \cdots \tag{8.10}$$

with

$$G_1 = (2\phi_P\phi_T)^{-1} \langle \phi_P (\boldsymbol{B} \wedge \nabla V) \cdot \nabla \zeta + \phi_T (\boldsymbol{B} \wedge \nabla V) \cdot \nabla \theta \rangle,$$

$$G_2 = \frac{\mathscr{B}_0^2}{2\phi_P\phi_T} \frac{\langle (\partial B/\partial\theta)^2 - q^2(\partial B/\partial\zeta)^2 \rangle}{\langle [(\partial B/\partial\theta) + q(\partial B/\partial\zeta)]^2 \rangle}. \tag{8.11}$$

The evaluation of all these terms is, in general, a rather complicated matter, especially for realistic geometries. It requires the detailed knowledge of the dependence of the magnetic fields intensity $B(\theta, \zeta)$ on the two angular coordinates. This function is usually represented as a Fourier series. The additional terms in (8.10) represent, among other things, the effect of the higher harmonics on the transport coefficients. Specific applications for model magnetic fields (representing stellarator, torsatron, rippled tokamak geometries) have been worked out explicitly by several authors, especially in the plateau regime (Shaing et al. 1986; Coronado and Wobig 1986, Rodriguez-Soland Ribeiro and Shaing 1987). The electrical coefficients have also been derived in these papers.

We do not pursue the discussion here. Our main point was to exhibit the structural unity of the neoclassical theory, in spite of the specific complications of the non-axisymmetric geometries.

B. Viscosity

The reader may be surprised to find no trace in the neoclassical theory of a set of viscosity coefficients, corresponding to a linear relation between the dissipative pressure tensor and the velocity gradient, such as eqs. (**5**.3.22) or (**5**.5.41). On the other hand, the pressure tensor (as well as the higher-order generalized stresses) plays a key role in the neoclassical theory. There is no paradox here, as will be briefly shown below.

We return to a comparison of eqs. (**5**.1.8) (with $\partial_t h = 0$) and (**12**.3.2). The crux of the argument lies in the ordering of the terms in ε: the fluid velocities are all of order ε, as results from the analysis in section **10**.5. As a result, in the determination of the pressure tensor through order ε, *the source terms involving the velocity gradients drop out of the tensor moment equations to this order*.

As can be seen from (**5**.1.8), eqs. (**12**.3.2) can also be *formally* obtained by setting $\tau_\alpha \to \infty$. This justifies the statement, often found in the literature, that the CGL pressure tensor represents a *collisionless plasma*. This statement must, however, be moderated in the present context, because we have amply insisted on the crucial role of the collisions in the neoclassical theory, even in the long mean free path regime. It must rather be interpreted as a dominance of the magnetic field effects (through $\Omega_\alpha \sim \varepsilon^{-1}$) over the collisional effects (measured by $\tau_\alpha^{-1} \sim \varepsilon^0$).

In this limit, the moment equations (together with the zero-trace constraint) reduce to a set of *homogeneous* (i.e. sourceless) *algebraic equations. Their structure is such that they allow a non-trivial solution*. This fact is basic for the neoclassical theory: it implies the CGL form (**12**.3.7) of the pressure tensor,

$$\pi^\alpha_{rs} = \tfrac{3}{2}\pi^\alpha_{\|}\left(b_r b_s - \tfrac{1}{3}\delta_{rs}\right), \tag{8.12}$$

where the single independent coefficient can be identified as

$$\pi^\alpha_{\|} = \boldsymbol{b}\cdot\boldsymbol{\pi}^\alpha\cdot\boldsymbol{b}. \tag{8.13}$$

This form of the pressure tensor is entirely determined by the magnetic field, which appears as the only origin of the anisotropy. The single coefficient $\pi^\alpha_{\|}$ cannot be determined by the tensor moment equations (because they are homogeneous!). Equations (8.12) cannot be obtained as a continuous limit of equations (**5**.3.22). Indeed, if τ_α is retained as a finite quantity, the homoge-

neous part of the linear equations (5.1.8) changes abruptly [by the inclusion of the terms $\tau_\alpha^{-1}(c_{22}^\alpha h_{\mathrm{rs}}^{\alpha(2)} + \cdots)$].

A stress tensor of the form (5.3.22) may appear in the neoclassical theory, i.e. in a magnetically confined plasma, if, for some reason, there exists an *enhanced velocity* $\boldsymbol{u}$ (of order ε^0). Such a situation naturally arises when *neutral beam injection* occurs. It is well known that one of the most efficient means of "additional" heating of a tokamak plasma consists of injecting a beam of very energetic neutral hydrogen or deuterium atoms into the plasma (see e.g. Stacey 1981, Raeder et al. 1986). Through collisions, these particles transmit their energy to the constituents of the plasma; the long-tail distribution formed in the first stage thermalizes to a new high-temperature state. But there is also an additional effect. The beam of particles is highly directional; it will therefore impart a large momentum to the plasma. This momentum points primarily in the *toroidal direction*. As a result, the velocity component u_ζ is strongly enhanced and results in a rotation of the plasma as a whole.

This phenomenon was studied experimentally in detail. The rotation should be damped by viscosity effects. The experimental results (Suckewer et al. 1979, 1981, Isler et al. 1983, Brau et al. 1983) seemed to indicate a disagreement of two orders of magnitude with the neoclassical predictions existing at that time (Grimm and Johnson 1972, Tsang and Frieman 1976) and led to many controversies in the literature.

In order to resolve the problem, it was necessary to calculate from the start the form of the *complete* pressure tensor in a magnetically confined plasma, with a strongly enhanced toroidal velocity. It turned out that the previously mentioned authors only calculated the parallel viscosity coefficient, i.e. the term $\eta_{\|}^\alpha$ in eq. (5.3.22).

The complete theory first involves a formulation of the pressure tensor in curvilinear coordinates (in particular, the velocity gradient must be determined in terms of covariant, rather than ordinary, derivatives). Next, an analysis of the resulting moment equations yields a set of transport equations of the form (5.3.22). This work was performed very recently (Hogan 1984, Mikhailovskii and Tsypin 1984a, b, Tsypin 1985, Catto 1986). The most conclusive work appears in the series of papers by Stacey and Sigmar (1984, 1985 and by Stacey et al. (1985). The paper by Stacey and Sigmar (1985) contains a complete determination of all five viscosity coefficients. It is also shown that the *gyro-viscosity coefficients* $\eta^{\alpha(1)}$, $\eta^{\alpha(3)}$ of eq. (5.5.42) are the dominant ones in determining the toroidal damping, and their magnitude is in agreement with the experiments. There is, however, still some controversy around this question (Connor et al. 1987).

The modified form of the pressure tensor also induces another, quite important effect. Indeed, the generalized stresses $h_{rs}^{\alpha(2p)}$ entering the vector flux equations (12.4.1)–(12.4.3) have now an altogether different form: the

whole neoclassical theory must be reconsidered. As a result, it is clear that the *average radial fluxes* $\langle \Gamma_\rho^\alpha \rangle$, $\langle q_\rho^\alpha \rangle$ will depend on the injected beam power (and direction) in a non-trivial way. In particular, the possibility of an impurity control, achieved by neutral beam injection, has been suggested: the injection-controlled impurity diffusion could be forced in the outward direction by injecting the neutrals in a well-chosen direction (co-injection). Here again, it is impossible for us to cover this interesting and important subject in this book. We refer the reader to the works by Stacey and Sigmar (1979a, b, 1984, 1985), Stacey et al. (1985), Hinton and Wong (1985) and Catto (1986).

C. Multispecies plasmas and impurity transport

The picture of a plasma as a system consisting of electrons and a single species of ions is an idealization. A tokamak plasma is, in reality, a mixture of electrons and *fuel ions*, i.e. deuterium and tritium, plus a certain amount of *impurity ions* originating from the walls (i.e. absorbed gases such as oxygen, nitrogen, helium, as well as metallic impurities) and present in various ionization levels, as well as – possibly – *reaction products*, i.e. α-particles produced by the fusion reactions.

For a global picture of the transport, the modelization of the mixture in terms of an average species of ions characterized by an effective mass and an effective charge number, is sufficient. But in many cases we need a much more detailed picture, in which the flux of each particle species and the associated heat flux is needed. For instance, it is well-known that high-Z impurities may be lethal to a fusion reactor, because they produce very strong radiation losses which therefore inhibit the ignition process. The control of their transport towards the central reaction zone is extremely important in practice.

The extension of the transport theory to a multi-component plasma poses no new problem of principle. The mechanisms at work are the same as in the two-component cases. But the algebraic complexity of the problem appears, a priori, formidable. It was the great merit of Hirshman and Sigmar to develop efficient methods for the calculation of transport coefficients in such systems. The key points explaining their success may be summarized as follows: the development of the moment approach to the neoclassical theory, the use of approximate collision operators of the type discussed in chapter 11, together with a number of efficient tricks for inverting large systems of linear algebraic equations. As a result, it can be said that the problem of impurity transport is completely (and analytically) solved at present, to the same degree as the two-component plasma transport theory. We can only refer the reader to the very comprehensive review paper by Hirshman and Sigmar, (1981), which contains a detailed account of the theory and an extensive list of additional references.

Appendix 16A.1. Some useful integrals

We give here some details which will help the reader in performing the integrations required in the calculation of the plateau pseudo-viscosity coefficients. For convenience we use here the simplified notation: $\bar{\nu}_\alpha \equiv \nu$.

A. Integrations over τ and ξ

We must evaluate the integral

$$J = -\int_{-1}^{1} d\xi\, \xi \frac{\partial}{\partial \xi} \int_0^\infty d\tau \cos\left(\nu^{-1/3}\xi\tau\right) e^{-\tau^3/6}.$$

Performing an integration by parts over ξ and substituting $s = \nu^{-1/3}\tau$, we get

$$J = \int_{-1}^{1} d\xi\, \nu^{1/3} \int_0^\infty ds \cos \xi s\, e^{-\nu s^3/6}.$$

In the plateau regime, $\nu \ll 1$; we thus expand the exponential and find

$$e^{-\nu s^3/6} = 1 - (\nu s^3/6) + \cdots.$$

Thus, neglecting contributions of relative order ν, we find that J reduces to

$$J \simeq \nu^{1/3} \int_{-1}^{1} d\xi \int_0^\infty ds \cos \xi s = \nu^{1/3} \int_{-1}^{1} ds\, 2\frac{\sin s}{s} = 2\nu^{1/3}\tfrac{1}{2}\pi.$$

Hence, finally,

$$J = \pi\nu^{1/3}. \tag{A1.1}$$

B. Integrations over x

All the integrations over x appearing in the calculation of the generalized stresses are of the form

$$K_{pq} = \int_0^\infty dx\, e^{-x} x^2 L_p^{3/2}(x)\, L_q^{3/2}(x). \tag{A1.2}$$

These integrals are quite elementary. The result, obtained by using table **G1**.3.2, is

$$K_{00} = \frac{4}{3}, \qquad K_{11} = \frac{26}{15},$$
$$K_{01} = \frac{2\sqrt{2}}{3\sqrt{5}}, \qquad K_{12} = \frac{23}{15\sqrt{7}},$$
$$K_{02} = -\frac{\sqrt{2}}{3\sqrt{35}}, \qquad K_{22} = \frac{433}{10}. \tag{A1.3}$$

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17

Entropy production and transport in magnetically confined plasmas

17.1. Introduction

In our investigation in chapters 12–15 we uncovered three distinct mechanisms of transport in a magnetically confined plasma (neglecting the small drift and modified drift mechanisms): classical, Pfirsch–Schlüter and banana transport. Each of the four relevant surface-averaged fluxes *

$$\left\langle h_\rho^{e(1)} \right\rangle \equiv J_1, \qquad \left\langle h_\rho^{e(3)} \right\rangle \equiv J_2,$$

$$\left\langle h_\rho^{i(3)} \right\rangle \equiv J_3, \qquad \mathscr{B}_0^{-1} \left\langle B h_\parallel^{(1)} \right\rangle \equiv J_4$$

appears as a sum of three terms, corresponding to the three mechanisms,

$$J_p = J_p^{CL} + J_p^{PS} + J_p^{B}, \quad p = 1, \cdots, 4. \tag{1.1}$$

Each separate component is related to the four thermodynamic forces

$$g_\rho^{(1)P} \equiv X_1, \qquad g_\rho^{e(3)} \equiv X_2,$$

$$g_\rho^{i(3)} \equiv X_3, \qquad \hat{g}_\parallel^{(1)A} \equiv X_4$$

through a *transport matrix* characteristic of each mechanism,

$$J_p^{A} = \sum_{q=1}^{4} \mathscr{L}_{pq}^{A} X_q, \quad A = CL, PS, B. \tag{1.2}$$

The matrices $\mathscr{L}_{pq}^{A}$ were displayed in eqs. (**15**.5.2)–(**15**.5.4). The transport

* We use the same compact notations as in section **15**.3, although we are dealing here with *dimensionless* fluxes, forces and transport coefficients. This should cause no confusion and avoids introducing yet another set of notations.

coefficients were obtained by solving the appropriate kinetic equation in the relevant regime.

A different approach to the transport theory that is, in principle, independent of the former, starts from the concept of *entropy production* and could be called the *thermodynamic approach* to transport theory. In this method one evaluates the entropy production (as in chapter 6) and realizes that, for a state not too far from equilibrium, this quantity is a quadratic form in *some* variable characterizing the departure from equilibrium. If these variables are taken to be the thermodynamic forces, the total surface-averaged entropy production is expressed as

$$\sum_{\alpha} \langle \Sigma^{\alpha} \rangle \stackrel{?}{=} \sum_{p,q=1}^{4} \mathscr{M}_{pq} X_p X_q. \tag{1.3}$$

Having obtained this form, everybody seems to take for granted that the coefficients $\mathscr{M}_{pq}$ are identical to the total transport coefficients of eq. 1.2,

$$\mathscr{M}_{pq} \stackrel{?}{=} \sum_{\mathrm{A}} \mathscr{L}_{pq}^{\mathrm{A}} \tag{1.4}$$

This point of view is very explicitly expressed in the paper by Hazeltine and Catto (1981, p. 296), where they introduce the following "recipe for the transport coefficients": "*for a complete solution to the radial transport problem it suffices to calculate the (entropy production) as a quadratic form in the forces*".

We have put a question mark on this equality, because the validity of this "recipe" is not obvious. Equality (1.4) is equivalent to requiring that the total average entropy production can be expressed in *thermodynamic form* as

$$\sum_{\alpha} \langle \Sigma^{\alpha} \rangle \stackrel{?}{=} \sum_{\mathrm{A}} \sum_{p=1}^{4} J_p^{\mathrm{A}} X_p. \tag{1.5}$$

This relation is based on a number of assumptions (de Groot and Mazur 1984) (in particular, the dominance of the collisions) which are *not* satisfied in the long mean free path regime of neoclassical theory.

Given this situation, it appears important to analyze the properties of the entropy production from the start. Our approach in this chapter is motivated by searching an answer to the following questions (which go farther than the global formulation described above):

(a) Is it possible to construct a thermodynamic formulation of the entropy production for each species α (electrons and ions)?

(b) Is it possible to associate a specific contribution to the entropy production with each of the mechanisms found in transport theory?

(c) If so, what is the relation between the "entropic coefficients" appearing in these contributions and the transport coefficients of eq. (1.2)?

It will be seen that the answers to these questions are by no means trivial. In particular, whenever the generalized stresses play a role (i.e. in the long mean free path regime) we shall find that the confined plasma behaves in a "non-thermodynamic" way.

Before starting this analysis, we mention that, to the best of our knowledge, the paper by Tessarotto (1982) is the only work that approaches the problem of the entropy production in a spirit similar to the one outlined above, in particular by trying to identify the portions of the entropy production associated with each transport mechanism. A summary of the results of the present chapter was published by Balescu (1987). A brief comparison with other relevant works is given in section 17.9.

17.2. The unaveraged entropy production

A fundamental remark is the fact that, even in a long mean free path regime, the distribution functions of a magnetically confined plasma are dominated by a local equilibrium reference state, see eq. (**12**.2.7). The deviations from this state can be expanded in a series of irreducible Hermite polynomials as in (**12**.2.8). In the present chapter (as in the preceding six chapters) we shall only be interested in the *vector fluxes*, disregarding the scalar and tensor moments; we thus write

$$\chi^{\alpha}(\boldsymbol{c};\, \rho,\, \theta,\, t) = \sum_{m=0}^{N} h_r^{\alpha(2m+1)}(\rho,\, \theta,\, t)\; H_r^{(2m+1)}(\boldsymbol{c}), \tag{2.1}$$

where N is the truncation level ($N = 1$ for 13M, $N = 2$ for 21M). *

In chapter 6, section **6**.2., we evaluated the *entropy production* σ^{α} of species α, a basic concept of non-equilibrium thermodynamics. It was shown there that the entropy production is intimately associated with the collision term. An intrinsic *kinetic form of the entropy production* was derived in eqs. (**6**.2.8)

* It should be noted that the dominant contributions to the tensor moments are actually included in the theory in the form of the generalized stresses. These are determined by the magnetic field and have the characteristic CGL form, discussed in section **12** 3. They enter the moment equations in the form of additional source terms (section **12**.4). The remaining part of the tensor fluxes (related to the velocity gradients) are negligible to dominant order in ϵ (see the discussion in section **16**.8B)

and (**6**.2.9), under the sole assumption that the state of the plasma is not too far from the reference local equilibrium state,

$$\Sigma^{\mathrm{e}} = \sum_{p=0}^{N} \sum_{q=0}^{N} c^{\mathrm{e}}_{2p+1,2q+1} h_r^{\mathrm{e}(2p+1)} h_r^{\mathrm{e}(2q+1)}, \qquad [\mathrm{CONV}]$$

$$\Sigma^{\mathrm{i}} = \sum_{p=1}^{N} \sum_{q=1}^{N} c^{\mathrm{i}}_{2p+1,2q+1} h_r^{\mathrm{i}(2p+1)} h_r^{\mathrm{i}(2q+1)}. \tag{2.2}$$

We introduced here the *dimensionless (density of) entropy production of species* α, defined as

$$\Sigma^{\alpha} \equiv \frac{\tau_{\alpha}}{n_{\alpha}} \sigma^{\alpha}. \tag{2.3}$$

We also recall that in eq. (2.2) the following convention is used:

$$h_r^{\mathrm{e}(1)} \equiv h_r^{(1)}. \qquad [\mathrm{CONV}] \tag{2.4}$$

In order to make a clear distinction between the electric current and the electron flux (which will also appear in the theory), the symbol [CONV] will be written after all the formulae in which this convention is used.

This kinetic form of the entropy production can be rewritten in the form (**6**.3.1) in terms of the *generalized frictions* of eq. (**4**.6.34). In our case (when we neglect the tensor moments) we obtain

$$\Sigma^{\alpha} = - \sum_{p=0(1)}^{N} h_r^{\alpha(2p+1)} \tau_{\alpha} Q_r^{\alpha(2p+1)}. \qquad [\mathrm{CONV}] \tag{2.5}$$

The lower limit for p is 0 for the electrons and 1 for the ions; the upper limit is actually ∞, but in practice the sum is truncated at $p = N$. The kinetic forms (2.2), (2.5) of the entropy production are completely general: they do not differ from the corresponding formulae of chapter 6. Indeed, they are based on the assumption of the dominance of the local equilibrium distribution.

The specificity of the neoclassical theory appears when we try to go over to the thermodynamic form of the entropy production, as in section **6**.3. In order to do so, we express the generalized frictions $Q_r^{\alpha(2p+1)}$ in terms of the thermodynamic forces $g_r^{\alpha(2p+1)}$, by making use of the linearized moment equations. However, we may no longer use eqs. (**5**.1.7)–(**5**.1.10) as in section **6**.3. Instead, we use eqs. (**12**.4.16)–(**12**.4.18), which differ from the former by the presence of the generalized source terms $\bar{g}_r^{\alpha(2p+1)}$, related to the gener-

alized stresses (**12**.4.8)–(**12**.4.10). Multiplying each one of eqs. (**12**.4.16)–(**12**.4.18) by the corresponding flux, we find

$$\Sigma^{e} = h_r^{(1)} g_r^{(1)} + h_r^{e(3)} g_r^{e(3)} - h_r^{(1)} \bar{g}_r^{e(1)} + \sum_{p=1}^{N} h_r^{e(2p+1)} \bar{g}_r^{e(2p+1)},$$

$$\Sigma^{i} = h_r^{i(3)} g_r^{i(3)} + \sum_{p=1}^{N} h_r^{i(2p+1)} \bar{g}_r^{i(2p+1)}. \tag{2.6}$$

Here we no longer use convention (2.4). The electron current thus contributes the term $h_r^{(1)}(g_r^{(1)} + \bar{g}_r^{(1)})$ to the entropy production. It is easily shown that the generalized stress associated with the electric current is related as follows to the generalized stress associated with the electron flux, $\bar{g}_r^{\alpha(1)}$, defined in (**12**.4.8):

$$\bar{g}_r^{(1)} = -\bar{g}_r^{e(1)}.$$

There is a fundamental difference between the thermodynamic form of the entropy production of eq. (**6**.3.4) and the form obtained in (2.6). The former contains a *finite number of terms:* the infinite quadratic form (2.2) was automatically cut down, because only the moments $h_r^{(1)}$, $h_r^{\alpha(3)}$ are *privileged moments.* Equation (**6**.3.4) thus only contains the thermodynamic source terms $g_r^{(1)}$, $g_r^{\alpha(3)}$. In the truncation of the hydrodynamic equations based on the smallness of the drift parameter ϵ (rather than of the parameter λ_H) there exists a generalized source term in *all* the vector moment equations. Strictly speaking, there exists no privileged moment any more, in the sense defined in section **4**.4 (although the privileged character of the moments $h_r^{(1)}$, $h_r^{e(3)}$ will reappear later in a subtler form). As a result, the entropy production becomes an *infinite bilinear form* (truncated in practice at $p = N$) which, besides the purely thermodynamic forces $g_r^{(1)}$, $g_r^{\alpha(3)}$, also involves the generalized stresses $\bar{g}_r^{\alpha(2p+1)}$. The form (2.6) will therefore be called the *quasi-thermodynamic form of the entropy production.* The elimination of the generalized stresses in terms of thermodynamic forces will be one of the objectives of the neoclassical theory.

Finally, repeating the procedure of section **6**.4, we relate the fluxes to the forces in order to express the entropy production as a quadratic form in the latter variables. Here again there appears an important difference with the transport form (**6**.4.2) of the entropy production. Indeed, because of the presence of the non-privileged fluxes in expressions (2.6), we must use the full set of *quasi-transport equations* (**12**.6.10), (**12**.6.11), (**12**.7.2), (**12**.7.3) in order to perform the transformation. We work, as usual, in the *21M approximation.*

The substitution of the former equations into (2.6) results in the following form of the *ion entropy production*:

$$\begin{aligned}\Sigma^{\mathrm{i}} = {} & \tilde{\kappa}^{\mathrm{i}}_{\parallel}\left(g^{\mathrm{i}(3)}_{\parallel} + \bar{g}^{\mathrm{i}(3)}_{\parallel}\right)^2 + \tilde{\epsilon}^{\mathrm{i}}_{\parallel}\left(\bar{g}^{\mathrm{i}(5)}_{\parallel}\right)^2 \\ & + 2\tilde{\delta}^{\mathrm{i}}_{\parallel}\left(g^{\mathrm{i}(3)}_{\parallel} + \bar{g}^{\mathrm{i}(3)}_{\parallel}\right)\bar{g}^{\mathrm{i}(5)}_{\parallel} \\ & + \tilde{\kappa}^{\mathrm{i}}_{\perp}\left[\left(g^{\mathrm{i}(3)}_{\wedge} + \bar{g}^{\mathrm{i}(3)}_{\wedge}\right)^2 + \left(g^{\mathrm{i}(3)}_{\rho} + \bar{g}^{\mathrm{i}(3)}_{\rho}\right)^2\right] \\ & + \tilde{\epsilon}^{\mathrm{i}}_{\perp}\left[\left(\bar{g}^{\mathrm{i}(5)}_{\wedge}\right)^2 + \left(\bar{g}^{\mathrm{i}(5)}_{\rho}\right)^2\right] \\ & + 2\tilde{\delta}^{\mathrm{i}}_{\perp}\left[\left(g^{\mathrm{i}(3)}_{\wedge} + \bar{g}^{\mathrm{i}(3)}_{\wedge}\right)\bar{g}^{\mathrm{i}(5)}_{\wedge} + \left(g^{\mathrm{i}(3)}_{\rho} + \bar{g}^{\mathrm{i}(3)}_{\rho}\right)\bar{g}^{\mathrm{i}(5)}_{\rho}\right]. \end{aligned} \tag{2.7}$$

A similar (but longer) formula holds for the electron entropy production. This formula is comparable to (**6**.4.6) (with the viscosity coefficients put equal to zero). The difference is obvious: instead of a finite quadratic form in the thermodynamic forces, we have now an *infinite quadratic form* (truncated for convenience) involving all the generalized stresses. For this reason, we see in eq. (2.7) not only the transport coefficients $\tilde{\kappa}^{\mathrm{i}}_{A}$, ($A = \parallel, \perp$) but also the *quasi-transport coefficients* $\tilde{\epsilon}^{\mathrm{i}}_{A}$, $\tilde{\delta}^{\mathrm{i}}_{A}$ defined in eqs. (**12**.6.11), (**12**.7.2), (**12**.7.3). Hence, (2.7) will be called the *quasi-transport form of the entropy production*.

There is, however, one important common feature with eq. (**6**.4.6): *the perp–tangential transport and quasi-transport coefficients $\tilde{L}^{\alpha}_{\wedge}$ do not contribute at all to the entropy production*. This fact confirms our earlier interpretation that the perp–tangential transport coefficients describe purely reversible fluxes; this is true not only for the privileged heat flux, but also for the non-privileged fluxes.

An important simplification of the quasi-transport form of the entropy production sets in when the *drift approximation* is properly taken into account. Assuming, as usual, that the drift parameter ϵ is small, we use the orderings of eqs. (**12**.7.4)–(**12**.7.6), to which we adjoin the following further condition [derived from eqs. (**10**.6.6), (**10**.6.7)]:

$$g^{\alpha(p)}_{\parallel} = \mathrm{O}(\epsilon), \qquad \bar{g}^{\alpha(p)}_{\parallel} = \mathrm{O}(\epsilon). \tag{2.8}$$

Introducing these scalings into the entropy production (2.7), we find that the dominant terms are of order ϵ^2. Neglecting all higher-order contributions,

we find that (2.7) reduces to

$$\Sigma^{\mathrm{i}} = \tilde{\kappa}^{\mathrm{i}}_{\parallel}\left(g^{\mathrm{i}(3)}_{\parallel} + \bar{g}^{\mathrm{i}(3)}_{\parallel}\right)^2 + \tilde{\epsilon}^{\mathrm{i}}_{\parallel}\left(\bar{g}^{\mathrm{i}(5)}_{\parallel}\right)^2 + 2\tilde{\delta}^{\mathrm{i}}_{\parallel}\left(g^{\mathrm{i}(3)}_{\parallel} + \bar{g}^{\mathrm{i}(3)}_{\parallel}\right)\bar{g}^{\mathrm{i}(5)}_{\parallel}$$
$$+ \tilde{\kappa}^{\mathrm{i}}_{\perp}\left(g^{\mathrm{i}(3)}_{\rho}\right)^2 + \mathrm{O}(\epsilon^3), \tag{2.9}$$

and an analogous expression for the electron entropy production.

Thus, the explicit use of the drift approximation leads to the following conclusions, valid through order ϵ^2:

- Not only the perp–tangential transport coefficients, but also *the perp–tangential components of the thermodynamic forces and of the generalized stresses drop out from the entropy production.*
- The only variables that enter the quadratic forms (2.9) (and the corresponding expression for the electrons) are
  (i) the parallel thermodynamic forces $g^{\alpha(p)}_{\parallel}$;
  (ii) the parallel generalized stresses $\bar{g}^{\alpha(p)}_{\parallel}$;
  (iii) the radial thermodynamic forces $g^{\alpha(p)}_{\rho}$.
- The terms involving the radial thermodynamic forces are identical to those of the classical, thermodynamic form (**6**.4.6). We may therefore split the entropy production as

$$\Sigma^{\alpha} = \Sigma^{\alpha}_{\mathrm{CL}} + \Sigma^{\alpha}_{(\parallel)}, \tag{2.10}$$

and we have the following expressions for the component terms (including now the electron entropy production):

$$\Sigma^{\mathrm{e}}_{\mathrm{CL}} = \tilde{\sigma}_{\perp}\left(g^{(1)P}_{\rho}\right)^2 + \tilde{\kappa}^{\mathrm{e}}_{\perp}\left(g^{\mathrm{e}(3)}_{\rho}\right)^2 - 2\tilde{\alpha}_{\perp}\, g^{(1)P}_{\rho} g^{\mathrm{e}(3)}_{\rho},$$

$$\Sigma^{\mathrm{i}}_{\mathrm{CL}} = \tilde{\kappa}^{\mathrm{i}}_{\perp}\left(g^{\mathrm{i}(3)}_{\rho}\right)^2, \tag{2.11}$$

and

$$\Sigma^{\mathrm{e}}_{(\parallel)} = \tilde{\sigma}_{\parallel}\left(g^{(1)}_{\parallel} - \bar{g}^{\mathrm{e}(1)}_{\parallel}\right)^2 + \tilde{\kappa}^{\mathrm{e}}_{\parallel}\left(g^{\mathrm{e}(3)}_{\parallel} + \bar{g}^{\mathrm{e}(3)}_{\parallel}\right)^2 + \tilde{\epsilon}^{\mathrm{e}}_{\parallel}\left(\bar{g}^{\mathrm{e}(5)}_{\parallel}\right)^2$$
$$+ 2\tilde{\alpha}_{\parallel}\left(g^{(1)}_{\parallel} - \bar{g}^{\mathrm{e}(1)}_{\parallel}\right)\left(g^{\mathrm{e}(3)}_{\parallel} + \bar{g}^{\mathrm{e}(3)}_{\parallel}\right)$$
$$+ 2\tilde{\gamma}_{\parallel}\left(g^{(1)}_{\parallel} - \bar{g}^{\mathrm{e}(1)}_{\parallel}\right)\bar{g}^{\mathrm{e}(5)}_{\parallel} + 2\tilde{\delta}^{\mathrm{e}}_{\parallel}\left(g^{\mathrm{e}(3)}_{\parallel} + \bar{g}^{\mathrm{e}(3)}_{\parallel}\right)\bar{g}^{\mathrm{e}(5)}_{\parallel},$$

$$\Sigma^{\mathrm{i}}_{(\parallel)} = \tilde{\kappa}^{\mathrm{i}}_{\parallel}\left(g^{\mathrm{i}(3)}_{\parallel} + \bar{g}^{\mathrm{i}(3)}_{\parallel}\right)^2 + \tilde{\epsilon}^{\mathrm{i}}_{\parallel}\left(\bar{g}^{\mathrm{i}(5)}_{\parallel}\right)^2$$
$$+ 2\tilde{\delta}^{\mathrm{i}}_{\parallel}\left(g^{\mathrm{i}(3)}_{\parallel} + \bar{g}^{\mathrm{i}(3)}_{\parallel}\right)\bar{g}^{\mathrm{i}(5)}_{\parallel}. \tag{2.12}$$

[Note that, in writing the classical entropy production, we made use of relation (**12**.7.11).]

For completeness, we also write down the corresponding *kinetic form* of the parallel entropy production,

$$\Sigma^{e}_{(\parallel)} = c^{e}_{13}\left(h^{(1)}_{\parallel}\right)^2 + c^{e}_{33}\left(h^{e(3)}_{\parallel}\right)^2 + c^{e}_{55}\left(h^{e(5)}_{\parallel}\right)^2$$
$$+ 2c^{e}_{13}\, h^{(1)}_{\parallel} h^{e(3)}_{\parallel} + 2c^{e}_{15}\, h^{(1)}_{\parallel} h^{e(5)}_{\parallel} + 2c^{e}_{35}\, h^{e(3)}_{\parallel} h^{e(5)}_{\parallel},$$

$$\Sigma^{i}_{(\parallel)} = c^{i}_{33}\left(h^{i(3)}_{\parallel}\right)^2 + c^{i}_{55}\left(h^{i(5)}_{\parallel}\right)^2 + 2c^{i}_{35}\, h^{i(3)}_{\parallel} h^{i(5)}_{\parallel}. \tag{2.13}$$

The main conclusion of the results obtained thus far is that *the neoclassical effects are entirely disguised in the parallel entropy production* $\Sigma^{\alpha}_{(\parallel)}$. Our next purpose will be to exhibit them explicitly.

17.3. Entropy production and quadratic forms

We now begin the systematic study of the properties of the entropy production. In order to abridge the notations, we introduce some special symbols. We know from algebra (e.g. Gantmacher 1959, Korn and Korn 1968, Smirnov 1970, Kurosh 1971) that with any symmetric $n \times n$ matrix one can associate a *quadratic form* in n variables, as well as a *bilinear form* in $2n$ variables.

We denote by $\boldsymbol{C}^{e}$ the *electron collision matrix in the 21M approximation*; its elements are c^{e}_{pq} $(p, q = 1, 3, 5)$. The associated *quadratic form* (in 3 variables) will be denoted by the symbol $\mathscr{C}^{e}(x_1, x_3, x_5)$. Thus, by definition,

$$\mathscr{C}^{e}(x_1, x_3, x_5) \equiv c^{e}_{11}x_1^2 + c^{e}_{33}x_3^2 + c^{e}_{55}x_5^2 + 2c^{e}_{13}x_1x_3 + 2c^{e}_{15}x_1x_5$$
$$+ 2c^{e}_{35}x_3x_5. \tag{3.1}$$

This definition holds whatever the meaning of the variables x_1, x_3, x_5. The *bilinear form* associated with the matrix $\boldsymbol{C}^{e}$ is denoted by

$$\mathscr{C}^{e}(x_1, x_3, x_5 \mid y_1, y_3, y_5) \equiv c^{e}_{11}x_1y_1 + c^{e}_{33}x_3y_3 + c^{e}_{55}x_5y_5$$
$$+ c^{e}_{13}(x_1y_3 + x_3y_1) + c^{e}_{15}(x_1y_5 + x_5y_1)$$
$$+ c^{e}_{35}(x_3y_5 + x_5y_3). \tag{3.2}$$

This definition also holds, whatever the values of the variables x_i, y_i. The following obvious relation exists between the bilinear and the quadratic forms:

$$\mathscr{C}^{\mathrm{e}}(x_1, x_3, x_5 | x_1, x_3, x_5) = \mathscr{C}^{\mathrm{e}}(x_1, x_3, x_5). \tag{3.3}$$

The quadratic form (in 2 variables) associated with the *ion collision matrix* $\boldsymbol{C}^{\mathrm{i}}$ $[\equiv c^{\mathrm{i}}_{pq},\ p, q = 3, 5]$ in the 21M approximation will be denoted by

$$\mathscr{C}^{\mathrm{i}}(x_3, x_5) \equiv c^{\mathrm{i}}_{33}x_3^2 + c^{\mathrm{i}}_{55}x_5^2 + 2c^{\mathrm{i}}_{35}x_3x_5, \tag{3.4}$$

with a corresponding, obvious definition of the bilinear form. We will also need the matrix $\boldsymbol{C}'^{\mathrm{e}}$ and the quadratic form $\mathscr{C}'^{\mathrm{e}}(x_1, x_3)$ (in 2 variables) corresponding to the 13M approximation,

$$\mathscr{C}'^{\mathrm{e}}(x_1, x_3) \equiv c^{\mathrm{e}}_{11}x_1^2 + c^{\mathrm{e}}_{33}x_3^2 + 2c^{\mathrm{e}}_{13}x_1x_3. \tag{3.5}$$

The quadratic form for the ion collision matrix in the 13M approximation reduces to a single term,

$$\mathscr{C}'^{\mathrm{i}}(x_3) \equiv c^{\mathrm{i}}_{33}x_3^2. \tag{3.6}$$

We now consider the matrices $\boldsymbol{T}^{\alpha}$ of the *parallel quasi-transport coefficients in the 21M approximation*, defined in eqs. (**12**.6.10) and (**12**.6.11),

$$\boldsymbol{T}^{\mathrm{e}} = \begin{pmatrix} \sigma & \alpha & \gamma \\ \alpha & \kappa^{\mathrm{e}} & \delta^{\mathrm{e}} \\ \gamma & \delta^{\mathrm{e}} & \epsilon^{\mathrm{e}} \end{pmatrix}, \qquad \boldsymbol{T}^{\mathrm{i}} = \begin{pmatrix} \kappa^{\mathrm{i}} & \delta^{\mathrm{i}} \\ \delta^{\mathrm{i}} & \epsilon^{\mathrm{i}} \end{pmatrix}. \tag{3.7}$$

From here on we use the convention introduced in section **15**.3: in the symbols denoting the dimensionless parallel quasi-transport coefficients we drop the tilda and the subscript $\|$, thus writing $\sigma \equiv \tilde{\sigma}_{\|}$, $\kappa^{\mathrm{e}} \equiv \tilde{\kappa}^{\mathrm{e}}_{\|}$, etc. The matrices $\boldsymbol{T}^{\alpha}$ are *symmetric* as a result of the Onsager property. The corresponding quadratic forms are

$$\mathscr{T}^{\mathrm{e}}(x_1, x_3, x_5) \equiv \sigma x_1^2 + \kappa^{\mathrm{e}}x_3^2 + \epsilon^{\mathrm{e}}x_5^2 + 2\alpha x_1x_3 + 2\gamma x_1x_5 + 2\delta^{\mathrm{e}}x_3x_5, \tag{3.8}$$

$$\mathscr{T}^{\mathrm{i}}(x_3, x_5) \equiv \kappa^{\mathrm{i}}x_3^2 + \epsilon^{\mathrm{i}}x_5^2 + 2\delta^{\mathrm{i}}x_3x_5. \tag{3.9}$$

For convenience, we also write down explicitly the corresponding bilinear forms

$$\mathscr{T}^{e}(x_1, x_3, x_5 \mid y_1, y_3, y_3) \equiv \sigma x_1 y_1 + \kappa^{e} x_3 y_3 + \epsilon^{e} x_5 y_5 + \alpha(x_1 y_3 + x_3 y_1) + \gamma(x_1 y_5 + x_5 y_1) + \delta^{e}(x_3 y_5 + x_5 y_3), \tag{3.10}$$

$$\mathscr{T}^{i}(x_3, x_5 \mid y_3, y_5) \equiv \kappa^{i} x_3 y_3 + \epsilon^{i} x_5 y_5 + \delta^{i}(x_3 y_5 + x_5 y_3). \tag{3.11}$$

With these new notations, the *parallel entropy production* in its *kinetic form* (2.13) is compactly rewritten as

$$\Sigma^{e}_{(\parallel)} = \mathscr{C}^{e}\left(h^{(1)}_{\parallel}, h^{e(3)}_{\parallel}, h^{e(5)}_{\parallel}\right),$$

$$\Sigma^{i}_{(\parallel)} = \mathscr{C}^{i}\left(h^{i(3)}_{\parallel}, h^{i(5)}_{\parallel}\right). \tag{3.12}$$

The corresponding quasi-transport form (2.12) is

$$\Sigma^{e}_{(\parallel)} = \mathscr{T}^{e}\left(g^{(1)}_{\parallel} - \bar{g}^{e(1)}_{\parallel}, g^{e(3)}_{\parallel} + \bar{g}^{e(3)}_{\parallel}, \bar{g}^{e(5)}_{\parallel}\right),$$

$$\Sigma^{i}_{(\parallel)} = \mathscr{T}^{i}\left(g^{i(3)}_{\parallel} + \bar{g}^{i(3)}_{\parallel}, \bar{g}^{i(5)}_{\parallel}\right). \tag{3.13}$$

The most important property for the thermodynamic interpretation of these quantities refers to their *sign*. This question was discussed in chapter 6. We recall (and complete) the conclusions of that chapter. The basic fact, derived in section 6.2, is the following:

Property 1. *The quadratic forms* $\mathscr{C}^{e}(x_1, x_3, x_5)$, $\mathscr{C}^{i}(x_3, x_5)$ *associated with the collision matrices in the* $21M$ *approximation are definite positive*:

$$\mathscr{C}^{e}(x_1, x_3, x_5) \geqslant 0, \qquad \mathscr{C}^{i}(x_3, x_5) \geqslant 0. \tag{3.14}$$

The adjective "definite" means that the equality is only satisfied when $x_1 = x_3 = x_5 = 0$. We stress the fact that (3.14) expresses an intrinsic property of the associated collision matrices $\boldsymbol{C}^{e}$, $\boldsymbol{C}^{i}$, independently of the meaning of the variables x_1, x_3, x_5. As shown in eqs. (**6**.2.11)–(**6**.2.13), these properties imply a certain number of inequalities involving the collision matrix elements; we now write them down explicitly:

– relations involving the diagonal elements,

$$c_{11}^{e} > 0, \qquad c_{33}^{e} > 0, \qquad c_{55}^{e} > 0,$$

$$c_{33}^{i} > 0, \qquad c_{55}^{i} > 0; \tag{3.15}$$

– relations involving 2×2 determinants,

$$D_{13}^{e} \equiv c_{11}^{e} c_{33}^{e} - c_{13}^{e2} > 0,$$

$$D_{15}^{e} \equiv c_{11}^{e} c_{55}^{c} - c_{15}^{e2} > 0,$$

$$D_{35}^{e} \equiv c_{33}^{e} c_{55}^{e} - c_{35}^{e2} > 0,$$

$$D_{35}^{i} \equiv c_{33}^{i} c_{55}^{i} - c_{35}^{i2} > 0; \tag{3.16}$$

– relation involving a 3×3 determinant,

$$F_{135}^{e} \equiv \| \boldsymbol{C}^{e} \| = c_{11}^{e} c_{33}^{e} c_{55}^{e} + 2 c_{13}^{e} c_{15}^{e} c_{35}^{e} - c_{11}^{e} c_{35}^{e2} - c_{33}^{e} c_{15}^{e2} - c_{55}^{e} c_{13}^{e2} > 0. \tag{3.17}$$

The notations used here are the same as those defined in (**15**.3.15).

It may be noted that eqs. (3.15)–(3.17) imply, a fortiori, the definite positive character of the 13M approximation collision matrix,

$$\mathscr{C}'^{e}(x_1, x_3) \geqslant 0, \qquad \mathscr{C}'^{i}(x_3) \geqslant 0. \tag{3.18}$$

When properties (3.14) are applied to the parallel entropy production, we find the fundamental result

$$\Sigma_{(\|)}^{e} \geqslant 0, \qquad \Sigma_{(\|)}^{i} \geqslant 0. \tag{3.19}$$

This result is stronger than the mere prediction of the second law of thermodynamics, which only states that the *total* entropy production must be positive. Here, we proved that the *parallel parts* of the electron entropy production and of the ion entropy production are *separately definite positive*.

The next important property is based on the following theorem:

Theorem 1. *If a quadratic form* $\mathscr{C}(x_1, x_3, x_5)$, *associated with the matrix* $\boldsymbol{C}$ *is definitive positive, the quadratic form associated with the inverse matrix* $\boldsymbol{C}^{-1}$ *is also definite positive.*

The proof of this theorem is given in Appendix 17A.1.

The relevance of this theorem for our problem results from the following important fact, which is obvious from a comparison of eqs. (**12**.6.9) and (**12**.6.10):

Property 2. *The parallel quasi-transport matrix* $\boldsymbol{T}^{\alpha}$ *is the inverse of the collision matrix* $\boldsymbol{C}^{\alpha}$:

$$\boldsymbol{T}^{e} = \boldsymbol{C}^{e\,-1}, \qquad \boldsymbol{T}^{i} = \boldsymbol{C}^{i\,-1}. \tag{3.20}$$

We then immediately deduce

Theorem 2. *The quadratic forms associated with the parallel quasi-transport matrices are definite positive*:

$$\mathscr{T}^{e}(x_1, x_3, x_5) \geqslant 0, \qquad \mathscr{T}^{i}(x_3, x_5) \geqslant 0. \tag{3.21}$$

This property then implies the following positivity criteria:

- relations involving the diagonal parallel quasi-transport coefficients,

$$\sigma > 0, \qquad \kappa^{e} > 0, \qquad \epsilon^{e} > 0,$$
$$\kappa^{i} > 0, \qquad \epsilon^{i} > 0; \tag{3.22}$$

- relations involving 2×2 determinants,

$$\Delta^{e}_{13} \equiv \sigma\kappa^{e} - \alpha^{2} > 0,$$
$$\Delta^{e}_{15} \equiv \sigma\epsilon^{e} - \gamma^{2} > 0,$$
$$\Delta^{e}_{35} \equiv \kappa^{e}\epsilon^{e} - \delta^{e2} > 0,$$
$$\Delta^{i}_{35} \equiv \kappa^{i}\epsilon^{i} - \delta^{i2} > 0; \tag{3.23}$$

- relation involving a 3×3 determinant,

$$\Phi^{e}_{135} \equiv \| \boldsymbol{T}^{e} \| = \sigma\kappa^{e}\epsilon^{e} + 2\alpha\gamma\delta^{e} - \sigma\delta^{e2} - \kappa^{e}\gamma^{2} - \epsilon^{e}\alpha^{2} > 0. \tag{3.24}$$

The symbols appearing here are the same as those defined in (**15**.3.17).

Theorem 2, applied to the transport form of the parallel entropy production (2.12) ensures its definite positive character, a fact which is not surprising. The latter form of the entropy production is, however, not very useful in the present context. Indeed, it is expressed in terms of two types of variables:

(1) the *parallel thermodynamic forces* $g_{\|}^{\alpha(p)}$, which are not directly accessible to measurement in a tokamak;

(2) the *parallel generalized stresses* $\bar{g}_{\|}^{\alpha(p)}$, which are as yet unknown. *

17.4. Decomposition of the parallel fluxes

The crux of the neoclassical theory lies in the geometrical constraint of *zero divergence of the vector fluxes* (to order ϵ). It was shown in detail in section **12**.8 how this constraint imposes a relationship between parallel and radial components of the various vectorial quantities. It can be said that the neoclassical enhancement of the radial transport originates in these relations. They will be further exploited in the present section. In section **12**.8, the following expressions were obtained for the parallel fluxes (see eqs. **12**.8.23, **12**.8.25):

$$
\begin{aligned}
h_{\|}^{(1)} &= -\frac{\mathscr{B}_0}{B}\mathscr{K}_e\, g_\rho^{(1)P} + \frac{B}{\mathscr{B}_0}\omega_1,\\
h_{\|}^{e(3)} &= \frac{\mathscr{B}_0}{B}\mathscr{K}_e\, g_\rho^{e(3)} + \frac{B}{\mathscr{B}_0}\omega_3^e,\\
h_{\|}^{e(5)} &= \frac{B}{\mathscr{B}_0}\omega_5^e,\\
h_{\|}^{i(3)} &= \frac{\mathscr{B}_0}{B}\mathscr{K}_i\, g_\rho^{i(3)} + \frac{B}{\mathscr{B}_0}\omega_3^i,\\
h_{\|}^{i(5)} &= \frac{B}{\mathscr{B}_0}\omega_5^i,
\end{aligned}
\tag{4.1}
$$

where

$$
\omega_1 \equiv a\omega_1^i - \omega_1^e. \tag{4.2}
$$

The quantities a and $\mathscr{K}_\alpha$ were defined in (**12**.6.3) and (**12**.8.18), respectively, and relation (**12**.6.7) was used.

The "poloidal fluxes" ω_n^α are determined by eqs. (**12**.8.30) and (**12**.8.31). It is useful to consider first, as a reference state, a situation in which *the*

* Remember that only the *surface-averaged* generalized stresses have been determined in chapter 14!

generalized stresses are strictly zero. In other words, this reference state is defined as a short mean free path regime. In this case, eqs. (12.8.30) reduce to a trivially simply form [and eq. (12.8.31) becomes "0 = 0"], with the solution

$$\omega_1^{(0)} = \mathscr{K}_e\, g_\rho^{(1)P} + \sigma\, \hat{g}_\parallel^{(1)A},$$

$$\omega_3^{(0)} = -\mathscr{K}_e\, g_\rho^{e(3)} + \alpha\, \hat{g}_\parallel^{(1)A},$$

$$\omega_5^{e(0)} = \gamma\, \hat{g}_\parallel^{(1)A},$$

$$\omega_3^{i(0)} = -\mathscr{K}_i\, g_\rho^{i(3)},$$

$$\omega_5^{i(0)} = 0. \tag{4.3}$$

The parallel fluxes then reduce to the following form, denoted by the symbols ξ_p^α:

$$\xi_1^e = \left(\frac{B}{\mathscr{B}_0} - \frac{\mathscr{B}_0}{B}\right)\mathscr{K}_e\, g_\rho^{(1)P} + \frac{B}{\mathscr{B}_0}\sigma\, \hat{g}_\parallel^{(1)A},$$

$$\xi_3^e = -\left(\frac{B}{\mathscr{B}_0} - \frac{\mathscr{B}_0}{B}\right)\mathscr{K}_e\, g_\rho^{e(3)} + \frac{B}{\mathscr{B}_0}\alpha\, \hat{g}_\parallel^{(1)A},$$

$$\xi_5^e = \frac{B}{\mathscr{B}_0}\gamma\, \hat{g}_\parallel^{(1)A},$$

$$\xi_3^i = -\left(\frac{B}{\mathscr{B}_0} - \frac{\mathscr{B}_0}{B}\right)\mathscr{K}_i\, g_\rho^{i(3)},$$

$$\xi_5^i = 0. \tag{4.4}$$

These quantities will be called the *reference parallel fluxes*. They can be further decomposed as follows, into a contribution of the *classical parallel fluxes* and of the *Pfirsch–Schlüter parallel fluxes*; the justification of these names will become clear below.

$$\xi_p^\alpha = h_{\parallel \mathrm{CL}}^{\alpha(p)} + h_{\parallel \mathrm{PS}}^{\alpha(p)}, \quad p = 1, 3, 5, \qquad [\mathrm{CONV}] \tag{4.5}$$

where

$$h_{\|\mathrm{CL}}^{(1)} = \frac{B}{\mathscr{B}_0}\sigma\,\hat{g}_{\|}^{(1)A},$$

$$h_{\|\mathrm{CL}}^{\mathrm{e}(3)} = \frac{B}{\mathscr{B}_0}\alpha\,\hat{g}_{\|}^{(1)A},$$

$$h_{\|\mathrm{CL}}^{\mathrm{e}(5)} = \frac{B}{\mathscr{B}_0}\gamma\,\hat{g}_{\|}^{(1)A},$$

$$h_{\|\mathrm{CL}}^{\mathrm{i}(3)} = 0,$$

$$h_{\|\mathrm{CL}}^{\mathrm{i}(5)} = 0, \tag{4.6}$$

and

$$h_{\|\mathrm{PS}}^{(1)} = \left(\frac{B}{\mathscr{B}_0} - \frac{\mathscr{B}_0}{B}\right)\mathscr{K}_{\mathrm{e}}\; g_{\rho}^{(1)P},$$

$$h_{\|\mathrm{PS}}^{\mathrm{e}(3)} = -\left(\frac{B}{\mathscr{B}_0} - \frac{\mathscr{B}_0}{B}\right)\mathscr{K}_{\mathrm{e}}\; g_{\rho}^{\mathrm{e}(3)},$$

$$h_{\|\mathrm{PS}}^{\mathrm{e}(5)} = 0,$$

$$h_{\|\mathrm{PS}}^{\mathrm{i}(3)} = -\left(\frac{B}{\mathscr{B}_0} - \frac{\mathscr{B}_0}{B}\right)\mathscr{K}_{\mathrm{i}}\; g_{\rho}^{i(3)},$$

$$h_{\|\mathrm{PS}}^{\mathrm{i}(5)} = 0. \tag{4.7}$$

We now come back to the complete parallel fluxes, defined in (4.1). The differences between these fluxes and the reference fluxes will be called the *parallel banana fluxes*, $h_{\|\mathrm{B}}^{\alpha(p)}$,

$$h_{\|}^{\alpha(p)} = h_{\|\mathrm{CL}}^{\alpha(p)} + h_{\|\mathrm{PS}}^{\alpha(p)} + h_{\|\mathrm{B}}^{\alpha(p)}. \qquad [\mathrm{CONV}] \tag{4.8}$$

It is easily shown that these (unaveraged) parallel banana fluxes are simply connected to the *average radial banana fluxes*. Consider, for instance, the parallel banana electric current which, by combining (4.1), (4.2), (4.6) and (4.7) is written as

$$h_{\|\mathrm{B}}^{(1)} = \frac{B}{\mathscr{B}_0}\left(\omega_1 - \mathscr{K}_{\mathrm{e}}\; g_{\rho}^{(1)P} - \sigma\,\hat{g}_{\|}^{(1)A}\right).$$

From eq. (**12**.8.30) we obtain

$$h_{\|B}^{(1)} = \frac{B}{\mathscr{B}_0}\left(-\sigma\left\langle \frac{B}{\mathscr{B}_0}\bar{g}_{\|}^{e(1)}\right\rangle + \alpha\left\langle \frac{B}{\mathscr{B}_0}\bar{g}_{\|}^{e(3)}\right\rangle + \gamma\left\langle \frac{B}{\mathscr{B}_0}\bar{g}_{\|}^{e(5)}\right\rangle\right).$$

By using the definitions of table **12**.9.1, the quantities on the right-hand side are directly related to the average radial banana fluxes. A similar argument applies to the other parallel banana fluxes:

$$h_{\|B}^{(1)} = \frac{B}{\mathscr{B}_0}\mathscr{K}_e^{-1}\left(-\sigma\left\langle h_\rho^{e(1)}\right\rangle_B + \alpha\left\langle h_\rho^{e(3)}\right\rangle_B + \gamma\left\langle h_\rho^{e(5)}\right\rangle_B\right),$$

$$h_{\|B}^{e(3)} = \frac{B}{\mathscr{B}_0}\mathscr{K}_e^{-1}\left(-\alpha\left\langle h_\rho^{e(1)}\right\rangle_B + \kappa^e\left\langle h_\rho^{e(3)}\right\rangle_B + \delta^e\left\langle h_\rho^{e(5)}\right\rangle_B\right),$$

$$h_{\|B}^{e(5)} = \frac{B}{\mathscr{B}_0}\mathscr{K}_e^{-1}\left(-\gamma\left\langle h_\rho^{e(1)}\right\rangle_B + \delta^e\left\langle h_\rho^{e(3)}\right\rangle_B + \epsilon^e\left\langle h_\rho^{e(5)}\right\rangle_B\right),$$

$$h_{\|B}^{i(3)} = \frac{B}{\mathscr{B}_0}\mathscr{K}_i^{-1}\left(\kappa^i\left\langle h_\rho^{i(3)}\right\rangle_B + \delta^i\left\langle h_\rho^{i(5)}\right\rangle_B\right),$$

$$h_{\|B}^{i(5)} = \frac{B}{\mathscr{B}_0}\mathscr{K}_i^{-1}\left(\delta^i\left\langle h_\rho^{i(3)}\right\rangle_B + \epsilon^i\left\langle h_\rho^{i(5)}\right\rangle_B\right). \tag{4.9}$$

Before discussing these relations, we derive, as a side-product, a set of remarkably simple formulae for the *parallel generalized frictions* $Q_{\|}^{\alpha(p)}$ defined in (**4**.6.34)–(**4**.6.35). These quantities, which are linearly related to the parallel fluxes, can obviously be decomposed in the same way as the latter:

$$Q_{\|}^{\alpha(p)} = Q_{\|CL}^{\alpha(p)} + Q_{\|PS}^{\alpha(p)} + Q_{\|B}^{\alpha(p)}. \qquad [\text{CONV}] \tag{4.10}$$

Using (4.9), the banana contribution can be expressed as

$$\begin{aligned} Q_{\|B}^{(1)} &= c_{11}^e h_{\|B}^{(1)} + c_{13}^e h_{\|B}^{e(3)} + c_{15}^e h_{\|B}^{e(5)} \\ &= \frac{B}{\mathscr{B}_0}\mathscr{K}_e^{-1}\Big[\left(-c_{11}^e\sigma - c_{13}^e\alpha - c_{15}^e\gamma\right)\left\langle h_\rho^{e(1)}\right\rangle_B \\ &\qquad + \left(c_{11}^e\alpha + c_{13}^e\kappa^e + c_{15}^e\delta^e\right)\left\langle h_\rho^{e(3)}\right\rangle_B \\ &\qquad + \left(c_{11}^e\gamma + c_{13}^e\delta^e + c_{15}^e\epsilon^e\right)\left\langle h_\rho^{e(5)}\right\rangle_B\Big]. \end{aligned}$$

Identities (A1.3) lead to a drastically simplified form for all the parallel banana frictions:

$$Q_{\|B}^{(1)} = -\frac{B}{\mathscr{B}_0}\mathscr{K}_e^{-1}\left\langle h_\rho^{e(1)}\right\rangle_B,$$

$$Q_{\|B}^{e(3)} = \frac{B}{\mathscr{B}_0}\mathscr{K}_e^{-1}\left\langle h_\rho^{e(3)}\right\rangle_B,$$

$$Q_{\|B}^{e(5)} = \frac{B}{\mathscr{B}_0}\mathscr{K}_e^{-1}\left\langle h_\rho^{e(5)}\right\rangle_B,$$

$$Q_{\|B}^{i(3)} = \frac{B}{\mathscr{B}_0}\mathscr{K}_i^{-1}\left\langle h_\rho^{i(3)}\right\rangle_B,$$

$$Q_{\|B}^{i(5)} = \frac{B}{\mathscr{B}_0}\mathscr{K}_i^{-1}\left\langle h_\rho^{i(5)}\right\rangle_B. \tag{4.11}$$

These formulae provide an alternative simple interpretation of the average radial banana fluxes.

We now discuss the general decomposition of the parallel fluxes (4.8) into three terms. Their common feature is that each of the components is related to a specific *surface-averaged quantity*. As a result, *the dependence of the parallel fluxes on the poloidal angle is completely explicit*: it is contained solely in the factors $B(\theta, \rho)$. This feature will be very useful in the forthcoming transformations. Each of the three components is characterized by a specific set of surface quantities appearing in its definition.

The *parallel classical fluxes* appear to be driven by the surface-averaged *external electric field*, through the source term $\hat{g}_\|^{(1)A}$. We know from the discussion following eq. (**12**.5.9) that the latter is the only parallel thermodynamic force having a non-vanishing surface average. Comparing eqs. (4.6) to (**12**.10.4), we see that the former lead, upon surface-averaging, precisely to the contributions to the average parallel fluxes that were called "classical" in section **12**.10.

The *parallel Pfirsch–Schlüter fluxes* are directly proportional to the *radial thermodynamic forces* (which, to leading order, are surface quantities). We see here the coefficients $[(B/\mathscr{B}_0) - (\mathscr{B}_0/B)]$ that are typical of the PS fluxes discussed in section **13**.3. The precise relation of the parallel PS fluxes to be radial ones will be discussed later.

The *parallel banana fluxes* are linear combinations of the *radial banana fluxes*, or, what amounts to the same, of the *average parallel generalized stresses*.

We have thus realized in eq. (4.8) a decomposition of the *unaveraged, parallel* fluxes that has the same characteristics as the decomposition obtained in section **12**.9 for the *averaged, radial* fluxes. These relations uncover the disguise of the neoclassical effects hidden in the parallel entropy production, as discussed at the end of section 17.2.

Note that the parallel fluxes, when expressed in their usual form (**12**.6.10), (**12**.6.11), appear to be driven by the *parallel* forces, to which they are related by the quasi-transport matrix. But the neoclassical part of the same fluxes, expressed in the form (4.7), (4.9) appear to be driven by the *radial* forces. There is no paradox involved here. The key to the "mystery" is the existence of the *geometrical constraint of zero divergence*. The latter introduces a relationship between parallel and radial components of the thermodynamic forces, expressed by eqs. (**12**.8.27). Hence, the parallel fluxes can be expressed in terms of either set of thermodynamic forces.

A last fact is somewhat puzzling. Whereas the reference fluxes (4.4) are explicitly related to the thermodynamic *forces*, the parallel banana fluxes are "naturally" related to the radial banana *fluxes*. The matrix interrelating these quantities is (up to a factor) simply the parallel quasi-transport matrix. It can be said that the average radial banana fluxes $(-\langle h_\rho^{e(1)}\rangle_B, \langle h_\rho^{e(3)}\rangle_B, \langle h_\rho^{e(5)}\rangle_B)$ play the role of *thermodynamic forces* conjugate to the parallel banana fluxes $(h_{\|B}^{(1)}, h_{\|B}^{e(3)}, h_{\|B}^{e(5)})$, respectively. This fact will bear some important consequences in the thermodynamic interpretation of the neoclassical transport theory.

Returning now to the entropy production, our present analysis allows us to exhibit the neoclassical effects explicitly. Indeed, substituting representation (4.8) into the parallel entropy production in its kinetic form (3.12), we find

$$\Sigma_{(\|)}^{e} = \mathscr{C}^{e}\left(h_{\|CL}^{(1)} + h_{\|PS}^{(1)} + h_{\|B}^{(1)},\ h_{\|CL}^{e(3)} + h_{\|PS}^{e(3)} + h_{\|B}^{e(3)},\ h_{\|CL}^{e(5)} + h_{\|PS}^{e(5)} + h_{B}^{e(5)}\right),$$

$$\Sigma_{(\|)}^{i} = \mathscr{C}^{i}\left(h_{\|CL}^{i(3)} + h_{\|PS}^{i(3)} + h_{\|B}^{i(3)},\ h_{\|CL}^{i(5)} + h_{\|PS}^{i(5)} + h_{\|B}^{i(5)}\right). \tag{4.12}$$

When the explicit forms of the variables, given in (4.6), (4.7) and (4.9) are introduced, these expressions become, of course, quite complicated. In particular, it appears that *the three transport mechanisms* (CL, PS, B) *are completely intermingled*, because they enter *non-linearly* the entropy production.

It will be shown in the next section that this complication can be seriously reduced when we realize that the description of a magnetically confined plasma requires a slightly modified formalism.

17.5. The surface-averaged entropy production

A characteristic feature of the transport theory of toroidally confined plasmas is the fact that a complete spatial localization of the physical quantities, i.e.

their definition at each point (ρ, θ, ζ), is a superfluous "luxury". On one hand, there exist no simple transport relationships between completely localized quantities. On the other hand, only *surface-averaged* quantities, depending solely on the radial coordinate ρ are experimentally accessible [typically, experiments provide us with temperature, density or pressure "*profiles*", i.e. the functions $T_\alpha(\rho)$, $n_\alpha(\rho)$, $P(\rho)$, and with average particle or heat fluxes]. The characteristic classical and neoclassical transport equations in toroidally confined plasmas derived in chapter 13, 15 and 16 are relations between surface-averaged fluxes and surface-averaged forces.

It is therefore tempting to define a surface-averaged entropy production $\langle \Sigma^\alpha \rangle$ and to investigate whether this quantity can provide a basis for the non-equilibrium thermodynamics of toroidally confined plasmas. Going back to eq. (2.10), which defines the entropy production in the drift approximation, we take the surface average of both sides and define the *surface-averaged* (or, simply *average*) *entropy production of species* α as

$$\langle \Sigma^\alpha \rangle = \langle \Sigma^\alpha \rangle_{\mathrm{CI}} + \langle \Sigma^\alpha \rangle_{(\parallel)}. \tag{5.1}$$

We first discuss the simpler problem of the *classical average entropy production*. In definitions (2.11) the thermodynamic forces $g_\rho^{(1)p}$, $g_\rho^{\alpha(3)}$, i.e. the radial pressure and temperature gradients, are (to leading order) surface quantities. Therefore, the averaging operation in (5.1) only involves the *perpendicular transport coefficients* $\tilde{\sigma}_\perp$, $\tilde{\alpha}_\perp$, $\tilde{\kappa}_\perp^\alpha$. Using their drift approximation values, i.e. the asymptotic expressions of eq. (**5**.6.2), we find (for the ion contribution)

$$\langle \Sigma^{\mathrm{i}} \rangle_{\mathrm{CI}} = \left\langle \tilde{\kappa}_\perp^{\mathrm{i}\infty} \right\rangle \left(g_\rho^{\mathrm{i}(3)} \right)^2, \tag{5.2}$$

where

$$\left\langle \tilde{\kappa}_\perp^{\mathrm{i}\infty} \right\rangle = \left\langle \frac{1}{(\Omega_{\mathrm{i}}\tau_{\mathrm{i}})^2} \right\rangle c_{33}^{\mathrm{i}} = \frac{1}{(\Omega_{\mathrm{i}0}\tau_{\mathrm{i}})^2}\, \mathscr{G}\, c_{33}^{\mathrm{i}}.$$

Thus, the transport coefficient entering (5.2) is precisely the *classical radial ion thermal conductivity*, as defined in eq. (**13**.2.13) [the symbol $\mathscr{G}$ was defined in (**13**.2.21)]. A similar conclusion holds for the average electron entropy production.

As a result, the *classical average entropy production* has the form

$$\langle \Sigma^{e} \rangle_{CL} = \left(\frac{1}{\Omega_{e0}\tau_e} \right)^2 \mathscr{G} \left[c^{e}_{11} \left(g^{(1)P}_{\rho} \right)^2 + c^{e}_{33} \left(g^{e(3)}_{\rho} \right)^2 - 2 c^{e}_{13}\, g^{(1)P}_{\rho} g^{e(3)}_{\rho} \right],$$

$$\langle \Sigma^{i} \rangle_{CL} = \left(\frac{1}{\Omega_{i0}\tau_i} \right)^2 \mathscr{G}\, c^{i}_{33} \left(g^{i(3)}_{\rho} \right)^2. \tag{5.3}$$

This can also be written more compactly as

$$\langle \Sigma^{e} \rangle_{CL} = \left(\frac{1}{\Omega_{e0}\tau_e} \right)^2 \mathscr{G}\, \mathscr{C}'^{e} \left(-g^{(1)P}_{\rho},\ g^{e(3)}_{\rho} \right),$$

$$\langle \Sigma^{i} \rangle_{CL} = \left(\frac{1}{\Omega_{i0}\tau_i} \right)^2 \mathscr{G}\, \mathscr{C}'^{i} \left(g^{i(3)}_{\rho} \right). \tag{5.4}$$

This is precisely the usual "transport form" of the radial contribution to the entropy production (eq. 2.11), in which the transport coefficients are replaced by their average values given in table **13**.2.1. The classical average entropy production of the electrons (ions) appears as a definite positive quadratic form in two (one) variables; these variables are simply the pressure gradient and the electron (ion) temperature gradient. It may be good to recall that not only the form of eq. (5.3) but also the values of the coefficients are *universal*, i.e. independent of the truncation level, as appears from the results of sections **5**.6 and **13**.2. The only assumption limiting this result is the drift approximation.

The classical signature in (5.3) is the direct appearance of the *collision matrix* in the transport form of the entropy production associated with the radial transport. The fact that only the *truncated* collision matrix (i.e. c^{e}_{11}, c^{e}_{33}, c^{e}_{13} for the electrons and c^{i}_{33} for the ions) defines the entropy production is *not* the result of an approximation, but results from the very structure of the transport equations, notably from the distinction between privileged and non-privileged fluxes, as explained in section **6**.4.

Next we note that, using the classical radial transport equations (**13**.2.3)–(**13**.2.5), the classical average entropy production can be cast into the form

$$\langle \Sigma^{e} \rangle_{CL} = \left\langle h^{e(1)}_{\rho} \right\rangle_{CL} g^{(1)P}_{\rho} + \left\langle h^{e(3)}_{\rho} \right\rangle_{CL} g^{e(3)}_{\rho},$$

$$\langle \Sigma^{i} \rangle_{Cl} = \left\langle h^{e(3)}_{\rho} \right\rangle_{CL} g^{i(3)}_{\rho}. \tag{5.5}$$

This is a typical *thermodynamic form* (in contrast to the quasithermodynamic form of the parallel entropy production): it is a bilinear form involving products of the classical *average* radial fluxes and their conjugate thermodynamic forces. No unprivileged moments contribute to the classical entropy production.

It is appropriate to discuss at this place a somewhat more general aspect of the neoclassical transport theory. It was stressed at the end of section **12**.4 that this theory offers a purely diffusive picture of the average radial electron flux. In such a picture it is necessary to pair the fluxes occurring in the problem with their corresponding forces. The criterion for such a pairing is the entropy production, in which a force–flux product appears for each irreversible process. According to this criterion, eq. (5.5) *clearly identifies the radial pressure gradient as the thermodynamic force associated with the average radial electron flux* (at least for the classical and, as will be shown in section 17.6, for the PS fluxes). It thus confirms the identification strongly suggested by the form (**15**.3.1)–(**15**.3.3) of the transport equations.

This identification is rather unusual from the point of view of the traditional theory of diffusion, at least at first sight. It is customary to think that a diffusion flux in a multicomponent system is driven by the concentration gradient of each species. It is, however, not difficult to understand that the present description of the electron diffusion is consistent with orthodox thermodynamics. Indeed, the concentration gradient is only an approximate form of the driving force, valid under some well-defined condition. The actual driving force of diffusion is the *gradient of chemical potential* (de Groot and Mazur, 1984). This thermodynamic function is, of course, a function of the composition variables x_α, but also of the temperature and the pressure. The gradient of the chemical potential can therefore be expressed as a combination of the gradients of the latter independent variables. As a result, one obtains the following general formula for the contribution of the diffusion to the entropy production of a multicomponent system (Hirschfelder et al. 1954, Cunningham and Williams 1980):

$$\sigma^{\text{diff}} = -T^{-1} \sum_\alpha \Gamma^\alpha \cdot \Lambda^\alpha,$$

where the force Λ^α associated with the diffusion flux is

$$\Lambda^\alpha = m_\alpha^{-1} \left(\sum_\beta \frac{\partial \mu^\alpha}{\partial x_\beta} \nabla x_\beta + \tilde{v}_\alpha \nabla P - \boldsymbol{f}_\alpha \right),$$

where μ^α is the chemical potential and $\tilde{v}_\alpha$ is the partial molal volume of

species α. Clearly, in a system of uniform pressure and in absence of external forces $\boldsymbol{f}_\alpha$ the driving force reduces to the concentration gradient. But in our confined plasma there is no concentration gradient (because of the electroneutrality constraint the ratio of electrons to ions is everywhere the same). Moreover the force $\boldsymbol{f}$, i.e. the radial electric field cancels from the final expressions (sections **12.7**, **12.9**). Thus *the gradient of chemical potential reduces to the effect of the pressure gradient*; this explains the form (5.5) and its relation to thermodynamics. This type of process is explicitly identified by Cunningham and Williams (1980) as *pressure diffusion*.

Consider now the *parallel contribution* to the average entropy production. From the analysis of section 17.3 we know that the unaveraged parallel entropy productions are positive definite quadratic forms in the parallel fluxes. When these expressions are surface-averaged, they are (of course) still positive, but they no longer appear to be quadratic forms. Instead, they are linear combinations of averages of products $\langle h_\parallel \; h_\parallel \rangle$. It will presently be shown that these averages have a very peculiar property.

Consider the basic decomposition (4.8) of the parallel fluxes, which is rewritten by grouping together the classical and the banana parts,

$$h_\parallel^{\alpha(p)} \equiv h_{\parallel \mathrm{PS}}^{\alpha(p)} + \left(h_{\parallel \mathrm{CL}}^{\alpha(p)} + h_{\parallel \mathrm{B}}^{\alpha(p)} \right)$$

$$= \sigma_p \left(\frac{B}{\mathscr{B}_0} - \frac{\mathscr{B}_0}{B} \right) \mathscr{K}_\alpha \; g_\rho^{\alpha(p)} + \frac{B}{\mathscr{B}_0} \zeta_p^\alpha, \qquad [\mathrm{CONV}] \tag{5.6}$$

where $g_\rho^{\alpha(1)} \equiv g_\rho^{(1)P}$, and

$$\sigma_p = \begin{cases} +1, & \text{for } p = 1, \\ -1, & \text{for } p = 3, \\ 0, & \text{for } p = 5. \end{cases} \tag{5.7}$$

The quantities ζ_p^α are defined as

$$\zeta_p^\alpha = \frac{\mathscr{B}_0}{B} \left(h_{\parallel \mathrm{CL}}^{\alpha(p)} + h_{\parallel \mathrm{B}}^{\alpha(p)} \right). \tag{5.8}$$

Their explicit form, obtained from (4.6) and (4.9) is rather complicated; but for the present argument it suffices to note that both $\mathscr{K}_\alpha g_\rho^{\alpha(p)}$ and ζ_p^α are *surface functions*. The dependence on the poloidal angle θ in (5.6) is introduced solely by the factors $B(\theta, \rho)$.

Consider now the average of the product of two parallel fluxes. Taking account of the previous remark, it is written as

$$\left\langle h_{\|}^{\alpha(p)} h_{\|}^{\alpha(q)} \right\rangle = \left\langle \left(\frac{B}{\mathscr{B}_0} - \frac{\mathscr{B}_0}{B} \right)^2 \right\rangle \sigma_p \sigma_q \mathscr{K}_\alpha^2 g_\rho^{\alpha(p)} g_\rho^{\alpha(q)} + \left\langle \frac{B^2}{\mathscr{B}_0^2} \right\rangle \zeta_p^\alpha \zeta_q^\alpha$$

$$+ 2 \left\langle \left(\frac{B}{\mathscr{B}_0} - \frac{\mathscr{B}_0}{B} \right) \frac{B}{\mathscr{B}_0} \right\rangle \mathscr{K}_\alpha \left(\sigma_p g_\rho^{\alpha(p)} \zeta_q^\alpha + \sigma_q g_\rho^{\alpha(q)} \zeta_p^\alpha \right).$$

We now note that, by a calculation similar to the one of section **13**.3, we find

$$\left\langle \left(\frac{B}{\mathscr{B}_0} - \frac{\mathscr{B}_0}{B} \right)^2 \right\rangle = \mathscr{G} - 1,$$

$$\left\langle \frac{B^2}{\mathscr{B}_0^2} \right\rangle = 1,$$

$$\left\langle \left(\frac{B}{\mathscr{B}_0} - \frac{\mathscr{B}_0}{B} \right) \frac{B}{\mathscr{B}_0} \right\rangle = 0. \tag{5.9}$$

We now introduce the following symbols:

$$\eta_p^\alpha = \sigma_p (\mathscr{G} - 1)^{1/2} \mathscr{K}_\alpha \, g_\rho^{\alpha(p)}. \tag{5.10}$$

We then obtain the following "**basic lemma**":

$$\left\langle h_{\|}^{\alpha(p)} h_{\|}^{\alpha(q)} \right\rangle = \eta_p^\alpha \eta_q^\alpha + \zeta_p^\alpha \zeta_q^\alpha. \qquad \text{[CONV]} \tag{5.11}$$

This result is quite important: it represents the next simplest situation compared to a factorization. Roughly speaking, it can expressed as follows: *the average of the product of two parallel fluxes equals the sum of two products.* Moreover, there appears a *decoupling* between terms of type η and terms of type ζ: there are no cross-terms of type $\eta\zeta$. The consequences and the physical interpretation of the basic lemma will progressively appear in the forthcoming arguments.

As a first application, we consider the simpler ion average parallel entropy production, by using (3.12) and the basic lemma (5.11),

$$\begin{aligned}\langle \Sigma^{\mathrm{i}} \rangle_{(\|)} &= \left\langle \mathscr{C}^{\mathrm{i}}\left(h_{\|}^{\mathrm{i}(3)},\ h_{\|}^{\mathrm{i}(5)}\right)\right\rangle \\ &= c_{33}^{\mathrm{i}}\left\langle h_{\|}^{\mathrm{i}(3)} h_{\|}^{\mathrm{i}(3)}\right\rangle + c_{55}^{\mathrm{i}}\left\langle h_{\|}^{\mathrm{i}(5)} h_{\|}^{\mathrm{i}(5)}\right\rangle + 2c_{35}^{\mathrm{i}}\left\langle h_{\|}^{\mathrm{i}(3)} h_{\|}^{\mathrm{i}(5)}\right\rangle \\ &= c_{33}^{\mathrm{i}}\,\eta_3^{\mathrm{i}}\,\eta_3^{\mathrm{i}} + c_{55}^{\mathrm{i}}\,\eta_5^{\mathrm{i}}\,\eta_5^{\mathrm{i}} + 2c_{35}^{\mathrm{i}}\,\eta_3^{\mathrm{i}}\,\eta_5^{\mathrm{i}} \\ &\quad + c_{33}^{\mathrm{i}}\,\zeta_3^{\mathrm{i}}\,\zeta_3^{\mathrm{i}} + c_{55}^{\mathrm{i}}\,\zeta_5^{\mathrm{i}}\,\zeta_5^{\mathrm{i}} + 2c_{35}^{\mathrm{i}}\,\zeta_3^{\mathrm{i}}\,\zeta_5^{\mathrm{i}}. \end{aligned} \tag{5.12}$$

Thus, *as a result of the basic lemma*, the average parallel entropy production is again expressed as a definite positive quadratic form. Alternatively, because of the absence of any coupling between the variables η and ζ, *the average parallel entropy production can be viewed as a sum of two separate, positive definite quadratic forms* in the variables η and ζ, respectively. Noting, moreover, that $\eta_5^{\mathrm{i}} = 0$, we see that the first of these forms reduces to a single term, i.e. to the quadratic form $\mathscr{C}'^{\mathrm{i}}(\eta_3^{\mathrm{i}})$. Summarizing these facts and performing the same calculation for the electron entropy production, we find

$$\langle \Sigma^{\alpha} \rangle_{(\|)} = \langle \Sigma^{\alpha} \rangle_{\mathrm{PS}} + \langle \Sigma^{\alpha} \rangle_{\mathrm{BCL}}, \tag{5.13}$$

with:

$$\begin{aligned}\langle \Sigma^{\mathrm{e}} \rangle_{\mathrm{PS}} &= \mathscr{C}'^{\mathrm{e}}(\eta_1^{\mathrm{e}},\ \eta_3^{\mathrm{e}}), \\ \langle \Sigma^{\mathrm{i}} \rangle_{\mathrm{PS}} &= \mathscr{C}'^{\mathrm{i}}\left(\eta_3^{\mathrm{i}}\right), \end{aligned} \tag{5.14}$$

and:

$$\begin{aligned}\langle \Sigma^{\mathrm{e}} \rangle_{\mathrm{BCL}} &= \mathscr{C}^{\mathrm{e}}(\zeta_1^{\mathrm{e}},\ \zeta_3^{\mathrm{e}},\ \zeta_5^{\mathrm{e}}), \\ \langle \Sigma^{\mathrm{i}} \rangle_{\mathrm{BCL}} &= \mathscr{C}^{\mathrm{i}}\left(\zeta_3^{\mathrm{i}},\ \zeta_5^{\mathrm{i}}\right). \end{aligned} \tag{5.15}$$

The interpretation of these results will be discussed in the forthcoming sections.

17.6. The Pfirsch–Schlüter average entropy production

Consider the contribution to the average entropy production denoted by the subscript PS. Upon explicit substitution of the variables η_p^{α} [eq. (5.10) into

(5.14)], we find

$$\langle \Sigma^{\mathrm{e}} \rangle_{\mathrm{PS}} = \mathscr{K}_{\mathrm{e}}^{2}(\mathscr{G}-1)\Big[c_{11}^{\mathrm{e}} \big(g_{\rho}^{(1)P} \big)^{2} + c_{33}^{\mathrm{e}} \big(g_{\rho}^{\mathrm{e}(3)} \big)^{2} - 2 c_{13}^{\mathrm{e}}\, g_{\rho}^{(1)P} g_{\rho}^{\mathrm{e}(3)} \Big],$$

$$\langle \Sigma^{\mathrm{i}} \rangle_{\mathrm{PS}} = \mathscr{K}_{\mathrm{i}}^{2}(\mathscr{G}-1)\, c_{33}^{\mathrm{i}} \big(g_{\rho}^{\mathrm{i}(3)} \big)^{2}. \tag{6.1}$$

These expressions are remarkably clear in the light of the results of section **13**.3. They represent the *average entropy production associated with the Pfirsch–Schlüter fluxes*. The entropy production emerged quite directly in the "transport form": the coefficients of the quadratic forms are simply the PS transport coefficients.

We also note that, taking into account the PS transport equations (**13**.3.6)–(**13**.3.8), the PS average entropy production can be cast into a *thermodynamic form*,

$$\langle \Sigma^{\mathrm{e}} \rangle_{\mathrm{PS}} = \big\langle h_{\rho}^{\mathrm{e}(1)} \big\rangle_{\mathrm{PS}}\, g_{\rho}^{(1)P} + \big\langle h_{\rho}^{\mathrm{e}(3)} \big\rangle_{\mathrm{PS}}\, g_{\rho}^{\mathrm{e}(3)},$$

$$\langle \Sigma^{\mathrm{i}} \rangle_{\mathrm{PS}} = \big\langle h_{\rho}^{\mathrm{i}(3)} \big\rangle_{\mathrm{PS}}\, g_{\rho}^{\mathrm{i}(3)}. \tag{6.2}$$

Four features are remarkable in this formula.

(1) The *factorization*: the average entropy production is a *sum of products* of the *average* PS fluxes by the conjugate forces.

(2) The *thermodynamic form*, as opposed to the quasi-thermodynamic form (2.6) of the complete, unaveraged entropy production. The average electron and ion entropy production appear as bilinear forms in a *finite* number of variables, viz. the truly thermodynamic average fluxes and their conjugate forces: average electron flux ⇔ pressure gradient, and electron (ion) heat flux ⇔ electron (ion) temperature gradient. *There is no contribution of the unprivileged moments* to the PS average entropy production. This is a *universal* feature, independent of the level of truncation; it results from the fact that the quantities η_{2p+1}^{α} are identically zero for $p \geqslant 2$.

(3) A striking fact is that the PS contribution to the average *parallel* entropy production, $\langle \Sigma^{\alpha} \rangle_{(\|)}$, is expressed entirely in terms of average *radial fluxes* and *radial forces*. This "inversion of roles" is a direct consequence of the zero-divergence geometrical constraint. The latter transforms the parallel average entropy production into a radial contribution, which adds to the classical average radial entropy production (5.3).

(4) We finally note the *PS amplification effect*. The PS average entropy production has the same form as the classical one (5.3), but with different coefficients. The latter are very simply related to the classical ones; hence,

comparing (6.1) and (5.3), we find

$$\langle \Sigma^{\alpha} \rangle_{\mathrm{PS}} = \mathscr{J}^2 \frac{\mathscr{G}-1}{\mathscr{G}} \langle \Sigma^{\alpha} \rangle_{\mathrm{CL}}. \tag{6.3}$$

The proportionality factor appearing here is precisely the PS amplification factor discussed in detail in section **13**.3.

An additional property of the PS entropy production will be discussed at the end of the present section.

We now turn to the second contribution (denoted by the subscript BCL) to the average parallel entropy production in (5.13), (5.15). In order to interpret it physically, we must first find the meaning of the quantities ζ_p^{α} introduced in (5.6). Multiplying both sides of that equation by $B/\mathscr{B}_0$ and surface-averaging the result, we find

$$\left\langle \frac{B}{\mathscr{B}_0} h_{\|}^{\alpha(p)} \right\rangle = \sigma_p \left\langle \frac{B}{\mathscr{B}_0} \left(\frac{B}{\mathscr{B}_0} - \frac{\mathscr{B}_0}{B} \right) \right\rangle \mathscr{K}_{\alpha} g_p^{\alpha(p)} + \left\langle \frac{B^2}{\mathscr{B}_0^2} \right\rangle \zeta_p^{\alpha}. \qquad [\mathrm{CONV}] \tag{6.4}$$

Using properties (5.9), we thus obtain the very simple result

$$\zeta_p^{\alpha} = \left\langle \frac{B}{\mathscr{B}_0} h_{\|}^{\alpha(p)} \right\rangle = \hat{h}_{\|}^{\alpha(p)}. \qquad [\mathrm{CONV}] \tag{6.5}$$

Thus, the quantities ζ_p^{α} are simply the *total average parallel fluxes*. From here on we shall use the more explicit notation $\hat{h}_{\|}^{\alpha(\mathrm{p})}$ instead of ζ_p^{α}. This notation is consistent with the symbol $\hat{g}_{\|}^{(1)A}$ used for the average parallel electric field (**12**.10.3). It may be noted at this point that the "natural" surface averages of all *parallel* components of vector quantities entering the neoclassical theory involves a factor $(B/\mathscr{B}_0)$ (see sections **12**.8, **12**.10 and **15**.1).

Next, we note that the *electronic* average parallel fluxes contain a *classical and a banana contribution*, whereas the *ionic* average parallel fluxes only contain a *banana contribution* [see eqs. (4.6) and (4.9)]:

$$\hat{h}_{\|}^{\mathrm{e}(p)} = \hat{h}_{\|\mathrm{CL}}^{\mathrm{e}(p)} + \hat{h}_{\|\mathrm{B}}^{\mathrm{e}(p)}; \qquad \hat{h}_{\|}^{\mathrm{i}(p)} = \hat{h}_{\|\mathrm{B}}^{\mathrm{i}(p)}, \qquad [\mathrm{CONV}] \tag{6.6}$$

where the classical parts are obtained from (4.6),

$$\begin{aligned} \hat{h}_{\|\mathrm{CL}}^{(1)} &= \sigma\, \hat{g}_{\|}^{(1)A}, \\ \hat{h}_{\|\mathrm{CL}}^{\mathrm{e}(3)} &= \alpha\, \hat{g}_{\|}^{(1)A}, \\ \hat{h}_{\|\mathrm{CL}}^{\mathrm{e}(5)} &= \gamma\, \hat{g}_{\|}^{(1)A}. \end{aligned} \tag{6.7}$$

It is also important to note that *the PS parallel fluxes have an identically vanishing surface average,*

$$\hat{h}_{\|\mathrm{PS}}^{\alpha(p)} = 0, \quad \forall p. \qquad [\mathrm{CONV}] \tag{6.8}$$

These results are in agreement with those of section **12**.10 (where only the parallel electric current was considered). The first equation (6.7) is identical to (**12**.10.4); eq. (**12**.10.5), combined with the definitions of table **12**.9.1, yields the average of (4.9).

We now see that the BCL contribution (5.15) to the average parallel entropy production is interpreted as a kinetic form of the parallel entropy production, eqs. (2.13) or (3.12), evaluated with the *average* parallel fluxes. Because of eq. (6.6), we see that the *ion* average BCL entropy production contains only banana fluxes; we therefore drop the index CL and write

$$\langle \Sigma^{\mathrm{i}} \rangle_{\mathrm{B}} = \mathscr{E}^{\mathrm{i}}\left(\hat{h}_{\|\mathrm{B}}^{\mathrm{i}(3)}, \hat{h}_{\|\mathrm{B}}^{\mathrm{i}(5)}\right). \tag{6.9}$$

The electron average BCL entropy production contains both classical and banana fluxes; we call it therefore the *banana–classical average entropy production*,

$$\langle \Sigma^{\mathrm{e}} \rangle_{\mathrm{BCL}} = \mathscr{E}^{\mathrm{e}}\left(\hat{h}_{\|\mathrm{CL}}^{\mathrm{e}(1)} + \hat{h}_{\|\mathrm{B}}^{\mathrm{e}(1)}, \hat{h}_{\|\mathrm{CL}}^{\mathrm{e}(3)} + \hat{h}_{\|\mathrm{B}}^{\mathrm{e}(3)}, \hat{h}_{\|\mathrm{CL}}^{\mathrm{e}(5)} + \hat{h}_{\|\mathrm{B}}^{\mathrm{e}(5)}\right). \tag{6.10}$$

The two effects cannot be separated, because they are non-linearly combined in the quadratic form. The average BCL entropy production will be analyzed in section 17.7.

This discussion throws a new light on the *significance of the Pfirsch–Schlüter effect*. Our basic lemma (5.11) can now be rewritten as follows, by using (5.10) and (6.5),

$$\left\langle h_{\|}^{\alpha(p)} h_{\|}^{\alpha(q)} \right\rangle - \hat{h}_{\|}^{\alpha(p)} \hat{h}_{\|}^{\alpha(q)} = \sigma_p \sigma_q (\mathscr{G} - 1) \mathscr{K}_{\alpha}^{2} \; g_{\rho}^{\alpha(p)} g_{\rho}^{\alpha(q)}$$

$$= \sigma_p \sigma_q \left\langle h_{\|\mathrm{PS}}^{\alpha(p)} h_{\|\mathrm{PS}}^{\alpha(q)} \right\rangle. \qquad [\mathrm{CONV}] \tag{6.11}$$

There exist non-vanishing fluctuations ($p = q$) *and correlations* ($p \neq q$) *of several parallel fluxes on each magnetic surface. These fluctuations and correlations are measured by the average of the product of the corresponding Pfirsch–Schlüter parallel fluxes.* Because of the special values of σ_p, eq. (5.7), we conclude that *only the privileged fluxes* $h_{\|}^{(1)}$, $h_{\|}^{\alpha(3)}$ *have non-vanishing fluctuations and correlations.* The non-privileged moments ($h_{\|}^{\alpha(5)}, \cdots$) have no fluctuations and are not correlated to any other moment.

The presence of these fluctuations is a specific source of entropy production. This is clearly seen when we rewrite eqs. (5.13)–(5.15) as

$$\langle \Sigma^{e} \rangle_{PS} \equiv \langle \Sigma^{e} \rangle_{(\parallel)} - \langle \Sigma^{e} \rangle_{BCL}$$

$$= \left\langle \mathscr{C}^{e}\left(h_{\parallel}^{(1)},\ h_{\parallel}^{e(3)},\ h_{\parallel}^{e(5)} \right) \right\rangle - \mathscr{C}^{e}\left(\hat{h}_{\parallel}^{(1)},\ \hat{h}_{\parallel}^{e(3)},\ \hat{h}_{\parallel}^{e(5)} \right)$$

$$\langle \Sigma^{i} \rangle_{PS} = \left\langle \mathscr{C}^{i}\left(h_{\parallel}^{i(3)},\ h_{\parallel}^{i(5)} \right) \right\rangle - \mathscr{C}^{i}\left(\hat{h}_{\parallel}^{i(3)},\ \hat{h}_{\parallel}^{i(5)} \right). \tag{6.12}$$

Thus, *the average PS entropy production is due to the fluctuations of the parallel fluxes on a magnetic surface.* In order to stress the importance of this result, we write it down explicitly for the simpler ion case,

$$\langle \Sigma^{i} \rangle_{PS} = c_{33}^{i}\left(\left\langle h_{\parallel}^{i(3)} h_{\parallel}^{i(3)} \right\rangle - \hat{h}_{\parallel}^{i(3)} \hat{h}_{\parallel}^{i(3)} \right)$$

$$+ c_{55}^{i}\left(\left\langle h_{\parallel}^{i(5)} h_{\parallel}^{i(5)} \right\rangle - \hat{h}_{\parallel}^{i(5)} \hat{h}_{\parallel}^{i(5)} \right)$$

$$+ 2c_{35}^{i}\left(\left\langle h_{\parallel}^{i(3)} h_{\parallel}^{i(5)} \right\rangle - \hat{h}_{\parallel}^{i(3)} \hat{h}_{\parallel}^{i(5)} \right)$$

$$= (\mathscr{G} - 1)\, c_{33}^{i}\, \mathscr{K}_{i}^{2}\left(g_{\rho}^{i(3)} \right)^{2} = c_{33}^{i} \left\langle h_{\parallel PS}^{i(3)}\, h_{\parallel PS}^{i(3)} \right\rangle. \tag{6.13}$$

The average PS entropy production provides a simple and universal measure of the parallel flux fluctuations in terms of the radial pressure and temperature profiles.

17.7. The average banana entropy production

The final purpose of the banana transport theory (chapters 15 and 16) was to provide a set of transport equations relating the average banana radial fluxes to the radial thermodynamic forces. This was achieved in eqs. (**15**.3.1)–(**15**.3.3). Our usual experience shows that whenever such a set of transport equations exists, its coefficients define a transport matrix; associated with the latter, there is a definite positive quadratic form that is identical to the entropy production in its "transport form" (remember the "recipe" quoted in section 17.1). It was shown in sections 17.5 and 17.6 that this identity also holds for *part* of the *average* entropy production: the average *classical* entropy production $\langle \Sigma^{\alpha} \rangle_{CL}$, [eq. (5.3) in relation to eqs. (**13**.2.3)–(**13**.2.5)] and the average *Pfirsch–Schlüter* entropy production $\langle \Sigma^{\alpha} \rangle_{PS}$, [eq. (6.1) in relation to eqs.

(**13**.3.6)–(**13**.3.8)]. There remains, however, a term in the average entropy production, $\langle \Sigma^{\alpha} \rangle_{\mathrm{BCL}}$, for which such a relation is not obvious. We now study this term.

Starting from eqs. (6.9), (6.10), combined with (6.7) and (4.9), a somewhat lengthy calculation, in which essential use is made of identities (A1.3) as well as of (**12**.10.5), leads to the result

$$\langle \Sigma^{\mathrm{e}} \rangle_{\mathrm{BCL}} = \mathscr{K}_{\mathrm{e}}^{-2} \mathscr{T}^{\mathrm{e}} \Big(-\big\langle h_{\rho}^{\mathrm{e}(1)} \big\rangle_{\mathrm{B}}, \big\langle h_{\rho}^{\mathrm{e}(3)} \big\rangle_{\mathrm{B}}, \big\langle h_{\rho}^{\mathrm{e}(5)} \big\rangle_{\mathrm{B}} \Big) + \sigma \big(\hat{g}_{\parallel}^{(1)A} \big)^2 + 2 \hat{g}_{\parallel}^{(1)A} \hat{h}_{\parallel \mathrm{B}}^{(1)}, \tag{7.1}$$

$$\langle \Sigma^{\mathrm{i}} \rangle_{\mathrm{B}} = \mathscr{K}_{\mathrm{i}}^{-2} \mathscr{T}^{\mathrm{i}} \Big(\big\langle h_{\rho}^{\mathrm{i}(3)} \big\rangle_{\mathrm{B}}, \big\langle h_{\rho}^{\mathrm{i}(5)} \big\rangle_{\mathrm{B}} \Big), \tag{7.2}$$

Consider first the ion average banana entropy production. It appears as a positive quadratic form in the average radial banana *fluxes*, whose coefficients are the parallel quasi-transport coefficients. Although this form ensures the basic thermodynamic properties of the entropy production (in particular, its positive sign), its structure is rather queer. Indeed, the entropy production is usually expressed either as a quadratic form in the *fluxes*, associated with the *collision matrix* (see eq. 2.13), or, alternatively, as a quadratic form in the *forces*, associated with the *quasi-transport matrix* (see eq. 2.9). In eq. (7.2), the average radial fluxes appear to play the role of thermodynamic forces. The explanation of this inversion is suggested by eq. (2.12): the quasi-transport form of the parallel entropy production contains the generalized stress terms, which are added to the source terms, and therefore play the role of additional "quasi-thermodynamic forces". But the average radial fluxes are proportional to the average parallel stresses: indeed, using eqs. (**15**.1.4) and (**15**.1.5), we may write

$$\langle \Sigma^{\mathrm{i}} \rangle_{\mathrm{B}} = \mathscr{T}^{\mathrm{i}} \left(\left\langle \frac{B}{\mathscr{B}_0} \bar{g}_{\parallel}^{\mathrm{i}(3)} \right\rangle, \left\langle \frac{B}{\mathscr{B}_0} \bar{g}_{\parallel}^{\mathrm{i}(5)} \right\rangle \right). \tag{7.3}$$

The electron average entropy production is more complicated, because of the special role played by the electric field $\hat{g}_{\parallel}^{(1)A}$. Consider the second term in (7.1): it may be called the *purely classical average parallel entropy production*. Indeed, σ is the classical ("Spitzer") parallel electrical conductivity. Recalling eq. (6.7) for the average classical parallel current (which is nothing other than the Ohm law!), this term can be written in the form

$$\sigma \big(\hat{g}_{\parallel}^{(1)A} \big)^2 = \hat{h}_{\parallel \mathrm{CL}}^{(1)} \hat{g}_{\parallel}^{(1)A},$$

which is a typical thermodynamic form of the entropy production, associated with the *Ohmic heating effect*. (Note that this expression involves a product of *average* quantities: all correlation effects have been isolated in the PS contribution). This is, however, not the complete Ohmic heating entropy production: the average parallel electric current also contains a banana contribution. The corresponding contribution to the entropy production appears in the third term of (7.1); but the latter contains a factor 2! Hence, the natural decomposition of the average BCL entropy production is

$$\langle \Sigma^e \rangle_{BCL} = \langle \Sigma^e \rangle_{Ohm} + \langle \Sigma^e \rangle_B . \tag{7.4}$$

The first term is the complete average entropy production associated with the *Ohmic heating*, expressed in terms of the *total* average parallel electric current, $\hat{h}_{\|}^{(1)} = \hat{h}_{\|CL}^{(1)} + \hat{h}_{\|B}^{(1)}$: *

$$\langle \Sigma^e \rangle_{Ohm} = \hat{h}_{\|}^{(1)} \hat{g}_{\|}^{(1)A} . \tag{7.5}$$

This term will be further discussed in section 17.8, where it will be shown that its physical meaning is not as straightforward as suggested by its simple form.

The remaining terms in (7.1) are grouped together into the electron *average banana entropy production*,

$$\langle \Sigma^e \rangle_B = \mathscr{K}_e^{-2} \mathscr{T}^e \left(- \left\langle h_\rho^{e(1)} \right\rangle_B , \left\langle h_\rho^{e(3)} \right\rangle_B , \left\langle h_\rho^{e(5)} \right\rangle_B \right) + \hat{h}_{\|B}^{(1)} \hat{g}_{\|}^{(1)A} . \tag{7.6}$$

Note that the addition of a term involving the banana parallel current to the entropy production to the terms containing the radial fluxes is not so surprising. We have seen in chapter 15, especially in the discussion in section **15.1**, that in the banana transport theory the parallel electric current must necessarily be adjoined to the radial fluxes in order to have a consistent and complete set of transport equations.

The next step consists of the transformation of the average banana entropy production (7.2), (7.6) into quadratic forms in the thermodynamic forces, i.e. into a "transport form" analogous to (5.3) or (6.1). There is no technical difficulty in doing this. The banana transport equations (**15**.3.1)–(**15**.3.3) are simply substituted into the quadratic forms (7.2) and (7.6), the result is

* We recall that the radial component of the average electric current is zero (ambipolar diffusion) and that the perp–tangential component is a non-dissipative current producing no entropy. Thus, eq. (7.5) can, indeed, be interpreted as the total average entropy production associated with the Ohmic heating, *provided* that the laws of irreversible thermodynamics are applicable to the banana regime, a point that is not obvious!

necessarily a set of two quadratic forms, whose coefficients are grouped into a matrix $\boldsymbol{P}$ for the electrons and a matrix $\boldsymbol{N}$ for the ions,

$$\boldsymbol{P}=\begin{pmatrix} p_{11}^{ee} & p_{13}^{ee} & p_{13}^{ei} & p_{1E}^{e} \\ p_{31}^{ee} & p_{33}^{ee} & p_{33}^{ei} & p_{3E}^{e} \\ p_{31}^{ie} & p_{33}^{ie} & p_{33}^{ii} & p_{3E}^{i} \\ p_{E1}^{e} & p_{E3}^{e} & p_{E3}^{i} & p_{EE} \end{pmatrix}, \tag{7.7}$$

$$\boldsymbol{N}=\begin{pmatrix} n_{11}^{ee} & n_{13}^{ee} & n_{13}^{ei} & n_{1E}^{e} \\ n_{31}^{ee} & n_{33}^{ee} & n_{33}^{ei} & n_{3E}^{e} \\ n_{31}^{ie} & n_{33}^{ie} & n_{33}^{ii} & n_{3E}^{i} \\ n_{E1}^{e} & n_{E3}^{e} & n_{E3}^{i} & n_{EE} \end{pmatrix}. \tag{7.8}$$

Both matrices are *symmetric*, thus

$$\begin{aligned} &p_{13}^{ee}=p_{31}^{ee}, \qquad p_{13}^{ei}=p_{31}^{ie}, \qquad p_{33}^{ei}=p_{33}^{ie}, \\ &p_{1E}^{e}=p_{E1}^{e}, \qquad p_{3E}^{e}=p_{E3}^{e}, \qquad p_{3E}^{i}=p_{E3}^{i}, \end{aligned} \tag{7.9}$$

with similar relations for the $n_{pq}^{\alpha\beta}$. The average electron and ion banana entropy productions are the quadratic forms associated with these matrices,

$$\begin{aligned} \langle \Sigma^{e} \rangle_{B} &= \mathscr{P}\left(g_{\rho}^{(1)P},\ g_{\rho}^{e(3)},\ g_{\rho}^{i(3)},\ \hat{g}_{\parallel}^{(1)A} \right), \\ \langle \Sigma^{i} \rangle_{B} &= \mathscr{N}\left(g_{\rho}^{(1)P},\ g_{\rho}^{e(3)},\ g_{\rho}^{i(3)},\ \hat{g}_{\parallel}^{(1)A} \right), \end{aligned} \tag{7.10}$$

where we used the compact notation for the quadratic forms, as defined in (3.1). To avoid misunderstandings, we write one of these formulae explicitly:

$$\begin{aligned} \langle \Sigma^{e} \rangle_{B} = {} & p_{11}^{ee}\left(g_{\rho}^{(1)P} \right)^{2} + p_{33}^{ee}\left(g_{\rho}^{e(3)} \right)^{2} + p_{33}^{ii}\left(g_{\rho}^{i(3)} \right)^{2} + p_{EE}\left(\hat{g}_{\parallel}^{(1)A} \right)^{2} \\ & + 2p_{13}^{ee}\, g_{\rho}^{(1)P} g_{\rho}^{e(3)} + 2p_{13}^{ei}\, g_{\rho}^{(1)P} g_{\rho}^{i(3)} + 2p_{1E}^{e}\, g_{\rho}^{(1)P} \hat{g}_{\parallel}^{(1)A} \\ & + 2p_{33}^{ei}\, g_{\rho}^{e(3)} g_{\rho}^{i(3)} + 2p_{3E}^{e}\, g_{\rho}^{e(3)} \hat{g}_{\parallel}^{(1)A} + 2p_{3E}^{i}\, g_{\rho}^{i(3)} \hat{g}_{\parallel}^{(1)A}. \end{aligned}$$

Before commenting these rather unusual formulae, we list the coefficients of the two matrices $\boldsymbol{P}$ and $\boldsymbol{N}$, which come out of the substitution mentioned above.

A. Electronic coefficients

Aa. *Diagonal "pure diffusive" coefficients*:

$$p_{qq}^{ee} = \mathscr{K}_e^2 \varphi^2 \mathscr{T}^e\left(-l_{1q}^{ee},\ l_{3q}^{ee},\ l_{5q}^{ee}\right), \quad q = 1, 3, 5. \tag{7.11}$$

These coefficients are simply expressed in terms of the parallel quasi-transport quadratic form (3.8) in which the variables x_1, x_3, x_5 are replaced by the banana transport coefficients $-l_{1q}^{ee}$, l_{3q}^{ee}, l_{5q}^{ee}, respectively. It is convenient to extend definition (7.11) to $q = 5$, thus defining a "quasi-entropic" coefficient p_{55}^{ee} [which does not appear in the matrix $\boldsymbol{P}$ of (7.7)].

Ab. *Non-diagonal "pure diffusive" coefficients*:

$$p_{qr}^{ee} = \mathscr{K}_e^2 \varphi^2 \mathscr{T}^e\left(-l_{1q}^{ee},\ l_{3q}^{ee},\ l_{5q}^{ee}\middle| -l_{1r}^{ee},\ l_{3r}^{ee},\ l_{5r}^{ee}\right),$$

$$q,\ r = 1, 3, 5. \tag{7.12}$$

These coefficients are expressed as bilinear forms in the banana transport coefficients, associated with the parallel quasi-transport matrix, as defined in (3.10).

Ac. *"Ionic coefficients" in the electron entropy production*:

$$p_{33}^{ii} = \mathscr{K}_e^2 (aA)^2 \varphi^2 \mathscr{T}^e\left(-l_{13}^{ei},\ l_{33}^{ei},\ l_{53}^{ei}\right), \tag{7.13}$$

$$p_{q3}^{ei} = \mathscr{K}_e^2 aA\, \varphi^2 \mathscr{T}^e\left(-l_{1q}^{ee},\ l_{3q}^{ee},\ l_{5q}^{ee}\middle| -l_{13}^{ei},\ l_{33}^{ei},\ l_{53}^{ei}\right). \tag{7.14}$$

An important property of these coefficients will be derived in section 17.8.

Ad. *Electric coefficients*.

$$p_{EE} = \varphi^2 \mathscr{T}^e\left(-l_{1E}^{e},\ l_{3E}^{e},\ l_{5E}^{e}\right) + \varphi\, l_{EE}, \tag{7.15}$$

$$p_{qE}^{e} = \mathscr{K}_e \varphi^2 \mathscr{T}^e\left(-l_{1q}^{ee},\ l_{3q}^{ee},\ l_{5q}^{ee}\middle| -l_{1E}^{e},\ l_{3E}^{e},\ l_{5E}^{e}\right) + \mathscr{K}_e \varphi\, l_{Eq}^{e}, \tag{7.16}$$

$$p_{3E}^{i} = \mathscr{K}_e aA\, \varphi^2 \mathscr{T}^e\left(-l_{13}^{ei},\ l_{33}^{ei},\ l_{53}^{ei}\middle| -l_{1E}^{e},\ l_{3E}^{e},\ l_{5E}^{e}\right) + \mathscr{K}_e aA\, \varphi\, l_{E3}^{i}. \tag{7.17}$$

B. Ionic coefficients

Ba. *Diagonal "pure diffusive" coefficients:*

$$n_{33}^{\mathrm{u}} = \mathscr{K}_{\mathrm{i}}^{2}\varphi^{2}\,\mathscr{T}^{\mathrm{i}}\left(l_{33}^{\mathrm{ii}},\, l_{53}^{\mathrm{u}}\right). \tag{7.18}$$

Bb. *Electronic coefficients in the ion entropy production:*

$$n_{pp}^{\mathrm{ee}} = \mathscr{K}_{\mathrm{e}}^{2}(aA)^{-2}\varphi^{2}\,\mathscr{T}^{\mathrm{i}}\left(l_{3p}^{\mathrm{ie}},\, l_{5p}^{\mathrm{ie}}\right), \tag{7.19}$$

$$n_{pq}^{\mathrm{ee}} = \mathscr{K}_{\mathrm{e}}^{2}(aA)^{-2}\varphi^{2}\,\mathscr{T}^{\mathrm{i}}\left(l_{3p}^{\mathrm{ie}},\, l_{5p}^{\mathrm{ie}} \middle| l_{3q}^{\mathrm{ie}},\, l_{5q}^{\mathrm{ie}}\right). \tag{7.20}$$

Bc. *Mixed diffusive coefficients:*

$$n_{3p}^{\mathrm{ie}} = \mathscr{K}_{\mathrm{e}}^{2}a^{-1}\varphi^{2}\,\mathscr{T}^{\mathrm{i}}\left(l_{33}^{\mathrm{ie}},\, l_{53}^{\mathrm{ie}} \middle| l_{3p}^{\mathrm{ee}},\, l_{5p}^{\mathrm{ee}}\right). \tag{7.21}$$

Bd. *Electrical coefficients:*

$$n_{\mathrm{EE}} = (aA)^{-2}\varphi^{2}\,\mathscr{T}^{\mathrm{i}}\left(l_{3\mathrm{E}}^{\mathrm{i}},\, l_{5\mathrm{E}}^{\mathrm{i}}\right), \tag{7.22}$$

$$n_{p\mathrm{E}}^{\mathrm{e}} = \mathscr{K}_{\mathrm{e}}(aA)^{-2}\varphi^{2}\,\mathscr{T}^{\mathrm{i}}\left(l_{3p}^{\mathrm{ie}},\, l_{5p}^{\mathrm{ie}} \middle| l_{3\mathrm{E}}^{\mathrm{i}},\, l_{5\mathrm{E}}^{\mathrm{i}}\right), \tag{7.23}$$

$$n_{3\mathrm{E}}^{\mathrm{i}} = \mathscr{K}_{\mathrm{e}}a^{-1}\varphi^{2}\,\mathscr{T}^{\mathrm{i}}\left(l_{33}^{\mathrm{u}},\, l_{53}^{\mathrm{u}} \middle| l_{3\mathrm{E}}^{\mathrm{i}},\, l_{5\mathrm{E}}^{\mathrm{i}}\right). \tag{7.24}$$

17.8. Transport coefficients and entropic coefficients

The result (eq. 7.10) for the average banana entropy production is quite unusual for various reasons, which we now discuss.

– We recall that the usual transport form of the entropy production is the *quadratic form in the thermodynamic forces associated with the matrix of the corresponding transport coefficients.* This structure was found not only in chap. 6, but also for the *average* classical and PS fluxes (eqs. 5.4, 6.1). Whenever this form is valid, the entropy production derives from a fundamental thermodynamic form (eqs. 5.5, 6.2) that is a simple bilinear form in the forces and their conjugate fluxes.

In order to illustrate the emergence of this structure in a situation analogous to the present one, we consider the passage from the (unaveraged) parallel ion entropy production (2.13) to the transport form (2.12). Starting from

$$\Sigma_{(\|)}^{\mathrm{i}} = \mathscr{C}^{\mathrm{i}}\left(h_{\|}^{\mathrm{i}(3)},\, h_{\|}^{\mathrm{i}(5)}\right)$$

we substitute the quasi-transport equations (**12**.6.11) and find a quadratic form, provisionally denoted by

$$\Sigma^{\mathrm{i}}_{(\|)} = \mathscr{S}^{\mathrm{i}}\left(g_{\|}^{\mathrm{i}(3)} + \bar{g}_{\|}^{\mathrm{i}(3)},\ \bar{g}_{\|}^{\mathrm{i}(5)}\right). \tag{8.1}$$

Its coefficients are

$$s^{\mathrm{ii}}_{33} = \mathscr{C}^{\mathrm{i}}(\kappa^{\mathrm{i}},\ \delta^{\mathrm{i}}), \qquad s^{\mathrm{ii}}_{55} = \mathscr{C}^{\mathrm{i}}(\delta^{\mathrm{i}},\ \epsilon^{\mathrm{i}}),$$

$$s^{\mathrm{u}}_{35} = \mathscr{C}^{\mathrm{i}}\left(\kappa^{\mathrm{i}},\ \delta^{\mathrm{i}} \mid \delta^{\mathrm{i}},\ \epsilon^{\mathrm{i}}\right). \tag{8.2}$$

These expressions are very similar to the expressions of the coefficients $n^{\alpha\beta}_{pq}$ of eqs. (7.18)–(7.24); however, the identities (A1.9), (A1.10) of Appendix 17A.1 enable us to identify the coefficients as

$$s^{\mathrm{ii}}_{33} = \kappa^{\mathrm{i}}, \qquad s^{\mathrm{u}}_{55} = \epsilon^{\mathrm{i}}, \qquad s^{\mathrm{ii}}_{35} = \delta^{\mathrm{i}}. \tag{8.3}$$

Hence, the form (8.1) is identical to the transport form (2.12). The crux in this derivation is that in (8.2) the quadratic form $\mathscr{C}^{\mathrm{i}}(x_3,\ x_5)$ is evaluated for the values, say, $x_3 = \kappa^{\mathrm{i}},\ x_5 = \delta^{\mathrm{i}}$ that are elements of the matrix $\boldsymbol{T}^{\mathrm{i}}$, inverse to the collision matrix $\boldsymbol{C}^{\mathrm{i}}$ (see section 17.3). Identification (8.3) results from this intimate relationship between the coefficients of the quadratic (or bilinear) forms and the values given to their variables.

It is precisely this relationship which is *absent* in the forms (7.11)–(7.24). Thus, one *cannot* make identifications such as

$$p^{\mathrm{ee}}_{11} \leftrightarrow l^{\mathrm{ee}}_{11}, \qquad p^{\mathrm{ee}}_{13} \leftrightarrow l^{\mathrm{ee}}_{13}, \qquad p_{\mathrm{EE}} \leftrightarrow l_{\mathrm{EE}}, \qquad n^{\mathrm{ii}}_{33} \leftrightarrow l^{\mathrm{ii}}_{33}, \quad \text{etc.}$$

The obvious conclusion is *the divorce between the banana transport equations and the average banana entropy production*. As a result, *the average banana entropy production does not derive from a thermodynamic form*, which would have been expected,

$$\langle \Sigma^{\mathrm{e}} \rangle_{\mathrm{B}} \neq \left\langle h^{\mathrm{e}(1)}_{\rho} \right\rangle_{\mathrm{B}} g^{(1)P}_{\rho} + \left\langle h^{\mathrm{e}(3)}_{\rho} \right\rangle_{\mathrm{B}} g^{\mathrm{e}(3)}_{\rho} + \left\langle \hat{h}^{(1)}_{\|} \right\rangle_{\mathrm{B}} \hat{g}^{(1)A}_{\|},$$

$$\langle \Sigma^{\mathrm{i}} \rangle_{\mathrm{B}} \neq \left\langle h^{\mathrm{i}(3)}_{\rho} \right\rangle_{\mathrm{B}} g^{\mathrm{i}(3)}_{\rho}. \tag{8.4}$$

The coefficients of the quadratic forms $\mathscr{P}$ ($\mathscr{N}$) of eq. (7.10) defining the average banana electron (ion) entropy production will be called the *electron (ion) entropic coefficients*. For the classical and PS case, the entropic coefficients are identically equal to the transport coefficients, but for the banana

case this identity breaks down. The entropic coefficients are combinations of the parallel quasi-transport coefficients and of the banana transport coefficients. It should also be noted, in order to avoid confusions, that the meanings of the subscripts and superscripts are not the same in the transport matrix and in the entropy production matrix. Thus, for instance, l_{33}^{ee} denotes the coefficient of the force $g_\rho^{e(3)}$ in the expression of the banana flux $\langle h_\rho^{e(3)} \rangle_B$; on the other hand, m_{33}^{ee} (n_{33}^{ee}) is the coefficient of $(g_\rho^{e(3)})^2$ in the electron (ion) banana entropy production.

A qualitative difference between the transport matrix and the entropy production matrices is due precisely to this different meaning of the coefficients. The entropic matrices $p_{qr}^{\alpha\beta}$ and $n_{qr}^{\alpha\beta}$ are *symmetric*, whereas the transport matrix $l_{qr}^{\alpha\beta}$ is not. We recal, indeed, that the electric transport coefficients are antisymmetric: $l_{qE}^{\alpha} = -l_{Eq}^{\alpha}$ (see eq. **15**.3.8) whereas the corresponding entropic coefficients are symmetric: $p_{Eq}^{\alpha} = p_{qE}^{\alpha}$, $n_{qE}^{\alpha} = n_{Eq}^{\alpha}$ (see eq. 7.9).

– Next, we note that *both the electron and the ion entropy productions depend on all four thermodynamic forces*. We no longer see a clear-cut separation between "electronic forces" producing the "electron entropy" and "ionic forces" producing the "ion entropy" as in (5.4), (6.1); instead, everything is coupled to everything.

A trace of such a separation can, however, be seen in the structure of the entropic coefficients. Thus, the electronic entropic coefficients $p_{jk}^{\alpha\beta}$ are exclusively combinations of the transport coefficients $l_{pq}^{e\alpha}$ ($p = 1, 3, 5$; $\alpha = e, i$; $q = 1, 3, 5, E$) entering the electron banana fluxes, whereas the ion entropic coefficients contain exclusively transport coefficients of the form $l_{pq}^{i\alpha}$ ($p = 3, 5$; $\alpha = e, i$; $q = 1, 3, E$). But even such a separation is illusory, because the mixed electron–ion transport coefficients are actually related to the electron coefficients by relations (**15**.2.11) and (**15**.2.16).

– The important relations just mentioned, involving the mixed electron–ion coefficients, as well as relations (**15**.2.12) and (**15**.2.19) involving the electrical coefficients induce a set of relations between the entropic coefficients as well. We now derive these relations.

(a) *Electronic entropic coefficients*. The expression of the diagonal ion coefficient p_{33}^{ii}, (7.13) is combined with (**15**.2.11) to yield

$$\begin{aligned} p_{33}^{ii} &= \mathscr{K}_e^2 (aA)^2 \varphi^2 \mathscr{T}^e \left(-m_3^i l_{11}^{ee},\ m_3^i l_{13}^{ee},\ m_3^i l_{15}^{ee} \right) \\ &= \mathscr{K}_e^2 (aA)^2 \varphi^2 \left(m_3^i \right)^2 \mathscr{T}^e \left(-l_{11}^{ee},\ l_{13}^{ee},\ l_{15}^{ee} \right) \\ &= (aA)^2 \left(m_3^i \right)^2 p_{11}^{ee}. \end{aligned} \tag{8.5}$$

Similarly, we find from eqs. (7.14), (7.17) and (**15**.2.11)

$$p_{q3}^{ei} = aA\, m_3^i p_{1q}^{ee},$$

$$p_{3E}^i = aA\, m_3^i p_{1E}^e. \tag{8.6}$$

On the other hand a slightly longer calculation shows that, upon combining eqs. (7.15), (7.16) and (**15**.2.12), (**15**.2.19) and (**15**.2.21), we find

$$p_{qE}^e = \sigma\left(p_{q1}^{ee} - \varphi l_{q1}^{ee}\right) - \alpha\left(p_{q3}^{ee} - \varphi l_{q3}^{ee}\right) - \gamma\left(p_{q5}^{ee} - \varphi l_{q5}^{ee}\right),$$

$$p_{FE} = \sigma p_{1F}^e - \alpha p_{3F}^e - \gamma p_{5E}^e. \tag{8.7}$$

(b) *Ionic entropic coefficients*. Relations (**15**.2.16) induce a lot of relationships between the ionic entropic coefficients. Typically, consider the coefficient n_{11}^{ee}, eq. (7.19),

$$n_{11}^{ee} = \mathscr{K}_e^2 (aA)^{-2} \varphi^2 \mathscr{T}^i\left(m_3^i l_{11}^{ee},\ m_5^i l_{11}^{ee}\right)$$

$$= \mathscr{K}_e^2 (aA)^{-2} \varphi^2 (l_{11}^{ee})^2 \mathscr{T}^i\left(m_3^i,\ m_5^i\right). \tag{8.8}$$

It turns out that all the ionic entropic coefficients, except n_{33}^{ii}, can be similarly expressed in terms of the electronic banana transport coefficients.

For the reader's convenience, all the electronic and ionic entropic coefficients are collected in table 8.1, by making full use of all the relations existing among them.

– We now study the properties of the quadratic forms $\mathscr{P}$ and $\mathscr{N}$, and in particular, their *sign*.

(A) *Electron average banana entropy production*. We should not be surprised to find that the electron average banana entropy production is positive, However, in view of the special role played by the parallel electric field, we should check at least some of the positivity criteria.

It is first noted that all the pure diagonal diffusive coefficients are manifestly positive. Indeed, they are of the form

$$p_{rr}^{ee} = \mathscr{K}_e^2 \varphi^2 \mathscr{T}^e\left(-l_{1r}^{ee},\ l_{3r}^{ee},\ l_{5r}^{ee}\right). \tag{8.9}$$

We know that $\mathscr{T}^e(x_1, x_3, x_5)$ is a positive definite quadratic form (see eq. 3.21) which can only vanish when $x_1 = x_3 = x_5 = 0$. Hence, $p_{rr}^{ee} > 0$, $r = 1, 3$. From (8.5) we conclude that $p_{33}^{ii} > 0$ as well. The positivity of the electric coefficient p_{EE} is less obvious, but a calculation given in Appendix 17A.2 shows that $p_{EE} > 0$.

Table 8.1
Electronic entropic coefficients (21M)

$$p^{ee}_{qr} = \mathscr{K}_e^2 \varphi^2 \mathscr{T}^e \left(-l^{ee}_{1q}, l^{ee}_{3q}, l^{ee}_{5q} \right| - l^{ee}_{1r}, l^{ee}_{3r}, l^{ee}_{5r} \Big), \quad q, r = 1, 3, 5$$

$$p^{ei}_{q3} = aAm^i_3 \, p^{ee}_{1q}, \quad q = 1, 3$$

$$p^{ii}_{33} = (aA)^2 \left(m^i_3 \right)^2 p^{ee}_{11}$$

$$p^e_{qE} = \sigma \left(p^{ee}_{q1} - \varphi l^{ee}_{q1} \right) - \alpha \left(p^{ee}_{q3} - \varphi l^{ee}_{q3} \right) - \gamma \left(p^{ee}_{q5} - \varphi l^{ee}_{q5} \right), \quad q = 1, 3, 5$$

$$p^i_{3E} = aAm^i_3 \, p^e_{1E}$$

$$p_{EE} = \sigma p^e_{1E} - \alpha p^v_{3E} - \gamma p^e_{5E}$$

Next, we consider the 4×4 determinant of the matrix $\boldsymbol{P}$. Taking account of relations (8.5), (8.6), we find

$$\| \boldsymbol{P} \| = \begin{vmatrix} p^{ee}_{11} & p^{ee}_{13} & aAm^i_3 p^{ee}_{11} & p^e_{1F} \\ p^{ee}_{13} & p^{ee}_{33} & aAm^i_3 p^{ee}_{13} & p^e_{3E} \\ aAm^i_3 p^{ee}_{11} & aAm^i_3 p^{ee}_{33} & \left(aAm^i_3 \right)^2 p^{ee}_{11} & aAm^i_3 p^e_{1E} \\ p^e_{1F} & p^e_{3E} & aAm^i_3 p^e_{1E} & p_{EE} \end{vmatrix}.$$

This determinant vanishes identically, because it has two columns that are proportional to each other. Clearly, several of its minors are also identically zero.

The conclusion is that the average electron banana entropy production is a *semi-definite positive quadratic form* (rather than a *definite* one) in the thermodynamic forces. This means that the electron banana entropy production vanishes for some non-zero values of the thermodynamic forces. *

* One can easily convince oneself that this property is *universal*, i.e. independent of the truncation level. Indeed, from the argument in section **15**.2 it can be seen that the *form* of eq. (**15**.2 6) is universal (although the *values* of the factors m^i_3, p_i depend on the truncation level) This implies the universal validity of proportionality (**15**.2.11), from which follows (8.6).

It is easy to find the linear combination of the forces which annuls the entropy production. A short calculation yields the following solution:

$$g_\rho^{e(3)} = 0, \qquad \hat{g}_\parallel^{(1)A} = 0, \qquad g_\rho^{i(3)} = -\frac{1}{aAm_3^i}\, g_\rho^{(1)P}. \tag{8.10}$$

It is, indeed, easily checked that

$$\langle \Sigma^e \rangle_B = \mathscr{P}\left(g_\rho^{(1)P},\ 0,\ -\left(aAm_3^i\right) g_\rho^{(1)P},\ 0\right) = 0. \tag{8.11}$$

Thus, the possibility exists of a *diffusive motion without dissipation*. How can this be imagined in reality?

First it should be realized that only the banana entropy production, and not the total one, is affected by this phenomenon. But in a long mean free path regime, the banana contribution is dominant, as we know. Next, it should be understood that condition (8.10) cannot be realized throughout the plasma, because it implies that the ion temperature gradient be everywhere opposite to the pressure gradient. In a tokamak, the temperatures and the pressure are always globally decreasing from the magnetic axis outward. Nevertheless, the condition could be realized *locally*, on some magnetic surface, if the ion temperature profile happens to have a "bump" as in fig. 8.1.

(B) *Ion average banana entropy production*. Here it is also easily checked that the diagonal coefficients are manifestly positive (being expressed as values of a positive definite quadratic form). The same phenomenon as for the electrons appears here, but is even stronger. Indeed, the determinant of matrix

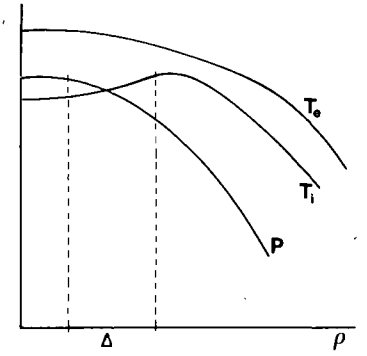

Fig. 8.1. The electron banana entropy production could vanish on some magnetic surface in the region Δ

Table 8.2
Ionic entropic coefficients (21M).

$n_{33}^{ii} = \mathscr{K}_e^2 \varphi^2 \mathscr{T}^i\left(l_{33}^{ii}, l_{53}^{ii}\right)$

$n_{qr}^{ee} = \mathscr{K}_e^2 (aA)^{-2} \varphi^2 C l_{1q}^{ee} l_{1r}^{ee}, \qquad q, r = 1, 3$

$n_{q3}^{ei} = \mathscr{K}_e^2 a^{-1} \varphi^2 c l_{1q}^{ee}, \quad q = 1, 3$

$n_{qF}^{e} = \mathscr{K}_e (aA)^{-2} \varphi^2 C l_{1E}^{e} l_{1q}^{e}, \quad q = 1, 3$

$n_{3E}^{i} = \mathscr{K}_e a^{-1} \varphi^2 c l_{1E}^{e}$

$n_{FE} = (aA)^{-2} \varphi^2 C \left(l_{1E}^{e}\right)^2$

Symbols·

$C \equiv \mathscr{T}^i\left(m_3^i, m_5^i\right)$

$c \equiv \mathscr{T}^i\left(m_3^i, m_5^i \middle| l_{33}^{ii}, l_{53}^{ii}\right)$

N can be written by substituting the relations of table 8.2 into (7.8) and taking out all common factors in the rows or columns:

$$\| \boldsymbol{N} \| = \mathscr{K}_e^6 a^{-6} A^{-4} l_{11}^{ee2} l_{13}^{ee2} l_{1E}^{e2} \begin{vmatrix} C & C & c & C \\ C & C & c & C \\ c & c & D & c \\ C & C & c & C \end{vmatrix} = 0, \tag{8.12}$$

where we used the abbreviations

$$C = \mathscr{T}^i\left(m_3^i, m_5^i\right),$$

$$c = \mathscr{T}^i\left(m_3^i, m_5^i \middle| l_{33}^{ii}, l_{53}^{ii}\right),$$

$$D = \mathscr{T}^i\left(l_{33}^{ii}, l_{53}^{ii}\right). \tag{8.13}$$

This determinant has three identical columns, hence is more strongly degenerate than its electron analog. The ion average banana entropy production is thus also a *positive semi-definite quadratic form*. We can now easily find *two*

independent non-trivial solutions annulling the ion banana entropy production,

$$g_\rho^{(1)P} = -\frac{l_{13}^{ee}}{l_{11}^{ee}}\, g_\rho^{e(3)}, \qquad g_\rho^{i(3)} = \hat{g}_\parallel^{(1)A} = 0,$$

$$g_\rho^{(1)P} = -\frac{l_{1E}^{e}}{\mathscr{K}_e l_{11}^{ee}}\, \hat{g}_\parallel^{(1)A}, \qquad g_\rho^{i(3)} = g_\rho^{e(3)} = 0. \tag{8.14}$$

The same comments apply here as for the electron case. Note, however, that l_{13}^{ee} is negative, hence the first condition can be more easily satisfied than (8.10).

– It is interesting to get some feeling of the *dependence of the entropic coefficients on the neoclassical factor* φ. For *small values of* φ, we know that the banana transport coefficients approach a limiting *linear* function $\varphi l_{pq}^{\alpha\beta}(0)$, as shown in table **15**.4.1. On the contrary, all the entropic coefficients [except the electrical coefficients $p_{q\mathrm{E}}^{\alpha}$, p_{EE}, eqs. (7.15)–(7.17)] start *quadratically* from the origin. Hence, for small values of φ, the difference between transport coefficients and entropic coefficients is quite clear cut (fig. 8.2).

If we now consider the opposite limit of *large values of* φ, we know that the banana transport coefficients tend toward finite asymptotic values, given in table **15**.4.2. Consider now the corresponding limiting values of the diffusive entropic coefficients, say, p_{11}^{ee},

$$p_{11}^{ee}(\varphi) = \mathscr{K}_e^2 \mathscr{T}^e(-\varphi l_{11}^{ee},\ \varphi l_{13}^{ee},\ \varphi l_{15}^{ee})$$

$$\underset{\varphi\to\infty}{\to} \mathscr{K}_e^2 \mathscr{T}^e(-c_{11}^{e},\ -c_{13}^{e},\ -c_{15}^{e}).$$

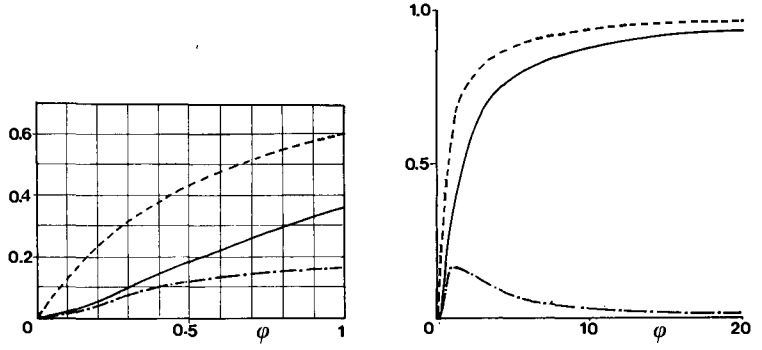

Fig. 8.2. Transport coefficient and entropic coefficients: - - - - - -, the transport coefficient $\varphi l_{11}^{ee}(\varphi)$; ———, the electronic entropic coefficient $\mathscr{K}_e^{-2}\varphi p_{11}^{ee}(\varphi)$; ·—·—, the ionic entropic coefficient $\mathscr{K}_e^{-2}\varphi n_{11}^{ee}(\varphi)$.

We note that c^{e}_{pq} are the elements of the inverse parallel quasi-transport matrix; thus theorem (A1.10) is applicable and we find

$$\lim_{\varphi\to\infty} p^{\mathrm{ee}}_{11}(\varphi) = \mathscr{K}_{\mathrm{e}}^2 c^{\mathrm{e}}_{11}.$$

As another example, we calculate the asymptotic value of the diagonal electrical entropic coefficient, using (7.15) and table **15**.4.2,

$$\lim_{\varphi\to\infty} p_{\mathrm{EE}}(\varphi) = \lim_{\varphi\to\infty} \mathscr{T}^{\mathrm{e}}(-\varphi l^{\mathrm{e}}_{1\mathrm{E}}, \varphi l^{\mathrm{e}}_{3\mathrm{E}}, \varphi l^{\mathrm{e}}_{5\mathrm{E}}) + \varphi l_{\mathrm{EE}}$$

$$= \mathscr{T}^{\mathrm{e}}(-1, 0, 0) - \sigma = \sigma - \sigma = 0.$$

Similar calculations lead to the following asymptotic forms of the entropic matrices:

$$\lim_{\varphi\to\infty} \mathscr{P}(\varphi) = \mathscr{K}_{\mathrm{e}}^2 \begin{pmatrix} c^{\mathrm{e}}_{11} & -c^{\mathrm{e}}_{13} & 0 & 0 \\ -c^{\mathrm{e}}_{13} & c^{\mathrm{e}}_{33} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\lim_{\varphi\to\infty} \mathscr{N}(\varphi) = \mathscr{K}_{\mathrm{i}}^2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & c^{\mathrm{i}}_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{8.15}$$

As a result, the average banana entropy production reduces, in this limit, to

$$\langle \Sigma^{\mathrm{e}} \rangle^{\mathrm{as}}_{\mathrm{B}} = \mathscr{K}_{\mathrm{e}}^2 \left[c^{\mathrm{e}}_{11} \left(g^{(1)P}_{\rho} \right)^2 + c^{\mathrm{e}}_{33} \left(g^{\mathrm{e}(3)}_{\rho} \right)^2 - 2 c^{\mathrm{e}}_{13} g^{(1)P}_{\rho} g^{\mathrm{e}(3)}_{\rho} \right],$$

$$\langle \Sigma^{\mathrm{i}} \rangle^{\mathrm{as}}_{\mathrm{B}} = \mathscr{K}_{\mathrm{i}}^2 c^{\mathrm{i}}_{33} \left(g^{\mathrm{i}(3)}_{\rho} \right)^2. \tag{8.16}$$

These are analogous to the "classical" forms (5.3). Moreover, we note that the entropic coefficients precisely equal the banana transport coefficients in this limit. Indeed, the transport equations (**15**.3.1) reduce to

$$\left\langle h^{\mathrm{e}(1)}_{\rho} \right\rangle_{\mathrm{B}} = \mathscr{K}_{\mathrm{e}}^2 \left(c^{\mathrm{e}}_{11} g^{(1)P}_{\rho} - c^{\mathrm{e}}_{13} g^{\mathrm{e}(3)}_{\rho} \right),$$

$$\left\langle h^{\mathrm{e}(1)}_{\rho} \right\rangle_{\mathrm{B}} = \mathscr{K}_{\mathrm{e}}^2 \left(-c^{\mathrm{e}}_{13} g^{(1)P}_{\rho} + c^{\mathrm{e}}_{33} g^{\mathrm{e}(3)}_{\rho} \right),$$

$$\left\langle h^{\mathrm{i}(3)}_{\rho} \right\rangle_{\mathrm{B}} = \mathscr{K}_{\mathrm{i}}^2 \; c^{\mathrm{i}}_{33} g^{\mathrm{i}(3)}_{\rho}, \quad \varphi \to \infty. \tag{8.17}$$

Thus, the asymptotic average banana entropy production derives from a *thermodynamic form*, like (5.5) or (6.2),

$$\langle \Sigma^{e} \rangle_{B}^{as} = \left\langle h_{\rho}^{e(1)} \right\rangle_{B} g_{\rho}^{(1)P} + \left\langle h_{\rho}^{e(3)} \right\rangle_{B} g_{\rho}^{e(3)},$$

$$\langle \Sigma^{i} \rangle_{B}^{as} = \left\langle h_{\rho}^{i(3)} \right\rangle_{B} g_{\rho}^{i(3)}, \quad \varphi \to \infty. \tag{8.18}$$

We thus found a surprisingly simple result: *in the limit of a very large neoclassical factor, the divorce between banana transport equations and banana entropy production fades out. The entropic coefficients reduce to the transport coefficients and the average banana entropy production assumes a thermodynamic form.* The reason for this simple asymptotic behaviour is the disappearance of the neoclassical pseudo-viscosity coefficients from the expression of the banana transport coefficients in this limit.

– *The Ohmic average entropy production.* We now return to the last term (7.5) in the electron entropy production. It is clearly related to the Joule dissipation of the average parallel electric current. It is expressed in a standard thermodynamic form, as a product of the current by its conjugate force, the average parallel electric field. However, this simplicity does not exclude some specific difficulties in the long mean free path regime.

Note, indeed, that whenever the generalized stresses $\bar{g}_{\|}^{\alpha(n)}$ can be neglected, the Ohmic entropy production reduces to (see eq. 7.4)

$$\langle \Sigma^{e} \rangle_{Ohm}^{clas} = \hat{h}_{\|CL}^{(1)} \hat{g}_{\|}^{(1)A} = \sigma \left(\hat{g}_{\|}^{(1)A} \right)^{2} > 0. \tag{8.19}$$

This is, clearly, the positive definite quadratic form in the forces, associated with the parallel transport matrix, which reduces to a single term in the present case.

In the long mean free path regime, the Ohmic entropy production takes a different form, obtained by substituting the banana parallel current from (**15**.3.3) into (7.5):

$$\langle \Sigma^{e} \rangle_{Ohm} = \left(\sigma + \varphi l_{E3}^{E} \right) \left(\hat{g}_{\|}^{(1)A} \right)^{2}$$

$$+ \mathscr{K}_{e} \varphi \left(l_{E1}^{e} g_{\rho}^{(1)P} + l_{E3}^{e} g_{\rho}^{e(3)} + aA l_{E3}^{i} g_{\rho}^{i(3)} \right) \hat{g}_{\|}^{(1)A}. \tag{8.20}$$

This is a very degenerate quadratic form, associated with the following matrix of *Ohmic entropic coefficients*:

$$\boldsymbol{O} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2}\mathscr{K}_e \varphi l^e_{E1} \\ 0 & 0 & 0 & \frac{1}{2}\mathscr{K}_e \varphi l^e_{E3} \\ 0 & 0 & 0 & \frac{1}{2}\mathscr{K}_e \varphi l^i_{E3} \\ \frac{1}{2}\mathscr{K}_e \varphi l^e_{E1} & \frac{1}{2}\mathscr{K}_e \varphi l^e_{E3} & \frac{1}{2}\mathscr{K}_e \varphi l^i_{E3} & \sigma + \varphi l_{EE} \end{pmatrix}. \tag{8.21}$$

Let us investigate the sign properties of this quadratic form. Its single non-zero diagonal element is positive, as we know from section **15**.5,

$$\sigma + \varphi l_{EE} > 0.$$

But the second-order determinants are either zero or *negative*; for instance,

$$\begin{vmatrix} 0 & \frac{1}{2}\mathscr{K}_e \varphi l^e_{E1} \\ \frac{1}{2}\mathscr{K}_e \varphi l^e_{E1} & \sigma + \varphi l_{EE} \end{vmatrix} = -\tfrac{1}{4}\mathscr{K}_e^2 \varphi^2 (l^e_{E1})^2 < 0.$$

As for the third- and fourth-order determinants, they are all zero.

We thus reach the conclusion that *the quadratic form* (8.20) *has no definite sign*.

This apparently unsatisfactory result should, however, not be surprising. It actually implies that (8.20) does not represent the complete entropy production associated with the Ohmic heating in the banana regime. Indeed, in the banana entropy production (7.10) we find all the terms of the form (8.20) appearing with additional coefficients. Thus, eq. (7.5) does not contain separately all the Ohmic effects: its suggestive thermodynamic form is an illusion (except in the short mean free path regime, when all the banana transport coefficients are negligible). The Ohmic effects cannot be disentangled from the banana effects. As a result, although $\langle \Sigma^e \rangle_B$ has a positive sign, $\langle \Sigma^e \rangle_{Ohm}$ has not. Their sum, however, is definite positive:

$$\langle \Sigma^e \rangle_B + \langle \Sigma^e \rangle_{Ohm} \geqslant 0. \tag{8.22}$$

This results from the fact that this sum is identical to the contribution called $\langle \Sigma^e \rangle_{BCL}$, which can be expressed as a manifestly definite-positive quadratic form in the fluxes, eq. (6.10). Note that a conclusion similar to (but weaker than) (8.22) was also reached by Tessarotto (1982).

The intimate tangling of the banana and Ohmic effects clearly results from the coupling of the radial fluxes to the parallel electric field and of the parallel electric current to the radial forces, which is typical of the banana regime.

17.9. Conclusions and comparison with other works

We now summarize and put together the main results obtained in the previous sections of this chapter.

We started from the expression of the total exact (unaveraged) entropy production in the drift approximation, which was obtained as the sum of two terms:

$$\Sigma^{\alpha} = \Sigma^{\alpha}_{\mathrm{CL}} + \Sigma^{\alpha}_{(\|)}. \tag{9.1}$$

There is a great difference between the two terms. The "classical" entropy production is clearly related to the classical transport in the radial direction: it appears in the standard transport form as a quadratic form in the radial forces, associated with the (asymptotic, drift approximation) matrix of the perpendicular transport coefficients (see eq. 2.11). On the contrary, the parallel entropy production appears in a quasi-transport quadratic form (2.12) associated with the parallel quasi-transport matrix and involving as variables the *sums* of the parallel *forces* and the parallel generalized *stresses*. The two terms in (9.1) include, by construction, *all* the dissipative mechanisms at hand in the confined plasma.

In order to exhibit, if possible, a specific contribution associated with each separate mechanism, we evaluated the *surface average* of the entropy production. The purpose of this operation was to introduce the *average* fluxes and forces that are characteristic of the neoclassical transport equations.

The classical term in (9.1) poses no problem: the surface averaging only affects the transport coefficients. On the contrary, the transformation of the average parallel entropy production into a sum of quadratic forms associated with the specific neoclassical transport mechanisms required a non-trivial analysis. The outcome of this analysis can be presented as follows:

Electron average entropy production:

$$\langle \Sigma^{\mathrm{e}} \rangle = \langle \Sigma^{\mathrm{e}} \rangle_{\mathrm{CL}} + \langle \Sigma^{\mathrm{e}} \rangle_{\mathrm{PS}} + \langle \Sigma^{\mathrm{e}} \rangle_{\mathrm{B}} + \langle \Sigma^{\mathrm{e}} \rangle_{\mathrm{Ohm}} \tag{9.2}$$

Ion average entropy production:

$$\langle \Sigma^{\mathrm{i}} \rangle = \langle \Sigma^{\mathrm{i}} \rangle_{\mathrm{CL}} + \langle \Sigma^{\mathrm{i}} \rangle_{\mathrm{PS}} + \langle \Sigma^{\mathrm{i}} \rangle_{\mathrm{B}}. \tag{9.3}$$

The first, simple result of our analysis is that both the classical and the Pfirsch–Schlüter contributions can be written in the standard transport quadratic form in the thermodynamic forces. The coefficients of these positive definite quadratic forms are identical with the classical and the Pfirsch–Schlüter

transport coefficients, respectively (see eqs. 5.3, 6.1). In other words, these particular contributions derive from a *thermodynamic form* eqs. (5.5, 6.2), i.e. a bilinear form in the forces and their conjugate fluxes.

These simple properties are *not* shared by the banana contributions. Their "natural" (i.e. simplest) form is a quadratic form in the average banana parallel fluxes, for the ions (eq. 6.9), or in the sums of average classical and banana parallel fluxes for the electrons (eq. 6.10); the matrices associated with these forms are the collision matrices. These forms can be brought into a quasi-transport form (7.1), (7.2) associated with the parallel quasi-transport matrix, and whose variables are the average radial banana fluxes. As a result, it can be said that the "natural" "thermodynamic" forces for the banana transport are the parallel generalized stresses (proportional to the banana fluxes).

Nothing prevents us from forcing these expressions into a quadratic form in the "ordinary" radial forces *plus* the average parallel electric field. But the associated matrices are definitely distinct from the banana transport matrices. This phenomenon was called the *divorce between entropy production and transport theory for the banana transport mechanism*. The entropy production, for each species, appears as a semi-definite (rather than definite) positive quadratic form in the forces, associated with an entropic matrix that is distinct from the transport matrix.

An additional difficulty results from the fact that in the electronic average entropy production, the parallel Ohmic dissipation term cannot be neatly separated from the banana contribution.

Thus, coming back to the programme outlined in section 17.1, we conclude that the question marks over the equality signs in eqs. (1.3) and (1.4) must be replaced by *inequality signs*. This shows that a statement such as the one quoted (from Hazeltine and Catto 1981) below eq. (1.4) is *incorrect*. There is no "passe-partout" recipe based on the entropy production: each particular case must be carefully analyzed before reaching a conclusion.

The entropy production has been used by many authors (in relatively older works) in the neoclassical theory as a basis for a *variational formulation* of transport theory (Rutherford 1970, Rosenbluth et al. 1972, Bernstein 1974, Hinton and Hazeltine 1976). The starting point is a theorem of Prigogine (1947) (see also de Groot and Mazur 1984) called the *minimum entropy production theorem*. It states that: *if* the transport equations relating fluxes and forces are *linear*, and the Onsager symmetry relations are satisfied, the steady-non-equilibrium state of a system under external constraint (expressed by non-vanishing thermodynamic forces) is characterized by a minimum of the entropy production. This extremal property is exploited for the construction of a variational principle whose Euler–Lagrange equations coincide with the drift kinetic equation plus the solubility constraints (discussed in chapter 14). When

carefully formulated in this way, there is no objection to this procedure, which has yielded many of the known results of neoclassical transport theory.

One should be careful, however, in not going beyond the limits of validity of Prigogine's theorem. For instance, the recent work by Catto and Myra (1986), in which the minimum of the entropy production is used as a criterion of choice among various branches of the solution of a *nonlinear* equation is definitely incorrect: it has been explicitly proved that the minimum entropy production theorem is invalid when the transport equations are nonlinear (Glansdorff and Prigogine 1954, Prigogine 1961).

It appears that since the advent of the work by Hirshman and Sigmar (1981) the neoclassical theory can be formulated in a much simpler and more transparent form. It is our feeling that the variational methods no longer appear as relevant as in former days. For this reason, we chose not to discuss them in this book.

Appendix 17A.1. Some properties of the collision matrix and of the parallel quasi-transport matrix

17A.1a. A theorem concerning the sign of quadratic forms

We will prove here Theorem 1 of section 17.3, using slightly different notations. Let $\boldsymbol{C} \equiv [c_{pq};\ p, q = 1, 3, 5]$ be a 3×3 symmetric matrix and $\Lambda \equiv [\lambda_{pq};\ p, q = 1, 3, 5]$ be its inverse matrix,

$$\boldsymbol{C} \cdot \Lambda = \boldsymbol{I}. \tag{A1.1}$$

We assume that the quadratic form $\mathscr{C}$ associated with $\boldsymbol{C}$ is definite positive. This means that all criteria (3.15)–(3.17) are satisfied. We now check the positivity criteria of the inverse matrix. Its diagonal elements are

$$\lambda_{11} = \frac{D_{35}}{F_{135}} > 0, \qquad \lambda_{33} = \frac{D_{15}}{F_{135}} > 0, \qquad \lambda_{55} = \frac{D_{13}}{F_{135}} > 0.$$

These quantities are all positive as a result of criteria (3.16), (3.17). We also note that, by a well-known theorem of algebra, the determinant of the inverse matrix Λ equals the inverse of the determinant of the matrix $\boldsymbol{C}$,

$$\Phi_{135} \equiv \| \Lambda \| = \frac{1}{\| \boldsymbol{C} \|} = \frac{1}{F_{135}} > 0.$$

This quantity is positive, as a result of (3.17). A direct calculation of the second-order determinant yields the following expressions:

$$\Delta_{13} = \frac{c_{55}}{F_{135}^2} > 0, \qquad \Delta_{15} = \frac{c_{33}}{F_{135}^2} > 0, \qquad \Delta_{35} = \frac{c_{11}}{F_{135}^2} > 0.$$

These quantities are also positive, as a result of (3.15). Thus, all the positivity criteria are satisfied for the inverse matrix (QED).

17A.1b. Some useful relations involving the collision matrix, the pseudo-viscosity matrix and the parallel quasi-transport matrix

The relations written below are simple consequences of the fact that the matrix $\boldsymbol{T}^{\alpha}$ of the parallel quasi-transport coefficients is the inverse of the collision matrix $\boldsymbol{C}^{\alpha}$ (Property 2, eq. (3.20) of section 17.3). Writing out this property explicitly, we find

$$\begin{pmatrix} c_{11}^{e} & c_{13}^{e} & c_{15}^{e} \\ c_{13}^{e} & c_{33}^{e} & c_{35}^{e} \\ c_{15}^{e} & c_{35}^{e} & c_{55}^{e} \end{pmatrix} \cdot \begin{pmatrix} \sigma^{e} & \alpha & \gamma \\ \alpha & \kappa^{e} & \delta^{e} \\ \gamma & \delta^{e} & \epsilon^{e} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{A1.2}$$

This equality is equivalent to a set of nine equations, *valid in the 21M approximation*,

$$c_{11}^{e}\sigma + c_{13}^{e}\alpha + c_{15}^{e}\gamma = 1,$$

$$c_{13}^{e}\alpha + c_{33}^{e}\kappa^{e} + c_{35}^{e}\delta^{e} = 1,$$

$$c_{15}^{e}\gamma + c_{35}^{e}\kappa^{e} + c_{55}^{e}\epsilon^{e} = 1,$$

$$c_{13}^{e}\sigma + c_{33}^{e}\alpha + c_{35}^{e}\gamma = 0,$$

$$c_{15}^{e}\sigma + c_{35}^{e}\alpha + c_{55}^{e}\delta^{e} = 0,$$

$$c_{11}^{e}\alpha + c_{13}^{e}\kappa^{e} + c_{35}^{e}\delta^{e} = 0,$$

$$c_{15}^{e}\alpha + c_{35}^{e}\kappa^{e} + c_{55}^{e}\delta^{e} = 0,$$

$$c_{11}^{e}\gamma + c_{13}^{e}\delta^{e} + c_{15}^{e}\epsilon^{e} = 0,$$

$$c_{13}^{e}\gamma + c_{33}^{e}\delta^{e} + c_{35}^{e}\epsilon^{e} = 0. \tag{A1.3}$$

A similar, but simpler set of relations holds for the ionic quasi-transport coefficients in the 21M approximation,

$$c^{i}_{33}\kappa^{i} + c^{i}_{35}\delta^{i} = 1,$$

$$c^{i}_{35}\delta^{i} + c^{i}_{55}\epsilon^{i} = 1,$$

$$c^{i}_{35}\kappa^{i} + c^{i}_{55}\delta^{i} = 0,$$

$$c^{i}_{33}\delta^{i} + c^{i}_{35}\epsilon^{i} = 0. \tag{A1.4}$$

These relations are very useful in the calculations involving the passage from the collision quadratic form $\mathscr{C}^{\alpha}$ to the transport quadratic form $\mathscr{T}^{\alpha}$. These same relations can be written in another guise by using the expressions in table **15**.3.1; they are then transformed into a set of relations among the collision matrix elements,

$$c^{e}_{11}D^{e}_{35} - c^{e}_{13}H^{e}_{153} - c^{e}_{15}H^{e}_{135} = F^{e}_{135},$$

$$c^{e}_{33}D^{e}_{15} - c^{e}_{13}H^{e}_{153} - c^{e}_{35}H^{e}_{315} = F^{e}_{135},$$

$$c^{e}_{55}D^{e}_{13} - c^{e}_{35}H^{e}_{315} - c^{e}_{15}H^{e}_{135} = F^{e}_{135},$$

$$c^{e}_{13}D^{e}_{35} - c^{e}_{33}H^{e}_{153} - c^{e}_{35}H^{e}_{135} = 0,$$

$$c^{e}_{15}D^{e}_{35} - c^{e}_{35}H^{e}_{153} - c^{e}_{55}H^{e}_{135} = 0,$$

$$c^{e}_{13}D^{e}_{15} - c^{e}_{11}H^{e}_{153} - c^{e}_{15}H^{e}_{315} = 0,$$

$$c^{e}_{35}D^{e}_{15} - c^{e}_{15}H^{e}_{153} - c^{e}_{55}H^{e}_{315} = 0,$$

$$c^{e}_{15}D^{e}_{13} - c^{e}_{11}H^{e}_{135} - c^{e}_{13}H^{e}_{315} = 0,$$

$$c^{e}_{35}D^{e}_{13} - c^{e}_{13}H^{e}_{135} - c^{e}_{33}H^{e}_{315} = 0. \tag{A1.5}$$

It is clear that these identities merely express the relation between a matrix and its inverse; as a result, the same relations hold for the combinations involving the pseudo-viscosity coefficients instead of the collision matrix elements. In other words, eqs. (A1.5) are still valid after the substitutions

$$c^{e}_{pq} \to \mu^{\alpha}_{pq}, \qquad D^{e}_{pq} \to \mathscr{D}^{\alpha}_{pq}, \qquad H^{e}_{pqr} \to \mathscr{H}^{\alpha}_{pqr}, \qquad F^{e}_{pqr} \to \mathscr{F}^{\alpha}_{pqr}$$

(See eq. **15**.3.16). Thus, for instance,

$$\mu_{11}^{\alpha}\mathscr{D}_{35}^{\alpha} - \mu_{13}^{\alpha}\mathscr{H}_{153}^{\alpha} - \mu_{15}^{\alpha}\mathscr{H}_{135}^{\alpha} = \mathscr{F}_{135}^{\alpha},$$

$$\mu_{13}^{\alpha}\mathscr{D}_{35}^{\alpha} - \mu_{33}^{\alpha}\mathscr{H}_{153}^{\alpha} - \mu_{35}^{\alpha}\mathscr{H}_{135}^{\alpha} = 0, \quad \text{etc.} \tag{A1.6}$$

Finally, the same type of relations hold for the parallel quasi-transport coefficients (in the 21M approximation) and their combinations defined in (**15**.3.17),

$$\sigma\Delta_{35}^{e} - \alpha\chi_{153}^{e} - \gamma\chi_{135}^{e} = \Phi_{135}^{e},$$

$$\kappa^{e}\Delta_{15}^{e} - \alpha\chi_{153}^{e} - \delta^{e}\chi_{315}^{e} = \Phi_{135}^{e},$$

$$\epsilon^{e}\Delta_{13}^{e} - \delta^{e}\chi_{315}^{e} - \gamma\chi_{135}^{e} = \Phi_{135}^{e},$$

$$\alpha\Delta_{35}^{e} - \kappa^{e}\chi_{153}^{e} - \delta^{e}\chi_{135}^{e} = 0,$$

$$\gamma\Delta_{35}^{e} - \delta^{e}\chi_{153}^{e} - \epsilon^{e}\chi_{135}^{e} = 0,$$

$$\alpha\Delta_{15}^{e} - \sigma\chi_{153}^{e} - \gamma\chi_{315}^{e} = 0,$$

$$\delta^{e}\Delta_{15}^{e} - \gamma\chi_{153}^{e} - \epsilon^{e}\chi_{315}^{e} = 0,$$

$$\gamma\Delta_{13}^{e} - \sigma\chi_{135}^{e} - \alpha\chi_{315}^{e} = 0,$$

$$\delta^{e}\Delta_{13}^{e} - \alpha\chi_{135}^{e} - \kappa^{e}\chi_{315}^{e} = 0. \tag{A1.7}$$

All these relations are very useful in the manipulations of expressions involving the transport coefficients.

17A.1c. A theorem on quadratic forms

We now state the following nice result, which is used in some of the calculations of the present chapter.

If p_{ij} *and* q_{ij} $(i, j = 1, 2, \cdots, M)$ *are elements of two symmetric matrices that are inverse to each other,*

$$p_{ij}q_{jk} = \delta_{ik}, \tag{A1.8}$$

then, the following relation holds for the bilinear form associated with these matrices:

$$\mathscr{P}(q_{1j}, q_{2j}, \cdots, q_{mj} \mid q_{1k}, q_{2k}, \cdots, q_{Mk}) = q_{jk}. \tag{A1.9}$$

In particular, for $j = k$, the result reduces to a property of the quadratic form

$$\mathscr{P}(q_{1j}, q_{2j}, \cdots, q_{Mj}) = q_{jj} \tag{A1.10}$$

(no summation over j!).

The proof is trivial, by using (A1.8) and the symmetry of the matrices,

$$\mathscr{P}(q_{1j}, \cdots, q_{Mj} \mid q_{1k}, \cdots, q_{Mk}) = p_{mn} q_{mj} q_{nk}$$

$$= q_{mj} \delta_{mk} = q_{kj} = q_{jk}.$$

Appendix 17A.2. Positivity of the entropic coefficient p_{EE}

We present the calculations for the simpler 13M approximation. In that case, the electric entropic coefficient p_{EE} is found from (7.15), combined with (**15**.2.21),

$$p_{EE} = \mathscr{T}'^{e}(-l^{e}_{1E}, l^{e}_{3E}) + l_{EE}$$

$$= \sigma (l^{e}_{1E})^2 + \kappa^{e} (l^{e}_{3E})^2 + 2\alpha l^{e}_{1E} l^{e}_{3E} - \sigma l^{e}_{1E} + \alpha l^{e}_{3E}. \tag{A2.1}$$

(In this appendix we set $\varphi = 1$, for convenience). The sign of this expression is not manifest, because of the presence of the linear terms. We therefore make the substitution

$$l^{e}_{1E} = \xi + u\eta, \qquad l^{e}_{3E} = \eta + v\xi. \tag{A2.2}$$

These values are substituted into (A2.1) and the coefficients u and v are determined in such a way as to annul the linear terms in ξ, η. The result is

$$u = \frac{\alpha}{\sigma}, \qquad v = \frac{\sigma}{\alpha}. \tag{A2.3}$$

The coefficient p_{EE} is now expressed as a true quadratic form,

$$p_{EE} \equiv \mathscr{Q}(\xi, \eta) = \Delta^{e}_{13} \left(\frac{\sigma}{\alpha^2} \xi^2 + \frac{1}{\sigma} \eta^2 + \frac{2}{\alpha} \xi\eta \right), \tag{A2.4}$$

where Δ^e_{13} was defined in (**15**.3.17). We now check the positivity criteria for this quadratic form. We find from (3.22) and (3.23)

$$\Delta^e_{13} > 0, \qquad \frac{\sigma}{\alpha^2} > 0, \qquad \frac{1}{\sigma} > 0.$$

Hence, the diagonal coefficients are positive; as for the 2×2 determinant,

$$D_{13} \equiv \frac{\sigma}{\alpha^2} \cdot \frac{1}{\sigma} - \left(\frac{1}{\alpha}\right)^2 = 0.$$

Hence, p_{EE} is a *semi-definite positive* quadratic form in ξ, η. This means that $\mathcal{Q}(\xi, \eta)$ may vanish for non-zero values of ξ and η. These values are easily found: $\mathcal{Q}(\xi, \eta)$ vanishes for

$$\xi = -\frac{\alpha}{\sigma}\eta. \tag{A2.5}$$

But, by eqs. (A2.2), (A2.3), this particular relationship implies that $l^e_{1E} = l^e_{3E} = 0$, which is obviously not true. Thus,

$$p_{EE} > 0.$$

The proof in the 21M approximation can be done along similar lines, but the calculations are more involved.

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18

Toroidal plasmadynamics

18.1. The problem of one-dimensional toroidal plasmadynamics

In the second part of this book (chapters 8–17) we have derived at length a set of *transport equations* relating the fluxes of matter and heat to the thermodynamic forces present in a magnetically confined plasma. For the completion of the macroscopic dynamical description of such a plasma, these transport equations must be incorporated into the *plasmadynamical balance equations* of chapter 3 in order to obtain closed equations of evolution for the densities and temperatures of the electrons and of the ions (in a two-fluid picture of the plasma).

In a *magnetically confined plasma* this final step is not as straightforward as in the unconstrained case studied in chapter 7, for the following reason. Consider, for instance, the simplest plasmadynamical equation, the continuity equation (3.4.5),

$$\partial_t n_\alpha(\boldsymbol{x};\, t) - \nabla \cdot \left[n_\alpha(\boldsymbol{x};\, t)\, \boldsymbol{u}^\alpha(\boldsymbol{x};\, t) \right]. \tag{1.1}$$

If we are to close this equation, we should express the particle flux $n_\alpha \boldsymbol{u}^\alpha \equiv \Gamma^\alpha(\boldsymbol{x};\, t)$ as a function of the thermodynamic forces. But in a magnetically confined plasma such a completely *local* description is unavailable. Transport theory provides only expressions for the *surface-averaged value of the radial component*, Γ^α_ρ, a function of the *radial coordinate* ρ alone, expressed in terms of the *radial components of the (averaged) thermodynamic forces*. Instead of a complete determination of the three components of a vector field depending on three spatial coordinates, transport theory (especially in the long mean free path regime) can only provide us with the expression of a single component of this field as a function of a single coordinate. The remaining information is, on one hand, theoretically unavailable; on the other hand, it is deemed *irrelevant* as being experimentally inaccessible.

A similar situation pertains for the scalar variables, i.e. the densities and the temperatures. The only relevant information is their dependence on the radial coordinate ρ [i.e. their "*profiles*" $n_\alpha(\rho;\, t)$, $T_\alpha(\rho;\, t)$]: the dependence on the poloidal and toroidal angles can be averaged out.

In conclusion, the completely local plasmadynamical equations (3.4.5), (3.4.11) must, in some consistent way, be reduced to *one-dimensional equations*. This reduction is the object of the present chapter.

An illuminating simple analogy is found by considering an ordinary hydrodynamical flow in a system having *cylindrical symmetry*. Using ordinary cylindrical coordinates r, φ, z, we assume that all quantities depend only on r. The continuity equation then reduces to the following form [using eq. G2.2.16 of General Appendix 2]:

$$\frac{\partial}{\partial t} n_\alpha(r, t) = -\frac{1}{l_r l_z l_\varphi}\left[\frac{\partial}{\partial r} l_z l_\varphi (n_\alpha u_r^\alpha) + 0 + 0\right],$$

the two zeroes originating from the independence of z and φ of the flux $n_\alpha u^\alpha$ as well as of the scale factors. Using $l_r = l_z = 1$, $l_\varphi = r$, we thus find

$$\frac{\partial}{\partial t} n_\alpha(r; t) = -\frac{1}{r}\frac{\partial}{\partial r} r n_\alpha u_r^\alpha. \tag{1.2}$$

The cylindrical symmetry has quite naturally produced a one-dimensional equation (involving the single spatial coordinate r), in which only the radial coordinate $n_\alpha u_r^\alpha$ of the particle flux appears. In our problem we wish to ultimately achieve a similar reduction, but we must tackle the more complicated toroidal (rather than cylindrical) geometry and, moreover, the independence of θ and ζ is not naturally given by the symmetry of the problem but must be obtained by suitable averaging.

The problem is further complicated by the "*electrical effects*" which are typical of the long mean free path transport theory: the radial fluxes are coupled to the average *parallel* electric field which, in turn, produces an average *parallel* electric current. These effects thus introduce "alien", non-radial components of vector quantities into the description. If a one-dimensional plasmadynamics is to exist at all, these quantities should be eliminated in favour of the radial forces. The possibility of such an elimination ultimately rests upon the fact that we are not dealing here with ordinary hydrodynamics, but rather with *plasmadynamics*. This implies that the hydrodynamical balance equations are coupled to the Maxwell equations. It will be shown that a consistent incorporation of the latter into transport theory provides the necessary expression of the parallel quantities in terms of radial forces.

Another point we have to deal with in order to obtain a general plasmadynamical picture is the following. In all the preceding chapters it was (explicitly or implicitly) assumed that the geometrical structure imposed by the confining magnetic field is a permanent one. More precisely, it was assumed in section 8.1 that the magnetic field $\boldsymbol{B}(\boldsymbol{x})$ is, at each point, tangent to a magnetic surface $\rho(\boldsymbol{x}) = \rho$, defined by property (8.1.2): $\boldsymbol{B}(\boldsymbol{x}) \cdot \nabla\rho(\boldsymbol{x}) = 0$.

(We use here the topological radius ρ as a magnetic surface label.) But in a real confined plasma, such as a tokamak, *the magnetic field is not stationary*. In particular, the toroidal field may be thought of as an external quasi-steady field but the poloidal component is self-consistently produced by the parallel electric field, hence by the parallel electric current; as a result of the Faraday law, such a field must be time-dependent (even though the time dependence might be "slow" in some sense).

As we now wish to construct a completely self-consistent formalism of confined plasmadynamics, we must take into account the non-stationary character of the magnetic field, hence of the magnetic surfaces. The latter are defined by

$$\rho(\boldsymbol{x};\ t) = \rho, \tag{1.3}$$

where the number ρ (the topological radius) on the right-hand side is the label of the magnetic surface. Function $\rho(\boldsymbol{x};\ t)$ must satisfy, at all time,

$$\boldsymbol{B}(\boldsymbol{x};\ t)\cdot\nabla\rho(\boldsymbol{x};\ t) = 0. \tag{1.4}$$

Clearly, the coordinates of the points $\boldsymbol{x}$ on each surface are now time-dependent. As a result of (1.3), we have

$$\frac{\mathrm{d}\rho[\boldsymbol{x}(t);\ t]}{\mathrm{d}t} = \frac{\partial\rho[\boldsymbol{x}(t);\ t]}{\partial t} + \frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t}\cdot\frac{\partial\rho(\boldsymbol{x};\ t)}{\partial t} = 0.$$

We define the *velocity of the magnetic surface* $\boldsymbol{w}$, as the derivative $\mathrm{d}\boldsymbol{x}/\mathrm{d}t$, where $\boldsymbol{x}$ is a point attached to the surface. The previous equation thus becomes

$$\frac{\partial\rho(\boldsymbol{x};\ t)}{\partial t} + \boldsymbol{w}\cdot\nabla\rho(\boldsymbol{x};\ t) = 0. \tag{1.5}$$

In the forthcoming sections we shall pursue the following goals (among others):
- incorporate the motion of the magnetic surfaces into the (reduced, one-dimensional) plasmadynamical equations of motion;
- evaluate the velocity $\boldsymbol{w}$ in terms of the physical quantities characterizing the plasma.

18.2. Toroidal plasmadynamics: fluid-dynamical aspects

We start by recalling, without proof, a lemma that is actually a well-known formula of integral vector calculus (Smirnov 1970, Hinton and Hazeltine 1976).

Let $\Delta(t)$ be a three-dimensional domain, varying in time, bounded by a surface $\Sigma(t)$; let $A(x; t)$ be a scalar function of space and time. The time-derivative of the volume integral of A is expressed in either of the following equivalent forms:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Delta(t)} \mathrm{d}^3x\, A(x; t) = \int_{\Delta(t)} \mathrm{d}^3x\, \frac{\partial}{\partial t} A(x; t) + \oint_{\Sigma(t)} \mathrm{d}\Sigma \cdot \left[w^{\Sigma} A(x; t)\right], \tag{2.1}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Delta(t)} \mathrm{d}^3x\, A(x; t) = \int_{\Delta(t)} \mathrm{d}^3x \left\{ \frac{\partial}{\partial t} A(x; t) + \nabla \cdot \left[w^{\Sigma} A(x; t)\right]\right\}, \tag{2.2}$$

where w^{Σ} is the velocity of the surface $\Sigma(t)$ defined by (1.5).

We consider the geometry illustrated in fig. 2.1. A toroidal (not necessarily circular) magnetic surface is labelled by its topological radius ρ defined in section **8**.8 and is determined by $\rho(x; t) = \rho$, as in eq. (1.3). The coordinates x

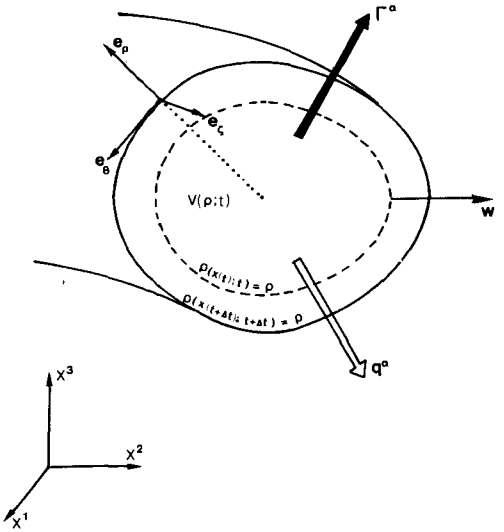

Fig. 2.1. Particle flux Γ^{α} and heat flux q^{α} through a magnetic surface $\rho(x; t) = \rho$, expanding at a rate w and bounding, volume $V(\rho; t)$. x denotes the coordinates referred to a fixed "laboratory" reference frame. $(e_{\rho}, e_{\theta}, e_{\zeta})$ is the geometrical triad of fig **12**.5.1, attached to the (moving) magnetic surface.

are referred to an arbitrary, fixed "laboratory" reference frame. As viewed from this frame, the surface (labeled by the topological radius ρ) is expanding with a velocity w, defined in (1.5). This magnetic surface encloses a volume $V(\rho; t)$ (see eq. **8**.3.1).

We wish to calculate the rate of change of the number of particles of species α contained in the volume $V(\rho; t)$. This can be achieved by using the continuity equation (**3**.4.5) (or eq. 1.1), integrated over the volume $V(\rho; t)$,

$$\int_{V(\rho;\, t)} \mathrm{d}^3x\, \partial_t n_\alpha(x; t) = -\int_{V(\rho,\, t)} \mathrm{d}^3x\, \nabla \cdot [n_\alpha(x; t) u^\alpha(x; t)]. \tag{2.3}$$

The quantity we are looking for is the time-derivative of the integral of n_α over the volume $V(\rho; t)$: it can be obtained from the left-hand side by using lemma (2.2),

$$\int_{V(\rho;\, t)} \mathrm{d}^3x\, \partial_t n_\alpha(x; t) = \frac{\partial}{\partial t}\int_{V(\rho,\, t)} \mathrm{d}^3x\, n_\alpha(x; t) - \int_{V(\rho,\, t)} \mathrm{d}^3x\, \nabla \cdot (n_\alpha w). \tag{2.4}$$

Combining eqs. (2.3) and (2.4), we find

$$\partial_t \int_{V(\rho;\, t)} \mathrm{d}^3x\, n_\alpha(x; t) = -\int_{V(\rho,\, t)} \mathrm{d}^3x\, \nabla \cdot [n_\alpha(u^\alpha - w)]. \tag{2.5}$$

We now evaluate the volume integrals by using the toroidal coordinates ρ, θ, ζ instead of the laboratory coordinates x [see sections **8**.2 and **12**.5]. With each point is associated a triad of basis vectors attached to the magnetic surface passing through it (see figs. **12**.5.1 and 2.1). Using the results of section **8**.8, we write (for an axisymmetric configuration where everything is independent of the toroidal angle ζ)

$$\int_{V(\rho;\, t)} \mathrm{d}^3x\, n_\alpha(x; t) = \int_0^\rho \mathrm{d}\rho' \int_0^{2\pi} \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\zeta\, l_{\rho'} l_\theta l_\zeta\, n_\alpha(\rho', \theta; t)$$

$$= \int_0^\rho \mathrm{d}\rho'\, V'(\rho'; t) \langle n_\alpha \rangle, \tag{2.6}$$

where we used definition (**8**.8.35) of the surface average and where $V'(\rho; t)$ is

defined by (**8**.8.36). The right-hand side can be evaluated in a similar way (using eq. **G2**.2.16),

$$\int_{V(\rho,\,t)} \mathrm{d}^3\boldsymbol{x}\, \nabla\cdot\left[n_\alpha(\boldsymbol{u}^\alpha - \boldsymbol{w})\right]$$

$$= \int_0^\rho \mathrm{d}\rho' \int_0^{2\pi} \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\zeta\, l_{\rho'} l_\theta l_\zeta \frac{1}{l_{\rho'} l_\theta l_\zeta}\left[\frac{\partial}{\partial\rho'} l_\theta l_\zeta n_\alpha(u^\alpha_{\rho'} - w_{\rho'}) + \cdots\right]$$

$$= \int_0^\rho \mathrm{d}\rho' \int_0^{2\pi} \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\zeta\, \frac{\partial}{\partial\rho'} l_\theta l_\zeta n_\alpha(u^\alpha_{\rho'} - w_{\rho'})$$

$$= \int_0^\rho \mathrm{d}\rho' \frac{\partial}{\partial\rho'} V'(\rho';\, t)\left\langle \frac{n_\alpha}{l_{\rho'}}(u^\alpha_{\rho'} - w_{\rho'})\right\rangle. \tag{2.7}$$

Substituting eqs. (2.6) and (2.7) into (2.5) and differentiating both sides with respect to ρ, we find

$$\frac{\partial}{\partial t}\left[V'(\rho;\, t)\langle n_\alpha\rangle\right] = -\frac{\partial}{\partial\rho}\left[V'(\rho;\, t)\left\langle l_\rho^{-1} n_\alpha(u^\alpha_\rho - w_\rho)\right\rangle\right]. \tag{2.8}$$

The average expression on the right-hand side is actually the average of the contravariant component of the particle flux (corrected by the velocity of the magnetic surface), see (**G2**.2.4) or (**12**.9.2),

$$l_\rho^{-1} u^\alpha_\rho \equiv \hat{u}^{\alpha\rho} \equiv \boldsymbol{u}^\alpha \cdot \nabla\rho. \tag{2.9}$$

We now note that, to leading order in ϵ, the density is a surface function, thus

$$\langle n_\alpha\rangle \equiv n_\alpha(\rho;\, t).$$

Moreover, we have always assumed in this book that the radial scale factor l_ρ only depends on ρ (eq. **12**.8.19). Equation (2.8) can therefore be rewritten in the simpler form

$$\frac{\partial}{\partial t}\left[V'(\rho;\, t)\; n_\alpha(\rho;\, t)\right] = -\frac{\partial}{\partial\rho}\left[V'(\rho;\, t)\; l_\rho^{-1} n_\alpha\left\langle u^\alpha_\rho - w_\rho\right\rangle\right]. \tag{2.10}$$

This equation is the *reduced form of the continuity equation* we were looking for. Indeed, it is an equation for the *density profile* $n_\alpha(\rho;\, t)$, expressed in terms

of the *average radial particle flux* $\langle n_\alpha u_\rho^\alpha \rangle$ (for which we have a transport equation) and of the *average radial surface velocity* $\langle w_\rho \rangle$ (for which an expression will be obtained later on). The specific effect of the toroidal geometry comes in through the factors $V'(\rho; t)$ and l_ρ.

We note that the two equations (2.10) for $\alpha = \mathrm{e}, \mathrm{i}$, are not independent. By recalling the local electroneutrality condition $n_\mathrm{e} = Zn_\mathrm{i}$, and the relation expressing the ambipolarity of the radial diffusion $\langle n_\mathrm{e} u_\rho^\mathrm{e} \rangle = Z\langle n_\mathrm{i} u_\rho^\mathrm{i} \rangle$, we see that a single equation is sufficient.

We now perform a similar treatment on the temperature equations (3.4.11) which, for convenience, are first transformed into *equations for the internal energy density of species* α,

$$n_\alpha \varepsilon_\alpha \equiv \tfrac{3}{2} n_\alpha T_\alpha \equiv \tfrac{3}{2} P_\alpha$$

(see eq. 3.2.9), by combining them with the continuity equations (3.4.5),

$$\begin{aligned} \tfrac{3}{2}\partial_t P_\alpha = &-\nabla \cdot \left(\boldsymbol{q}^\alpha + \tfrac{5}{2} T_\alpha n_\alpha \boldsymbol{u}^\alpha + \boldsymbol{\pi}^\alpha \cdot \boldsymbol{u}^\alpha \right) \\ &+ \boldsymbol{u}^\alpha \cdot \nabla P_\alpha + \boldsymbol{u}^\alpha \cdot (\nabla \cdot \boldsymbol{\pi}^\alpha) + Q_\alpha . \end{aligned} \tag{2.11}$$

We first transform these equations in order to automatically satisfy the constraints of total momentum and energy conservation, expressed by relations (3.4.14) and (3.4.15). Scalar multiplication of the steady-state momentum balance equations (3.4.10) by $\boldsymbol{u}^\alpha$, and neglect of the nonlinear inertial terms yields

$$\boldsymbol{u}^\alpha \cdot \nabla P_\alpha + \boldsymbol{u}^\alpha \cdot (\nabla \cdot \boldsymbol{\pi}^\alpha) = \boldsymbol{u}^\alpha \cdot (e_\alpha n_\alpha \boldsymbol{E} + \boldsymbol{R}^\alpha). \tag{2.12}$$

Summing these relations for $\alpha = \mathrm{e}, \mathrm{i}$, and using (3.4.14), (3.4.15) we obtain

$$\begin{aligned} \sum_\alpha \left[\boldsymbol{u}^\alpha \cdot \nabla P_\alpha + \boldsymbol{u}^\alpha \cdot (\nabla \cdot \boldsymbol{\pi}^\alpha) + Q_\alpha \right] &= \boldsymbol{j} \cdot \boldsymbol{E} + (\boldsymbol{u}^\mathrm{e} - \boldsymbol{u}^\mathrm{i}) \cdot \boldsymbol{R}^\mathrm{e} + Q_\mathrm{e} + Q_\mathrm{i} \\ &= \boldsymbol{j} \cdot \boldsymbol{E}. \end{aligned}$$

We thus conclude that

$$\boldsymbol{u}^\mathrm{e} \cdot (\nabla P_\mathrm{e} + \nabla \cdot \boldsymbol{\pi}^\mathrm{e}) + Q_\mathrm{e} = -\boldsymbol{u}^\mathrm{i} \cdot (\nabla P_\mathrm{i} + \nabla \cdot \boldsymbol{\pi}^\mathrm{i}) + \boldsymbol{j} \cdot \boldsymbol{E} - Q_\mathrm{i}. \tag{2.13}$$

The two energy equations are rewritten in the form

$$\tfrac{3}{2}\partial_t P_i = -\nabla\cdot\left(\boldsymbol{q}^i + \tfrac{5}{2}P_i\boldsymbol{u}^i + \boldsymbol{\pi}^i\cdot\boldsymbol{u}^i\right) + \boldsymbol{u}^i\cdot\nabla P_i + \boldsymbol{u}^i\cdot(\nabla\cdot\boldsymbol{\pi}^i) + Q_i, \tag{2.14}$$

$$\tfrac{3}{2}\partial_t P_e = -\nabla\cdot\left(\boldsymbol{q}^e + \tfrac{5}{2}P_e\boldsymbol{u}^e + \boldsymbol{\pi}^e\cdot\boldsymbol{u}^e\right) - \boldsymbol{u}^i\cdot\nabla P_i - \boldsymbol{u}^i\cdot(\nabla\cdot\boldsymbol{\pi}^i) + \boldsymbol{j}\cdot\boldsymbol{E} - Q_i. \tag{2.15}$$

We now proceed exactly as for the derivation of the continuity equation (2.10) and obtain, for the ions,

$$\begin{aligned}\tfrac{3}{2}\partial_t\left[V'P_i(\rho;\,t)\right] = &-\frac{\partial}{\partial\rho}\left\{V'l_\rho^{-1}\left\langle q_\rho^i + \tfrac{5}{2}P_i u_\rho^i - \tfrac{3}{2}P_i w_\rho\right\rangle\right\}\\ &- V'\left[\nabla\cdot(\boldsymbol{\pi}^i\cdot\boldsymbol{u}^i)\right] + V'\left\langle u_\rho^i\right\rangle\nabla_\rho P_i\\ &+ V'\langle\boldsymbol{u}^i\cdot(\nabla\cdot\boldsymbol{\pi}^i)\rangle + V'\langle Q_i\rangle.\end{aligned} \tag{2.16}$$

The terms on the right-hand side are transformed in several steps.

(i) In the first term, the contribution $(\partial/\partial\rho)\,(V'l_\rho^{-1}P_i w_\rho)$ is added and subtracted; this term becomes

$$\begin{aligned}-\frac{\partial}{\partial\rho}\{\cdots\} &= -\frac{\partial}{\partial\rho}\left[V'l_\rho^{-1}\left\langle q_\rho^i + \tfrac{5}{2}P_i\left(u_\rho^i - w_\rho\right)\right\rangle\right] - \frac{\partial}{\partial\rho}\left(V'l_\rho^{-1}P_i\langle w_\rho\rangle\right)\\ &= -\frac{\partial}{\partial\rho}[\cdots] - V'l_\rho^{-1}\frac{\partial P_i}{\partial\rho}\langle w_\rho\rangle - P_i\frac{\partial}{\partial\rho}\left(V'l_\rho^{-1}\langle w_\rho\rangle\right).\end{aligned}$$

We then note that lemma (2.2), applied to $A \equiv 1$, yields

$$\partial_t V'(\rho;\,t) = \frac{\partial}{\partial\rho}V'l_\rho^{-1}\langle w_\rho\rangle. \tag{2.17}$$

The previous formula is thus rewritten as

$$\begin{aligned}&-\frac{\partial}{\partial\rho}\left\{V'l_\rho^{-1}\left\langle q_\rho^i + \tfrac{5}{2}P_i u_\rho^i - \tfrac{3}{2}P_i w_\rho\right\rangle\right\}\\ &= -\frac{\partial}{\partial\rho}\left[V'l_\rho^{-1}\left\langle q_\rho^i + \tfrac{5}{2}P_i\left(u_\rho^i - w_\rho\right)\right\rangle\right]\\ &\quad - V'l_\rho^{-1}\langle w_\rho\rangle\frac{\partial P_i}{\partial\rho} - P_i\,\partial_t V'.\end{aligned} \tag{2.18}$$

(ii) The divergence of the dissipative pressure tensor is evaluated by a straightforward extension of the argument used in section **12**.9 for obtaining eq. (**12**.9.5). Indeed, using (**8**.6.9), we have

$$\langle \nabla \cdot (\boldsymbol{\pi} \cdot \boldsymbol{u}^\alpha) \rangle = \frac{1}{V'} \frac{\mathrm{d}}{\mathrm{d}\rho} V' l_\rho^{-1} \langle (\boldsymbol{\pi}^\alpha \cdot \boldsymbol{u}^\alpha) \cdot \boldsymbol{e}_\rho \rangle .$$

We note that, to leading order in ϵ, $\boldsymbol{\pi}^\alpha$ has the CGL form (**12**.3.7), whereas $\boldsymbol{u}^\alpha$ has only parallel and perp–tangential components; therefore the expression under the average necessarily involves non-diagonal components of the stress tensor in the dynamical reference frame, which are identically zero; thus

$$\langle \nabla \cdot (\boldsymbol{\pi}^\alpha \cdot \boldsymbol{u}^\alpha) \rangle = 0. \tag{2.19}$$

In order to evaluate the term $\langle \boldsymbol{u}^{\mathrm{i}} \cdot (\nabla \cdot \boldsymbol{\pi}^{\mathrm{i}}) \rangle$, we note that the leading-order expression of the ion velocity is given by eqs. (**10**.6.10) and (**12**.8.22); using also (**12**.5.7), we obtain

$$\begin{aligned} \boldsymbol{u}^\alpha &= -\frac{1}{n_\alpha m_\alpha \Omega_\alpha} (P_\alpha' + e_\alpha n_\alpha \Phi') \left(l_\rho^{-1} \boldsymbol{e}_\wedge + \frac{\mathscr{J}}{R_0 \mathscr{B}_P} \boldsymbol{b} \right) + J^{\alpha(1)} \boldsymbol{B} \\ &= -\frac{\mathscr{B}_0}{n_\alpha m_\alpha \Omega_{\alpha 0} R_0 \mathscr{B}_{\mathrm{P}}} (P_\alpha' + e_\alpha n_\alpha \Phi') l_\zeta \boldsymbol{e}_\zeta + J^{\alpha(1)} \boldsymbol{B}. \end{aligned} \tag{2.20}$$

Relations (**12**.9.3) and (**12**.9.5) imply that the first term on the right-hand side contributes nothing to the term we are considering,

$$\left\langle l_\zeta \boldsymbol{e}_\zeta \cdot (\nabla \cdot \boldsymbol{\pi}^{\mathrm{i}}) \right\rangle = 0,$$

and we are left with

$$\langle \boldsymbol{u}^{\mathrm{i}} \cdot (\nabla \cdot \boldsymbol{\pi}^{\mathrm{i}}) \rangle = J^{\mathrm{i}(1)} \langle \boldsymbol{B} \cdot (\nabla \cdot \boldsymbol{\pi}^{\mathrm{i}}) \rangle. \tag{2.21}$$

(iii) The right-hand side of eq. (2.21) is an old acquaintance: it is the typical ingredient of the long mean free path banana fluxes. It can be written in dimensionless form by using table 3.2.1 and eqs. (**12**.6.4) and (**12**.8.24),

$$\langle \boldsymbol{u}^{\mathrm{i}} \cdot (\nabla \cdot \boldsymbol{\pi}^{\mathrm{i}}) \rangle = -\frac{n_{\mathrm{i}} T_{\mathrm{i}}}{\tau_{\mathrm{i}}} \omega_1^{\mathrm{i}} \left\langle \frac{B}{\mathscr{B}_0} \bar{g}_{\parallel}^{\mathrm{i}(1)} \right\rangle. \tag{2.22}$$

A first expression can be obtained (in the 21M approximation) in terms of the "poloidal fluxes" ω_p^i and the pseudo-viscosity coefficients introduced in (**15**.1.1),

$$\langle \boldsymbol{u}^i \cdot (\nabla \cdot \boldsymbol{\pi}^i) \rangle = \frac{n_i T_i}{\tau_i} \varphi \left(\mu_{11}^i \omega_1^i + \mu_{13}^i \omega_3^i + \mu_{15}^i \omega_5^i \right) \omega_1^i. \tag{2.23}$$

This term is thus a quadratic form in the "poloidal fluxes" and is therefore a typical long mean free path contribution to the energy balance.

Hirshman and Sigmar (1981) give in their eq. (9.5) the "approximate result" (expressed in our notation)

$$-\langle \boldsymbol{\pi}^\alpha : \nabla \boldsymbol{u}^\alpha \rangle \sim -\mu_{11}^\alpha (\omega_1^\alpha)^2.$$

The right-hand side of this equation has the wrong sign (it is inconsistent with their eqs. (4.18) and (9.4)!). They interpret the effect as a damping of the "poloidal flow" ω_1^α by the parallel viscosity, i.e. as a "decay of the poloidal plasma rotation". They relate this effect to the work by Bickerton (1972) (we confess of not seeing a very close connection between the results of the latter paper and the present problem!). We pointed out in section **12**.8 that the identification of the surface quantities ω_1^α with the poloidal fluxes $h_\theta^{\alpha(1)}$ is inaccurate (see eq. **12**.8.26); we therefore do not retain this interpretation.

In the following sentence of the same paragraph, Hirshman and Sigmar note

$$-\langle \boldsymbol{\pi}^\alpha : \nabla \boldsymbol{u}^\alpha \rangle \sim -\omega_1^\alpha \left\langle \Gamma_\rho^{\alpha(1)} \right\rangle_B,$$

which is correct (see our forthcoming eq. 2.24) and is interpreted as "viscous heating" (a reasonable interpretation), with a reference to Ware (1980).

An alternative, more explicit form of the above eq. (2.23) is obtained by using table **12**.9.1 and also eqs. (**12**.9.23) and (**12**.6.3),

$$\langle \boldsymbol{u}^i \cdot (\nabla \cdot \boldsymbol{\pi}^i) \rangle = -\frac{n_i T_i}{\tau_i} \frac{1}{\mathscr{K}_i} \omega_1^i \left\langle h_\rho^{i(1)} \right\rangle_B = -\frac{n_i T_i}{\tau_i} \frac{1}{a \mathscr{K}_i} \omega_1^i \left\langle h_\rho^{e(1)} \right\rangle_B. \tag{2.24}$$

This quantity is easily expressed in terms of the thermodynamic forces by

using eq. (**15**.2.6) [in which the small term proportional to $(aA)^{-1}$ is neglected] and the banana transport equation (**15**.3.1),

$$\langle \boldsymbol{u}^{\mathrm{i}} \cdot (\nabla \cdot \boldsymbol{\pi}^{\mathrm{i}}) \rangle = -\frac{n_{\mathrm{i}} T_{\mathrm{i}}}{\tau_{\mathrm{i}}} g_{\rho}^{\mathrm{i}(3)} \frac{m_3^{\mathrm{i}}}{a} \times \Big\{ \mathscr{K}_{\mathrm{e}}^2 \varphi \Big[l_{11}^{\mathrm{ee}}(\phi)\, g_{\rho}^{(1)P} + l_{13}^{\mathrm{ee}}(\varphi)\, g_{\rho}^{\mathrm{e}(3)} + aA\, l_{13}^{\mathrm{ei}}(\varphi)\, g_{\rho}^{\mathrm{i}(3)} \Big] + \mathscr{K}_{\mathrm{e}} \varphi\, l_{1\mathrm{E}}^{\mathrm{e}}(\varphi)\, \hat{g}_{\parallel}^{(1)A} \Big\}. \tag{2.25}$$

We now make use of relations (**15**.2.16) involving the mixed ion–electron banana transport coefficients,

$$\langle \boldsymbol{u}^{i} \cdot (\nabla \cdot \boldsymbol{\pi}^{i}) \rangle = -\frac{n_{\mathrm{i}} T_{\mathrm{i}}}{\tau_{\mathrm{i}}} g_{\rho}^{i(3)} a^{-1} \times \Big\{ \mathscr{K}_{\mathrm{e}}^2 \varphi \Big[l_{31}^{\mathrm{ie}}(\varphi)\, g_{\rho}^{(1)P} + l_{33}^{\mathrm{ie}}(\varphi)\, g_{\rho}^{\mathrm{e}(3)} \Big] + \mathscr{K}_{\mathrm{e}} \varphi\, l_{3\mathrm{E}}^{\mathrm{i}}(\varphi)\, \hat{g}_{\parallel}^{(1)A} + \mathscr{K}_{\mathrm{e}}^2 \varphi\, aA\, m_3^{\mathrm{i}}\, l_{13}^{\mathrm{ei}}(\varphi)\, g_{\rho}^{\mathrm{i}(3)} \Big\}. \tag{2.26}$$

Thus, because of relations (**15**.2.16), the right-hand side can be "almost" identified with the *banana radial ion heat flux* (**15**.3.2); it differs from the latter only in the coefficient of the ion temperature gradient. Thus

$$\langle \boldsymbol{u}^{\mathrm{i}} \cdot (\nabla \cdot \boldsymbol{\pi}^{\mathrm{i}}) \rangle = -\frac{n_{\mathrm{i}} T_{\mathrm{i}}}{\tau_{\mathrm{i}}} g_{\rho}^{\mathrm{i}(3)} \Big\{ \big\langle h_{\rho}^{\mathrm{i}(3)} \big\rangle_{\mathrm{B}} + \varphi \big[-\mathscr{K}_{\mathrm{i}}^2 l_{33}^{\mathrm{ii}}(\varphi) + \mathscr{K}_{\mathrm{e}}^2 A\, m_3^{\mathrm{i}}\, l_{13}^{\mathrm{ei}}(\varphi) \big] g_{\rho}^{\mathrm{i}(3)} \Big\}. \tag{2.27}$$

Moreover, we note that, because of (**15**.2.2), the term proportional to $\mathscr{K}_{\mathrm{e}}^2 A$ is of order A^{-1}, hence negligible compared to the term proportional to $\mathscr{K}_{\mathrm{i}}^2$; thus, we finally obtain, whenever $A^{-1} \ll 1$,

$$\langle \boldsymbol{u}^{\mathrm{i}} \cdot (\nabla \cdot \boldsymbol{\pi}^{\mathrm{i}}) \rangle = -\frac{n_{\mathrm{i}} T_{\mathrm{i}}}{\tau_{\mathrm{i}}} g_{\rho}^{\mathrm{i}(3)} \Big[\big\langle h_{\rho}^{\mathrm{i}(3)} \big\rangle_{\mathrm{B}} - \mathscr{K}_{\mathrm{i}}^2 \varphi\, l_{33}^{\mathrm{ii}}(\varphi)\, g_{\rho}^{\mathrm{i}(3)} \Big]. \tag{2.28}$$

This expression can also be written in terms of dimensional quantities, by using (**12**.4.6) and table **15**.3.4,

$$\langle \boldsymbol{u}^{\mathrm{i}} \cdot (\nabla \cdot \boldsymbol{\pi}^{\mathrm{i}}) \rangle = T_{\mathrm{i}}^{-1} (\nabla_\rho T_{\mathrm{i}}) \Big[\big\langle q_\rho^{\mathrm{i}} \big\rangle_{\mathrm{B}} + L_{33}^{\mathrm{ii\,B}} (\nabla_\rho T_{\mathrm{i}}) \Big]. \tag{2.29}$$

This remarkably simple expression was found by making full use of the *relations existing among the banana transport coefficients*. To the best of our knowledge, it was never published before. In the paper by Hinton and Hazeltine (1976) this term is expressed in terms of rather ill-defined quantities ($\mu_{\mathrm{i}} A_{\mathrm{ii}}$); in the paper by Hirshman and Sigmar (1981) the term is not written in a fully explicit form (see the remark above).

We now collect all the partial results obtained so far, by substituting (2.18), (2.19) and (2.29), as well as eqs. (**4**.5.5), (**4**.6.41) for the ion–electron heat exchange, into the ion energy equation (2.16),

$$\begin{aligned}
&\tfrac{3}{2} V'^{-2/3} \, \partial_t \big[V'^{5/3} P_{\mathrm{i}}(\rho; t) \big] \\
&\quad = -\frac{\partial}{\partial \rho} \Big\{ V' l_\rho^{-1} \Big[\Big\langle q_\rho^{\mathrm{i}} + \tfrac{5}{2} P_{\mathrm{i}} \big(u_\rho^{\mathrm{i}} - w_\rho \big) \Big\rangle \Big] \Big\} \\
&\qquad + V' \big\langle u_\rho^{\mathrm{i}} - w_\rho \big\rangle \nabla_\rho P_{\mathrm{i}} + V' T_{\mathrm{i}}^{-1} (\nabla_\rho T_{\mathrm{i}}) \Big[\big\langle q_\rho^{\mathrm{i}} \big\rangle_{\mathrm{B}} + L_{33}^{\mathrm{ii\,B}} (\nabla_\rho T_{\mathrm{i}}) \Big] \\
&\qquad - 3V' \frac{m_{\mathrm{e}}}{m_{\mathrm{i}}} \frac{n_{\mathrm{e}}}{\tau_{\mathrm{c}}} (T_{\mathrm{i}} - T_{\mathrm{e}}).
\end{aligned} \tag{2.30}$$

A similar treatment applied to eq. (2.15) leads to the following form for the electron energy equation:

$$\begin{aligned}
&\tfrac{3}{2} V'^{-2/3} \, \partial_t \big[V'^{5/3} P_{\mathrm{e}}(\rho; t) \big] \\
&\quad = -\frac{\partial}{\partial \rho} \Big\{ V' l_\rho^{-1} \Big[\Big\langle q_\rho^{\mathrm{e}} + \tfrac{5}{2} P_{\mathrm{e}} \big(u_\rho^{\mathrm{e}} - w_\rho \big) \Big\rangle \Big] \Big\} \\
&\qquad - V' \Big[\big\langle u_\rho^{\mathrm{i}} \big\rangle \nabla_\rho P_{\mathrm{i}} + \langle w_\rho \rangle \nabla_\rho P_{\mathrm{e}} \Big] + V' \langle \boldsymbol{j} \cdot \boldsymbol{E} \rangle \\
&\qquad - V' T_{\mathrm{i}}^{-1} (\nabla_\rho T_{\mathrm{i}}) \Big[\big\langle q_\rho^{\mathrm{i}} \big\rangle_{\mathrm{B}} + L_{33}^{\mathrm{ii\,B}} (\nabla_\rho T_{\mathrm{i}}) \Big] \\
&\qquad + 3V' \frac{m_{\mathrm{e}}}{m_{\mathrm{i}}} \frac{n_{\mathrm{e}}}{\tau_{\mathrm{e}}} (T_{\mathrm{i}} - T_{\mathrm{e}}).
\end{aligned} \tag{2.31}$$

We have gone a long way towards a one-dimensional plasmadynamics, but the work is not finished, as we still have two quantities that are not yet explicit: the Joule heating term $\langle \boldsymbol{j} \cdot \boldsymbol{E} \rangle$ and the velocity w_ρ to the magnetic surface. Both quantities are related to the dynamics of the electromagnetic fields, an aspect that will be studied in the next section.

18.3. Toroidal plasmadynamics: electrodynamical aspects

Our first objective will be the determination of $\boldsymbol{w}$, the velocity (in the laboratory frame) of a surface of constant topological radius ρ, defined by eq. (1.5). According to definition (**8**.8.26), a surface of constant topological radius is also a surface of constant toroidal flux, thus $\boldsymbol{w}$ can also be defined as

$$\partial_t \Phi_T(\boldsymbol{x};\, t) + \boldsymbol{w} \cdot \nabla \Phi_T(\boldsymbol{x};\, t) = 0. \tag{3.1}$$

The determination of $\boldsymbol{w}$ thus involves the use of the Faraday law, that deals with the rate of change of the magnetic fluxes. The calculation is, however, somewhat tricky and must be done with great care. We closely follow here the treatment of Hinton and Hazeltine (1976).

The calculation involves several changes of coordinates between laboratory coordinates and flux coordinates; therefore, when a partial derivative with respect to time appears in a formula, the variables that are held constant must be carefully specified. In order to avoid heavy notations, we make the convention that a time-derivative is always calculated by keeping constant the particular spatial coordinate written explicitly in the symbol of the function. Thus, $\partial_t \Phi_T(\boldsymbol{x};\, t)$ in eq. (3.1) denotes a time-derivative at constant $\boldsymbol{x}$, i.e. at a fixed point in the laboratory frame. If the toroidal flux is expressed in terms of the poloidal flux [or, equivalently, of the flux coordinate $\psi = (2\pi)^{-1}\Phi_P$ used in section **8**.8],

$$\Phi_T(\boldsymbol{x};\, t) \to \tilde{\Phi}_T[\psi(\boldsymbol{x},\, t);\, t] \equiv \Phi_T(\psi;\, t),$$

then $\partial_t \Phi_T(\psi;\, t)$ denotes a time-derivative at constant ψ. With this convention, we note that

$$\partial_t \psi(\boldsymbol{x};\, t) = \partial_t \psi(\Phi_T;\, t) + \frac{\partial \psi(\Phi_T;\, t)}{\partial \Phi_T}\, \partial_t \Phi_T(\boldsymbol{x};\, t).$$

Thus

$$\partial_t \Phi_T(\boldsymbol{x};\, t) = \left(\frac{\partial \psi}{\partial \Phi_T}\right)^{-1} \left[\partial_t \psi(\boldsymbol{x};\, t) - \partial_t \psi(\Phi_T;\, t)\right]. \tag{3.2}$$

Using also the obvious relation

$$\nabla \Phi_T = \left(\frac{\partial \Phi_T}{\partial \rho} \right) \nabla \rho,$$

we find from (3.1)

$$\begin{aligned} \boldsymbol{w} \cdot \nabla \rho &= \left(\frac{\partial \Phi_T}{\partial \rho} \frac{\partial \psi}{\partial \Phi_T} \right)^{-1} \left[\partial_t \psi(\rho; t) - \partial_t \psi(\boldsymbol{x}; t) \right] \\ &= \frac{1}{R_0 \mathscr{B}_P} \left[\partial_t \psi(\rho; t) - \partial_t \psi(\boldsymbol{x}; t) \right], \end{aligned} \tag{3.3}$$

where we used definitions (**8**.8.26) and (**8**.8.30) and expressed ψ as a function of ρ rather than of Φ_T.

We now evaluate the two terms on the right-hand side of (3.3). We start from definition (**8**.3.6) and choose for Σ_P a *fixed* surface, as defined in fig. **8**.3.1. Using the Faraday law (**3**.4.24), we find

$$\partial_t \psi(\boldsymbol{x}; t) = \partial_t (2\pi)^{-1} \int_{\Sigma_P} \mathrm{d}\boldsymbol{S}_\theta \cdot \boldsymbol{B} = -\frac{c}{2\pi} \int_{\Sigma_P} \mathrm{d}\boldsymbol{S}_\theta \cdot (\nabla \wedge \boldsymbol{E}).$$

Applying Stokes' theorem, we transform the surface integral into a line integral taken along the magnetic axis which, for an axisymmetric system, is simply a circle,

$$\partial_t \psi(\boldsymbol{x}; t) = -\frac{c}{2\pi} \int \mathrm{d}\boldsymbol{s}_\zeta \cdot \boldsymbol{E}_\zeta = \frac{c}{2\pi} \int_0^{2\pi} \mathrm{d}\zeta \, l_\zeta E_\zeta,$$

and finally

$$\partial_t \psi(\boldsymbol{x}; t) = c l_\zeta E_\zeta. \tag{3.4}$$

The evaluation of the first term in (3.3) must be done in a somewhat indirect way. We start with the trivial identity

$$\partial_t \Phi_T(\psi; t) + \frac{\partial \Phi_T}{\partial \psi} \partial_t \psi(\Phi_T; t) \equiv \partial_t \Phi_T(\Phi_T; t) = 0.$$

(The time derivative of Φ_T at constant Φ_T is, of course, zero!) Thus, using also definition (**8**.4.19) of the safety factor q,

$$\partial_t \psi(\Phi_T; t) = -(2\pi q)^{-1} \partial_t \Phi_T(\psi; t). \tag{3.5}$$

We now calculate the rate of change of the toroidal flux through a surface of constant poloidal flux (this surface is moving in the laboratory frame). Using definition (**8**.3.3) and lemma (2.1), we find

$$\partial_t \Phi_T(\psi;\, t) = (2\pi)^{-1}\, \partial_t \int_\psi \mathrm{d}^3\boldsymbol{x}\, \boldsymbol{B}\cdot\nabla\zeta$$

$$= (2\pi)^{-1} \int_\psi \mathrm{d}^3\boldsymbol{x}\, (\partial_t \boldsymbol{B})\cdot\nabla\zeta + (2\pi)^{-1} \int \mathrm{d}\boldsymbol{S}_\psi \cdot \boldsymbol{W}(\boldsymbol{B}\cdot\nabla\zeta)$$

$$\equiv T_1 + T_2, \tag{3.6}$$

where $\boldsymbol{W}$ is the velocity of a surface of constant poloidal flux,

$$\partial_t \psi(\boldsymbol{x};\, t) + \boldsymbol{W}\cdot\nabla\psi = 0. \tag{3.7}$$

Comparing this equation with (3.4), we find

$$\boldsymbol{W}\cdot\nabla\psi \equiv l_\psi^{-1} W_\psi = -c l_\zeta E_\zeta.$$

Recalling also (eq. **G2**.2.13) that $\mathrm{d}\boldsymbol{S}_\psi = l_\theta l_\zeta\, \mathrm{d}\theta\, \mathrm{d}\zeta\, \boldsymbol{e}_\psi$, we find the expression of the term T_2,

$$T_2 = (2\pi)^{-1} \int \mathrm{d}\theta\, \mathrm{d}\zeta\, l_\theta l_\zeta (-c l_\psi l_\zeta E_\zeta) l_\zeta^{-1} B_\zeta,$$

or

$$T_2 = -\frac{c}{2\pi} \int \mathrm{d}\theta\, \mathrm{d}\zeta\, l_\psi l_\theta l_\zeta E_\zeta B_\zeta. \tag{3.8}$$

The evaluation of the term T_1 starts with the toroidal component of the Faraday equation,

$$(\partial_t \boldsymbol{B})\cdot\nabla\zeta = -c(\nabla\wedge\boldsymbol{E})\cdot\nabla\zeta = -c\nabla\cdot(\boldsymbol{E}\wedge\nabla\zeta).$$

This equation is integrated over a surface of constant ψ and transformed by using the divergence theorem,

$$T_1 = -\frac{c}{2\pi} \int_\psi \mathrm{d}^3\boldsymbol{x}\, \nabla\cdot(\boldsymbol{E}\wedge\nabla\zeta) = -\frac{c}{2\pi} \int \mathrm{d}\boldsymbol{S}_\psi \cdot (\boldsymbol{E}\wedge\nabla\zeta)$$

$$= -\frac{c}{2\pi} \int \mathrm{d}\theta\, \mathrm{d}\zeta\, l_\theta l_\zeta\, \boldsymbol{e}_\psi \cdot \left(\boldsymbol{E}\wedge l_\zeta^{-1}\boldsymbol{e}_\zeta\right) = -\frac{c}{2\pi} \int \mathrm{d}\zeta\, \mathrm{d}\theta\, l_\theta E_\theta$$

$$= -\frac{c}{2\pi} \int \mathrm{d}\theta\, \mathrm{d}\zeta\, l_\psi l_\theta l_\zeta\, (l_\psi l_\zeta)^{-1} E_\theta.$$

Recalling (**8**.8.2) we find

$$T_1 = -\frac{c}{2\pi}\int \mathrm{d}\theta\, \mathrm{d}\zeta\, l_\psi l_\theta l_\zeta E_\theta B_\theta.$$

We now substitute this result, together with (3.8) into (3.6) and find

$$\partial_t \Phi_T(\psi;\, t) = -\frac{c}{2\pi}\int \mathrm{d}\theta\, \mathrm{d}\zeta\, l_\psi l_\theta l_\zeta \boldsymbol{B}\cdot\boldsymbol{E} = -\frac{c}{2\pi} V'\langle BE_\parallel\rangle.$$

Hence, using (3.5), we find

$$\partial_t \psi(\rho;\, t) = \frac{c}{(2\pi)^2 q} V'\langle BE_\parallel\rangle. \tag{3.9}$$

Finally, with (**8**.8.24), we get

$$\partial_t \psi(\rho;\, t) = \frac{c}{\mathscr{J}\left\langle l_\zeta^{-2}\right\rangle}\langle BE_\parallel\rangle. \tag{3.10}$$

This result generates another interesting formula by differentiation with respect to ρ and use of definition (**8**.8.30),

$$\partial_t \mathscr{B}_P(\rho;\, t) = c\frac{\partial}{\partial\rho}\frac{1}{\mathscr{J}R_0\left\langle l_\zeta^{-2}\right\rangle}\left\langle BE_\parallel^{(A)}\right\rangle. \tag{3.11}$$

This equation has the form of the Faraday law in cylindrical coordinates, corrected by the appropriate geometrical factors.

The two results (3.4) and (3.10) are now substituted into (3.3) and lead to the final result

$$\boldsymbol{w}\cdot\nabla\rho \equiv l_\rho^{-1} w_\rho = \frac{c}{R_0\mathscr{B}_P}\left(\frac{\left\langle BE_\parallel^{(A)}\right\rangle}{\mathscr{J}\left\langle l_\zeta^{-2}\right\rangle} - l_\zeta E_\zeta^{(A)}\right). \tag{3.12}$$

In these formulae we replaced $\boldsymbol{E}$ by $\boldsymbol{E}^{(A)}$ because we know that, to leading order in ϵ, both the parallel and the toroidal components of the electric field are solely due to the externally induced field. The result (3.12) can be found in the article by Bernstein (1974), but first appeared in completely explicit form in the work by Hinton and Hazeltine (1976).

Our next objective is the evaluation of the Joule heating term $\langle \boldsymbol{j}\cdot\boldsymbol{E}\rangle$. We note again that, as $j_\rho = 0$ to leading order, the radial potential electric field $(-\mathrm{d}\Phi/\mathrm{d}\rho)$ does not contribute to this quantity. We thus have

$$\langle \boldsymbol{j}\cdot\boldsymbol{E}\rangle = \left\langle j_\theta E_\theta^{(A)}\right\rangle + \left\langle j_\zeta E_\zeta^{(A)}\right\rangle. \tag{3.13}$$

But j_ζ is related to j_θ through the MHD equilibrium relation (**8**.8.11),

$$j_\zeta = B_\theta^{-1}\left[j_\theta B_\zeta - c l_\rho^{-1} P'(\rho)\right]. \tag{3.14}$$

Substituting into (3.13) we find

$$\langle \boldsymbol{j}\cdot\boldsymbol{E}\rangle = \left\langle j_\theta \frac{B}{B_\theta} E_\parallel^{(A)}\right\rangle - \left\langle \frac{E_\zeta^{(A)}}{l_\rho B_\theta}\right\rangle cP'. \tag{3.15}$$

Finally, using (**8**.8.32) and (**8**.8.34) we find

$$\langle \boldsymbol{j}\cdot\boldsymbol{E}\rangle = -\frac{c}{4\pi R_0 \mathscr{B}_\mathrm{P}}\left(\mathscr{I}'(\rho)\left\langle BE_\parallel^{(A)}\right\rangle + 4\pi P'(\rho)\left\langle l_\zeta E_\zeta^{(A)}\right\rangle\right). \tag{3.16}$$

But the toroidal component of the electric field is, in turn, related to the parallel component through (3.12),

$$\left\langle l_\zeta E_\zeta^{(A)}\right\rangle = \frac{\left\langle BE_\parallel^{(A)}\right\rangle}{\mathscr{I}\left\langle l_\zeta^{-2}\right\rangle} - \frac{R_0\mathscr{B}_\mathrm{P}}{cl_\rho}\langle w_\rho\rangle. \tag{3.17}$$

The Joule heating term thus becomes

$$\langle \boldsymbol{j}\cdot\boldsymbol{E}\rangle = -\frac{c}{4\pi R_0 \mathscr{B}_\mathrm{P}\mathscr{I}}\left(\frac{4\pi}{\left\langle l_\zeta^{-2}\right\rangle}P' + \mathscr{I}\mathscr{I}'\right)\left\langle BE_\parallel^{(A)}\right\rangle + l_\rho^{-1}\langle w_\rho\rangle P'. \tag{3.18}$$

The last term can be interpreted as the work done by the expanding magnetic surface against the pressure gradient. In order to transform the first term into a more explicit form, we use (3.14) in combination with (**8**.8.32), (**8**.8.34) and find

$$l_\zeta^{-1} j_\zeta = -\frac{c}{4\pi R_0 \mathscr{B}_\mathrm{P}}\left(l_\zeta^{-2}\mathscr{I}\mathscr{I}' + 4\pi P'\right). \tag{3.19}$$

Surface-averaging this relation and substituting the result into (3.18), we get the final form

$$\langle \boldsymbol{j}\cdot\boldsymbol{E}\rangle = \frac{\langle l_\zeta^{-1} j_\zeta\rangle}{\mathscr{I}\langle l_\zeta^{-2}\rangle}\langle BE_\parallel^{(A)}\rangle + l_\rho^{-1}\langle w_\rho\rangle\, P'(\rho). \tag{3.20}$$

This result is substituted into the electron energy equation (2.31), which becomes

$$\begin{aligned}
&\tfrac{3}{2}V'^{-2/3}\,\partial_t\big[V'^{5/3}P_e(\rho;\,t)\big] \\
&\quad = -\frac{\partial}{\partial\rho}\Big[V' l_\rho^{-1}\big\langle q_\rho^e + \tfrac{5}{2}P_e(u_\rho^e - w_\rho)\big\rangle\Big] \\
&\qquad - V'\big(\langle u_\rho^i - w_\rho\rangle\nabla_\rho P_i\big) + V'\frac{\langle l_\zeta^{-1} j_\zeta\rangle}{\mathscr{I}\langle l_\zeta^{-2}\rangle}\langle BE_\parallel^{(A)}\rangle \\
&\qquad - V' T_i^{-1}(\nabla_\rho T_i)\big[\langle q_\rho^i\rangle_B + L_{33}^{iiB}(\nabla_\rho T_i)\big] \\
&\qquad + 3V'\frac{m_e}{m_i}\frac{n_e}{\tau_e}(T_i - T_e).
\end{aligned} \tag{3.21}$$

There is one important feature that is apparent in this final form, as well as in the continuity equation (2.10) and in the ion energy equation (2.30): the radial electron and ion fluxes $n_\alpha\langle u_\rho^\alpha\rangle$ always appear in the combination $n_\alpha\langle u_\rho^\alpha - w_\rho\rangle$, i.e. as *the fluxes relative to the moving magnetic surface*.

We now calculate the relative electron flux, by taking into account the *total* average radial particle flux (table **12**.9.1), including also the electric drift and modified drift contributions, which were left out of the discussion till now. Putting all these terms together by using eqs. (**13**.1.1), (**13**.1.6) and (3.12), we find for the relative electron flux

$$\begin{aligned}
n_e\big[\langle u_\rho^e\rangle - \langle w_\rho\rangle\big] &= \langle\Gamma_\rho^e\rangle_{CL} + \langle\Gamma_\rho^e\rangle_{PS} + \langle\Gamma_\rho^e\rangle_B \\
&\quad + n_e c\left[\left\langle\frac{E_\wedge^{(A)}}{B}\right\rangle + \frac{\mathscr{I} l_\rho}{R_0\mathscr{B}_P}\left\langle BE_\parallel^{(A)}\left(\frac{1}{B^2} - \frac{1}{\mathscr{B}_0^2}\right)\right\rangle\right. \\
&\quad \left. + \frac{l_\rho}{R_0\mathscr{B}_P}\frac{\langle BE_\parallel^{(A)}\rangle}{\mathscr{I}\langle l_\zeta^{-2}\rangle} + \frac{l_\rho}{R_0\mathscr{B}_P}\langle l_\zeta E_\zeta^{(A)}\rangle\right].
\end{aligned}$$

Using (**12**.5.7), we note that

$$\left\langle l_\zeta E_\zeta^{(A)} \right\rangle = \mathscr{J} \left\langle \frac{E_\parallel^{(A)}}{B} \right\rangle + \frac{R_0 \mathscr{B}_P}{l_\rho} \left\langle \frac{E_\wedge^{(A)}}{B} \right\rangle, \tag{3.22}$$

hence

$$n_e \left\langle u_\rho^e - w_\rho \right\rangle = \left\langle \Gamma_\rho^e \right\rangle + n_e c \mathscr{J} \left[\frac{1}{\mathscr{B}_0^2} - \frac{1}{\mathscr{J}^2 \left\langle l_\zeta^{-2} \right\rangle} \right] \left\langle B E_\parallel^{(A)} \right\rangle$$

$$\equiv \left\langle \Gamma_\rho^e \right\rangle + \langle \gamma_\rho \rangle. \tag{3.23}$$

[The symbol $\mathscr{J}$ was defined in eq. (**13**.3.12).] Let us stress the following convention which is used here in order to simplify the notations.

– We write the *total average radial electron flux, **including** the electric drift and modified drift fluxes* as $n_e \langle u_\rho^e \rangle$.

– Consistently with the practice of the previous chapters, we denote by $\langle \Gamma_\rho^e \rangle$ the *sum of the classical, PS and banana fluxes*, for which we possess explicit transport equations,

$$\left\langle \Gamma_\rho^e \right\rangle \equiv \left\langle \Gamma_\rho^e \right\rangle_{CL} + \left\langle \Gamma_\rho^e \right\rangle_{PS} + \left\langle \Gamma_\rho^e \right\rangle_B. \tag{3.24}$$

– Thus, $\langle \gamma_\rho \rangle$ denotes the difference between the total average radial flux *relative to the magnetic surface* and the sum of the three classical and neoclassical fluxes; $\langle \gamma_\rho \rangle$ is entirely controlled by the parallel electric field. It can be written in the form (see eq. **12**.5.3)

$$\langle \gamma_\rho \rangle = n_e c \mathscr{J} \left(\frac{1}{\mathscr{B}_0^2} - \frac{1}{\left\langle B_\zeta^2 \right\rangle} \right) \left\langle B E_\parallel^{(A)} \right\rangle. \tag{3.25}$$

Clearly, the term between parentheses is quite small, because the magnetic field is directed almost in the toroidal direction; this smallness is partly compensated by the factor $\mathscr{J}$, which represents the ratio B_ζ / B_θ.

We see that the total average electron flux relative to the magnetic surface is entirely expressed in terms of the four thermodynamic forces $\nabla_\rho P$, $\nabla_\rho T_e$, $\nabla_\rho T_i$ and $\langle B E_\parallel^{(A)} \rangle$. In particular, the components $\langle E_\wedge^{(A)}/B \rangle$, $\langle E_\parallel^{(A)}/B \rangle$ have cancelled out of the expressions.

The only quantity remaining in eq. (3.21) that has not yet been expressed in terms of thermodynamic forces is the contravariant toroidal component of the

electric current, $\langle l_\zeta^{-1} j_\zeta \rangle$. Indeed, the quantity entering the transport equations is the parallel component of the current, or more precisely, $\langle B j_\parallel \rangle$.

The electric current is, among the other fluxes, a somewhat special quantity that is "*sitting between two chairs*". On one hand, it is related to the electric field and to the spatial pressure and temperature gradients through a transport equation, i.e. a generalized Ohm law; on the other hand, it is related to the magnetic field inhomogeneities through the Ampère law. We now explore the latter aspect.

Projecting the Ampère equation (**4**.1.9) on the toroidal direction, we obtain, using in succession eqs. (**G2**.2.17), (**12**.5.3), (**G2**.2.16), (**8**.6.9),

$$\left\langle l_\zeta^{-1} j_\zeta \right\rangle \equiv \langle \boldsymbol{j} \cdot \nabla \zeta \rangle = \frac{c}{4\pi} \langle (\nabla \wedge \boldsymbol{B}) \cdot \nabla \zeta \rangle$$

$$= \frac{c}{4\pi} \left\langle \frac{1}{l_\rho l_\theta} \left(\frac{\partial}{\partial \rho} l_\theta B_\theta \right) \frac{1}{l_\zeta} \right\rangle = \frac{c}{4\pi} \left\langle \frac{1}{l_\rho l_\theta l_\zeta} \frac{\partial}{\partial \rho} l_\theta \frac{R_0 \mathscr{B}_P}{l_\zeta l_\rho} \right\rangle$$

$$= \frac{c}{4\pi} \left\langle \nabla \cdot \left(\frac{R_0 \mathscr{B}_P}{l_\zeta^2 l_\rho} \boldsymbol{e}_\rho \right) \right\rangle = \frac{c}{4\pi} \frac{1}{V'} \frac{\mathrm{d}}{\mathrm{d}\rho} V' \left\langle \frac{1}{R_0 \mathscr{B}_P} \left(\frac{R_0 \mathscr{B}_P}{l_\rho l_\zeta} \right)^2 \right\rangle.$$

Finally, using (**12**.5.3),

$$\left\langle l_\zeta^{-1} j_\zeta \right\rangle = \frac{c}{4\pi R_0 V'} \frac{\mathrm{d}}{\mathrm{d}\rho} \frac{V'}{\mathscr{B}_P} \left\langle B_\theta^2 \right\rangle. \tag{3.26}$$

The poloidal component could be evaluated in a similar way; we rather calculate, however, the more directly required quantity $\langle B j_\parallel \rangle$ by using (**12**.5.6),

$$\langle B j_\parallel \rangle = R_0 \mathscr{B}_P \left\langle \frac{j_\theta}{l_\rho l_\zeta} \right\rangle + \mathscr{I} \left\langle l_\zeta^{-1} j_\zeta \right\rangle. \tag{3.27}$$

The first term is evaluated by using (**8**.8.34) and (**8**.8.32):

$$R_0 \mathscr{B}_P \left\langle \frac{j_\theta}{l_\rho l_\zeta} \right\rangle = -\frac{c}{4\pi} \frac{\mathscr{I}'}{R_0 \mathscr{B}_P} \left\langle \left(\frac{R_0 \mathscr{B}_P}{l_\rho l_\zeta} \right)^2 \right\rangle = -\frac{c}{4\pi} \frac{\mathscr{I}'}{R_0 \mathscr{B}_P} \left\langle B_\theta^2 \right\rangle. \tag{3.28}$$

These equations can be exploited in two ways. Combining them with (3.26), we obtain

$$\langle Bj_{\|}\rangle = \frac{c}{4\pi}\left(-\frac{\mathscr{I}'}{R_0\mathscr{B}_P}\langle B_\theta^2\rangle + \frac{\mathscr{I}}{R_0V'}\frac{\mathrm{d}}{\mathrm{d}\rho}\frac{V'}{\mathscr{B}_P}\langle B_\theta^2\rangle\right),$$

or, more compactly,

$$\langle Bj_{\|}\rangle = \frac{c}{4\pi}\frac{\mathscr{I}^2}{R_0V'}\frac{\mathrm{d}}{\mathrm{d}\rho}\frac{V'}{\mathscr{I}\mathscr{B}_P}\langle B_\theta^2\rangle. \tag{3.29}$$

This result is due to Hinton and Hazeltine (1976). Alternatively, one obtains a relation between toroidal and parallel components of the electric current,

$$\langle l_\zeta^{-1}j_\zeta\rangle = \mathscr{I}^{-1}\langle Bj_{\|}\rangle + \frac{c}{4\pi}\frac{\mathscr{I}'}{\mathscr{I}R_0\mathscr{B}_P}\langle B_\theta^2\rangle. \tag{3.30}$$

This relation completes the closure problem of the one-dimensional plasmadynamical equations; $\langle Bj_{\|}\rangle$ obeys a transport equation and the remaining term is related to the magnetic field geometry that is self-consistently determined by the Ampère law and by the equilibrium MHD relations (i.e. the Grad–Shafranov equation).

Because of the "two-chairs" position of the electric current, there have been many discussions in the plasma physics literature about the interpretation of the pair of quantities $\langle Bj_{\|}\rangle$ and $\langle BE_{\|}^{(A)}\rangle$. It has been proposed (Braginskii 1965, Hinton and Hazeltine 1976) that the electric current be given the status of a thermodynamic force and the electric field the status of a flux. The argument in favour of this interpretation is that the externally created magnetic field determines the electric current through eq. (3.29); this relation is "close to" Hinton and Hazeltine's (1976, p. 301) definition of a force as "the derivative of an even moment with respect to the topological radius" (note, however, that $\langle B_\theta^2\rangle$ is *not* a moment of the distribution function!). The electric field is then determined by inverting the transport equations (the electric field does not correspond at all to the Hinton–Hazeltine definition of a flux as "an average of certain odd moments over a magnetic surface"!).

I personally believe that such discussions are only marginally relevant, as they are more related to terminology than to physics. Nevertheless, in my opinion, the classification of the electric field as a force and of the electric current as a flux is the only one that makes physical sense. We deal here with the only thermodynamic force that is a "real" force (in the mechanical sense); the gradients of pressure and temperature are only "generalized" forces.

Moreover, in connection with kinetic theory, it is clear that the electric current is a simple linear combination of the particle fluxes and should not be treated differently from the latter (on the other hand, the average electric field is not a moment of the distribution function, but an external parameter).

Table 3.1
Equations of one-dimensional plasmadynamics in toroidal geometry

Continuity equation

$$\partial_t\left(V' n_e(\rho, t)\right) = -\frac{\partial}{\partial\rho}\left[V' l_\rho^{-1}\left(\left\langle \Gamma_\rho^e\right\rangle + \langle \gamma_\rho\rangle\right)\right] \tag{1}$$

Energy equations

$$\begin{aligned}
&\tfrac{3}{2} V'^{-2/3}\,\partial_t\left[V'^{5/3} P_e(\rho, t)\right] \\
&= -\frac{\partial}{\partial\rho}\left\{V' l_\rho^{-1}\left[\left\langle q_\rho^e\right\rangle + \tfrac{5}{2} T_e\left(\left\langle \Gamma_\rho^e\right\rangle + \langle \gamma_\rho\rangle\right)\right]\right\} \\
&\quad - \frac{V'}{Z n_i}\left(\left\langle \Gamma_\rho^e\right\rangle + \langle \gamma_\rho\rangle\right)\nabla_\rho P_i + \frac{V'}{\mathscr{J}^2\left\langle l_\zeta^{-2}\right\rangle}\left[\langle B j_\parallel\rangle + \frac{c}{4\pi R_0 \mathscr{B}_P}\left\langle B_\theta^2\right\rangle\right]\left\langle B E_\parallel^{(A)}\right\rangle \\
&\quad - V' T_i^{-1}\left(\nabla_\rho T_i\right)\left[\left\langle q_\rho^i\right\rangle_B + L_{33}^{iiB}\left(\nabla_\rho T_i\right)\right] + 3V'\frac{m_e}{m_i}\frac{n_e}{\tau_e}\left(T_i - T_e\right)
\end{aligned} \tag{2}$$

$$\begin{aligned}
\tfrac{3}{2} V'^{-2/3}\,\partial_t\left[V'^{5/3} P_i(\rho, t)\right] &= -\frac{\partial}{\partial\rho}\left\{V' l_\rho^{-1}\left[\langle q_\rho^i\rangle + \tfrac{5}{2} Z^{-1} T_i\left(\left\langle \Gamma_\rho^e\right\rangle + \langle \gamma_\rho\rangle\right)\right]\right\} \\
&\quad + \frac{V'}{Z n_i}\left(\left\langle \Gamma_\rho^e\right\rangle + \langle \gamma_\rho\rangle\right)\nabla_\rho P_i + V' T_i^{-1}\left(\nabla_\rho T_i\right)\left[\left\langle q_\rho^i\right\rangle_B\right. \\
&\quad \left. + L_{33}^{iB}\left(\nabla_\rho T_i\right)\right] - 3V'\frac{m_e}{m_i}\frac{n_e}{\tau_e}\left(T_i - T_e\right)
\end{aligned} \tag{3}$$

Fluxes

$$\left\langle \Gamma_\rho^e\right\rangle = \left\langle \Gamma_\rho^e\right\rangle_{CL} + \left\langle \Gamma_\rho^e\right\rangle_{PS} + \left\langle \Gamma_\rho^e\right\rangle_B \tag{4}$$

$$\left\langle q_\rho^\alpha\right\rangle = \left\langle q_\rho^\alpha\right\rangle_{CL} + \left\langle q_\rho^\alpha\right\rangle_{PS} + \left\langle q_\rho^\alpha\right\rangle_B, \quad \alpha = e, i \tag{5}$$

$$\langle B j_\parallel\rangle = \langle B j_\parallel\rangle_{CL} + \langle B j_\parallel\rangle_B \tag{6}$$

$$\langle \gamma_\rho\rangle = n_e c \mathscr{J}\left(\frac{1}{\mathscr{B}_0^2} - \frac{1}{\left\langle B_\zeta^2\right\rangle}\right)\left\langle B E_\parallel^{(A)}\right\rangle \tag{7}$$

Once the true transport equations are obtained, together with the Ampère law, nothing prevents us from considering *macroscopic problems* in which the electric current is given and the electric field is adjusted to the latter in agreement with the transport equations.

In conclusion, we collect all the equations of motion of the one-dimensional plasmadynamics of toroidally confined plasmas in table 3.1.

18.4. Discussion of the toroidal plasmadynamical equations

We have now reached our goal. We have obtained the plasmadynamical equations for a toroidally confined plasma: a set of three partial differential equations for the scalar quantities $n_e(\rho; t)$, $P_e(\rho; t)$, $P_i(\rho; t)$, i.e. the density and pressure (or temperature) profiles.

The first striking feature is that the equations have been reduced to *a single spatial dimension*, the topological radius ρ. Note that, besides the assumption of *axisymmetry*, no assumption was made about the shape of the cross-section of the magnetic surfaces. Thus, the topological radius allows us to make the best use of the specific symmetry of the problem. Note, however, that we deal here with a *topological symmetry* (which is also a *physical symmetry*), rather than an "Euclidian symmetry". Only by using the topological radius (or any other coordinate defined by the magnetic flux through some section of the magnetic surface) can we reduce the plasmadynamical description to one dimension. Had we used a fixed (laboratory) coordinate system, even a well adapted one such as a cylindrical coordinate system (R, z, ζ) around the symmetry axis, the transformation formula would be of the form $\rho \rightarrow \rho(R, z; t)$ where the function $\rho(R, z; t)$ should be obtained from the Grad–Shafranov equation. The point is that the resulting plasmadynamical equations necessarily depend on *two* "laboratory" coordinates, R, z; only the toroidal angle ζ could be eliminated by using the "Euclidian" axisymmetry.

The next important feature is that the average radial electron flux always appears in the combination

$$\langle \Gamma_\rho^e \rangle + \langle \gamma_\rho \rangle \equiv \langle n_e (u_\rho^e - w_\rho) \rangle .$$

In other words, the electron diffusive flux is always referred to the velocity of the moving magnetic surface. This is another feature demonstrating the consistency of the neoclassical transport theory.

It is interesting in this connection to recall that all the neoclassical fluxes were derived in chapter 12 under the condition of *mechanical equilibrium*. The latter entered as an explicit solubility constraint in eqs. (**12**.6.6), (**12**.6.15),

(**12**.7.1) and (**12**.8.31). We then quote a theorem due to Prigogine (1947) (see also de Groot & Mazur 1982, ch. V): *Under the condition of mechanical equilibrium, the diffusion fluxes Γ^α can be referred to an arbitrary reference velocity* $\boldsymbol{w}$ *without changing the entropy production.*

The origin of the various terms of the plasmadynamical equations in table 3.1 can be easily understood. The continuity equation contains the familiar divergence of the electron flux. On the right-hand side of the electron energy equation we find in succession:
- the divergence of the total enthalpy flux;
- the work done against the pressure gradient;
- the Joule heating term;
- the specific effect of the banana fluxes, i.e. of the dissipative pressure tensor;
- the electron-ion collisional heat exchange.

The overall energy conservation is automatically ensured by the two equations (2) and (3) of table 3.1.

We now resume the discussion initiated at the end of section **12**.4. It is clearly apparent that the structure of the set of equations (1)–(3) given in table 3.1 is markedly different from the set (7.1.1)–(7.1.6). In the latter case we have essentially a *continuity equation*, a *momentum balance equation* (i.e. a generalized Navier–Stokes equation) and *two energy balance equations*, completed with the appropriate transport equations. In table 3.1, *the momentum balance equation is absent.* There is no Navier–Stokes equation in this picture, hence no need of a transport equation for the pressure tensor (i.e. a set of viscosity coefficients).

Both radial particle fluxes are *diffusive* and are determined by two *algebraic transport equations* in terms of the thermodynamic forces (or, equivalently, by a transport equation for the electron flux and the ambipolarity constraint). There is, of course, no contradiction between the two points of view: the transport equations are obtained by a separate *asymptotic solution* of the momentum balance equations [combined with the equations for the other vector moments see chapter 12)]; this asymptotic solution is then used for the closure of the plasmadynamical equations (see also the discussion in section **5**.2).

As a result of this description, *the viscosity effects are included in the diffusive particle and heat fluxes.*

We now devise the strategy for the solution of the transport problem. In principle, the scheme goes as follows:

– The electron flux and the two heat fluxes are determined by eqs. (4) and (5) (table 3.1) in which the explicit transport equations are substituted from eqs. (**13**.2.8), (**13**.3.9) and (**15**.3.11), with the transport coefficients taken from

tables **13**.2.1, **13**.3.1 and **15**.3.4. Alternatively, for small inverse aspect ratio η, one may use the interpolation formulae of section **16**.6, in order to cover the entire range of temperatures and densities. We thus obtain the expression of the fluxes in terms of the three 'truly thermodynamic" forces, the *total radial pressure gradient* and the two *radial temperature gradients*, as well as the *parallel electric field*.

– We now need to eliminate the electric field: this is done by using the generalized Ohm law, i.e. the transport equation (6) (table 3.1), combined with eqs. (**13**.2.8) and (**15**.3.11). This introduces the *parallel electric current* as a new unknown.

–The two relevant components of the electric current are determined by the Ampère law: eqs. (3.26), (3.29) in terms of the magnetic field configuration.

– The latter is essentially determined by the poloidal flux $2\pi\psi$ which obeys eq. (3.10) and the Grad–Shafranov equation.

The closure scheme is thus completed. It may be noted that the strategy of the closure outlined above is essentially the same as the one described in section 7.2 for the resistive MHD problem.

Clearly, the implementation of this program for the actual calculation of the density and temperature profiles is very complex and can only be done by devising refined computer codes for solving the set of equations in table 3.1. Several such codes have been developed and are an indispensable tool for the interpretation of the fusion experiments. This aspect lies outside of the scope of this book.

There is one case in which the equations are considerably simplified: it is (not surprisingly!) the case of the *standard model*, i.e. the limit of a very small inverse aspect ratio, $\eta \ll 1$. The simplifications are the following.

– The topological radius ρ radius reduces to the geometrical radius r and the scale factor $l_\rho = 1$ (see section **8**.9).

– The motion of the magnetic surfaces is negligible (see section **13**.4). As a result, $w_\rho \approx 0$ and the volume V' is time-independent: $V' = V'(r) = 4\pi^2 R_0 r$.

– The Joule heating term in eq. (3.21) is easily evaluated by using the results of section **8**.9 and **13**.4.

The plasmadynamical equations then reduce to the form given in table 4.1. Not surprisingly, they have the general form of the balance equations for *cylindrical symmetry* (see eq. (1.2). In the small inverse aspect ratio limit, a very thin torus is, indeed, locally approximated by a piece of a cylinder. It should be stressed, however, that the non-trivial effects of the toroidal curvature are retained, even in this limit, through the form of the Pfirsch–Schlüter and the banana transport equations.

18.5. The neoclassical confinement times

Having achieved the construction of the neoclassical transport theory, a discussion of the experimental evidence is necessary. This discussion will be rather brief, because the situation is still in a rather confusing stage and the gap between the theoretical and experimental results is pretty wide.

The ideal confrontation between theory and experiment would require *local* measurements of the particle and heat fluxes and of the pressure and temperature profiles, as well as of the magnetic field configuration. Although much work has been done (and is still in progress) in the diagnostic techniques of the tokamaks (for instance), these techniques are not yet adequate for such a detailed description. As a result, there exist at present detailed experimental data on the profiles, but not on the fluxes. It should be added, as a caution to be kept in mind whenever detailed comparisons are made, that most of the diagnostic techniques are *indirect*. By this statement, I mean that the inference of the final quantity from the raw experimental data requires a theoretical interpretation that contains many simplifying assumptions, whose validity should be checked in each case.

Table 4 1
Equations of one-dimensional plasmadynamics in the standard model.

Continuity equation

$$\partial_t n_e(r;t) = -\frac{1}{r}\frac{\partial}{\partial r} r\langle \Gamma_r^e \rangle$$

Energy equations

$$\tfrac{3}{2}\partial_t P_e(r;t) = -\frac{1}{r}\frac{\partial}{\partial r} r\Big(\langle q_r^e\rangle + \tfrac{5}{2}T_e\langle \Gamma_r^e\rangle\Big) - \frac{1}{Zn_i}\langle \Gamma_r^e\rangle \nabla_r P_i + \frac{cE_0}{4\pi r}\frac{\partial}{\partial r} r\mathscr{B}_P - T_i^{-1}(\nabla_r T_i)\Big(\langle q_r^i\rangle_B + L_{33}^{iiB}\nabla_r T_i\Big) + 3\frac{m_e}{m_i}\frac{n_e}{\tau_e}(T_i - T_e)$$

$$\tfrac{3}{2}\partial_t P_i(r;t) = -\frac{1}{r}\frac{\partial}{\partial r} r\big(\langle q_r^i\rangle + \tfrac{5}{2}Z^{-1}T_e\langle \Gamma_r^e\rangle\big) + \frac{1}{Zn_i}\langle \Gamma_r^e\rangle \nabla_r P_i + T_i^{-1}(\nabla_r T_i)\Big(\langle q_r^i\rangle_B + L_{33}^{iiB}\nabla_r T_i\Big) - 3\frac{m_e}{m_i}\frac{n_e}{\tau_e}(T_i - T_e)$$

Given this situation, the test is made by comparing the experimental results of some *global* (rather than *local*) quantities with the theoretical predictions from the transport equations. The most extensively studied quantities of this kind are the *particle and energy confinement times*, usually denoted by τ_P and τ_E, respectively. These quantities are of considerable practical importance, as a figure of merit of the efficiency of the magnetic confinement in a certain configuration. They are expressed in terms of some global quantities, such as the major radius R_0, the minor radius a, the central density n_0, the central temperatures $T_{\alpha 0}$, the externally induced current I, etc. Such relations between the confinement times and the global characteristics of the plasma are called *scaling laws*.

The *particle confinement time* τ_P is rather naturally defined as

$$\tau_P = \frac{\text{number of ions in the plasma}}{\text{ion loss rate through the plasma boundary}}. \tag{5.1}$$

The *energy confinement time for the electrons (ions)* $\tau_{E\alpha}$ is correspondingly defined as

$$\tau_{E\alpha} = \frac{\text{electron (ion) thermal energy in the plasma}}{\text{electron (ion) energy loss through the boundary}}. \tag{5.2}$$

The *global energy confinement time* τ_E is

$$\tau_E = \frac{\text{total thermal energy in the plasma}}{\text{total energy loss rate through the boundary}}. \tag{5.3}$$

The particle and energy content are uniquely defined whenever the density and pressure profiles are known. For the denominators, the situation is not so simple. The particle loss rate is due to the presence of a radial particle flux $\langle \Gamma_\rho^\alpha \rangle$ across the magnetic surfaces and is thus directly obtainable from transport theory; but it may also contain other sources and sinks, such as ionization and recombination processes or neutral beam injection. We disregard such processes, which were not discussed in this book. Similarly, the energy loss rate is due to radial energy fluxes, but also to radiative losses and to additional (non-Ohmic) heating sources (such as RF heating); the latter factors will also be disregarded here. An interesting brief discussion of the difficulties and the ambiguities encountered in the definition of these quantities can be found in the article by Furth (1981). With the simplifications

defined above, definitions (5.1)–(5.3) are translated as

$$\tau_{\mathrm{P}} = \frac{\int \mathrm{d}^3\boldsymbol{x}\, n_i}{\int \mathrm{d}\boldsymbol{S}_\rho \cdot \boldsymbol{\Gamma}^i}, \tag{5.4}$$

$$\tau_{\mathrm{E}\alpha} = \frac{\int \mathrm{d}^3\boldsymbol{x}\, \frac{3}{2} P_\alpha}{\int \mathrm{d}\boldsymbol{S}_\rho \cdot \left(\boldsymbol{q}^\alpha + \frac{5}{2} T_\alpha \boldsymbol{\Gamma}^\alpha\right)}, \tag{5.5}$$

$$\tau_{\mathrm{E}} = \frac{\int \mathrm{d}^3\boldsymbol{x}\, \frac{3}{2}(P_{\mathrm{e}} + P_{\mathrm{i}})}{\int \mathrm{d}\boldsymbol{S}_\rho \cdot \left[\boldsymbol{q}^{\mathrm{e}} + \boldsymbol{q}^{\mathrm{i}} + \frac{5}{2}(T_{\mathrm{e}} + Z^{-1} T_{\mathrm{i}})\boldsymbol{\Gamma}^{\mathrm{e}}\right]}. \tag{5.6}$$

The volume integrals extend over the whole region occupied by the plasma; the surface integrals are taken over the boundary of that region. In terms of our usual toroidal coordinates, and recalling definitions (**G2**.2.14), (**G2**.2.13), we obtain

$$\tau_{\mathrm{P}} = \frac{\int_0^a \mathrm{d}\rho \int \mathrm{d}\theta\, \mathrm{d}\zeta\, l_\rho l_\theta l_\zeta\, n_{\mathrm{i}}(\rho)}{\int \mathrm{d}\theta\, \mathrm{d}\zeta\, l_\theta l_\zeta\, \Gamma^{\mathrm{i}}_\rho(\rho = a,\, \theta)},$$

or else, using (**8**.8.35),

$$\tau_{\mathrm{P}} = \frac{\int_0^a \mathrm{d}\rho\, V'(\rho)\, n_{\mathrm{i}}(\rho)}{V'(\rho) \left\langle l_\rho^{-1} \Gamma^{\mathrm{i}}_\rho \right\rangle \Big|_{\rho = a}}. \tag{5.7}$$

Similarly, we have

$$\tau_{\mathrm{E}} = \frac{\int_0^a \mathrm{d}\rho\, V'(\rho)\, \frac{3}{2} P(\rho)}{V'(\rho) \sum_\alpha \left\langle l_\rho^{-1} \left(q^\alpha_\rho + \frac{5}{2} T_\alpha \Gamma^\alpha_\rho\right)\right\rangle \Big|_{\rho = a}}. \tag{5.8}$$

The calculation of the numerators in these expressions requires the knowledge of the density and pressure profiles, i.e. the solution of the plasmadynamical equations given in table 3.1. This can only be precisely done by using a sophisticated computer code. Such codes have been developed in all the fusion laboratories, and expressions (5.7), (5.8) are the main output of these calculations. Here we shall limit ourselves to simple, but very rough estimates of the confinement times (see Stacey 1981, Dolan 1982). We assume that the standard model is applicable.

In order to estimate the numerators, we make the very usual approximation of a parabolic profile for the scalar quantities,

$$n_i(r) \approx n_{i0}\left[1 - \gamma_n (r^2/a^2)\right],$$

$$T_\alpha(r) \approx T_{\alpha 0}\left[1 - \gamma_{T\alpha} (r^2/a^2)\right],$$

$$P(r) \approx P_0\left[1 - \gamma_P (r^2/a^2)\right], \tag{5.9}$$

where γ_n, $\gamma_{T\alpha}$, γ_P are constants, close to one. We then easily find, using (**8**.9.11),

$$\int_0^a \mathrm{d}r\, V'(r)\, n_i(r) = 4\pi^2 R_0 \tfrac{1}{2}\left(1 - \tfrac{1}{2}\gamma_n\right) a^2 n_{i0},$$

$$\int_0^a \mathrm{d}r\, V'(r)\, P(r) = 4\pi^2 R_0 \tfrac{1}{2}\left(1 - \tfrac{1}{2}\gamma_P\right) a^2 P_0. \tag{5.10}$$

Following the usual practice, we make a very rough simplification of the transport equations, by retaining only the *diagonal processes* and neglecting all cross-effects. We then find, from (**15**.3.11) (with $l_\rho = 1$), using again (5.9) and assuming for simplicity $Z = 1$ and $T_e = T_i$:

$$V'\left\langle l_r^{-1} \Gamma_r^e \right\rangle\Big|_{r=a} \approx V' L_{11}^{ee}\, X_1^e|_{r=a} = -V' L_{11}^{ee} \frac{2}{P} \frac{\mathrm{d}}{\mathrm{d}r} P(r)\,|_{r=a}$$

$$= 4\pi^2 R_0 a \frac{4\gamma_P}{1-\gamma_P} L_{11}^{ee} \frac{1}{a} = 4\pi^2 R_0 \frac{4\gamma_P}{1-\gamma_P} L_{11}^{ee}. \tag{5.11}$$

For the energy loss rate, we assume that it is dominated by the ion heat transport (see section **15**.5) and retain again only the diagonal ion thermal conductivity,

$$V' \sum_\alpha \left\langle l_\rho^{-1}\left(q_\rho^\alpha + \tfrac{5}{2} T_\alpha \Gamma_r^\alpha\right)\right\rangle \approx 4\pi^2 R_0 \frac{4\gamma_{Ti}}{1-\gamma_{Ti}} L_{33}^{ii}. \tag{5.12}$$

Putting all these results into (5.7), (5.8) we find the following rough estimates:

$$\tau_P \sim a^2 \frac{n_{i0}}{L_{11}^{ee}(r=a)}, \qquad \tau_E \sim a^2 \frac{P_0}{L_{33}^{ii}(r=a)}. \tag{5.13}$$

If we use the expressions given in table **15**.3.4 for the banana transport coefficients, combined with (**15**.5.8) in the standard model, together with definitions (**13**.2.20) for ρ_{e0}^2, and (**4**.6.4) for τ_e, we find (disregarding numerical factors)

$$L_{11}^{ee} \sim q^2 \eta^{-3/2} \frac{nT}{m_e \Omega_{e0}^2} \frac{Z^2 e^4 \ln \Lambda}{m_e^{1/2} T^{3/2}} n,$$

$$L_{33}^{ii} \sim q^2 \eta^{-3/2} \frac{nT}{m_i \Omega_{i0}^2} \frac{Z^4 e^4 \ln \Lambda}{m_i^{1/2} T^{3/2}} n.$$

We thus obtain, by extracting only the relevant factors,

$$\tau_P \sim \mathscr{B}_0^2 \eta^{3/2} a^2 T^{1/2} n^{-1},$$

$$\tau_E \sim \mathscr{B}_0^2 \eta^{3/2} a^2 T^{1/2} n^{-1}. \tag{5.14}$$

In these two formulae, the proportionality constants are different, but the combinations of relevant parameters is the same. Equations (5.14) define the *neoclassical scaling law*. Relations (5.13) have been evaluated with the banana transport coefficients; we note, however, that *the Pfirsch–Schlüter coefficients* (table **13**.4.1) *yield the same dependence on* $\mathscr{B}_0$, a, T and n.

The neoclassical theory thus predicts the following features:

- the confinement times increase like $\mathscr{B}_0^2$;
- a long confinement time requires a large and fat torus (proportional to $a^2 \eta^{3/2}$);
- the confinement times go like the square root of the temperature and are inversely proportional to the density.

The comparison with the experiments is not very favourable. The scaling of τ_E as $\mathscr{B}_0^2$, which was a great hope of the first generation of fusion physicists, does not seem to be supported by experiment: the data appear to be closer to a scaling proportional to $\mathscr{B}_0$ (*Bohm scaling*). This implies the necessity of much larger magnetic fields for reaching the critical Lawson criterion, $n\tau_E \approx 10^{14}\,\mathrm{cm}^{-3}$ s for the start-up of a fusion reactor.

The dependence on temperature and density also seems to be wrong. An empirical scaling law that appears to cover pretty well many experimental results from Ohmically heated tokamaks is the *Alcator scaling law*,

$$\tau_E \sim a^2 n. \tag{5.15}$$

It should be added, however, that if we consider separately the *ion energy confinement time*, the neoclassical scaling law appears to be in agreement with the experimental data within a factor of order one. Thus the discrepancy is due to an *anomalously large electron thermal conductivity*.

The conclusion of this brief confrontation is that *the neoclassical transport theory covers only partly the variety of possible transport mechanisms, especially for the electrons*. It was clear from the beginning that the main assumptions of this theory are very restrictive. In particular, the existence of a quiet plasma confined in a magnetic field having a regular structure of nested toroidal magnetic surfaces imposes very stringent constraints.

A real plasma tends to develop instabilities which, eventually, drive it into a *turbulent state*. These instabilities affect the shape of the magnetic surfaces which may eventually be completely disrupted and reconnected in a topologically quite different way (*tearing modes*). Such a strong distortion of the magnetic surfaces necessarily introduces dramatic changes in the transport phenomena. The mobile electrons will, of course, be the first "victims" of these catastrophies.

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19

The limitations of the classical and neoclassical transport theories. The runaway effect

19.1. The conceptual framework of classical and neoclassical transport theories

We now turn back to have a brief global look at the corpus of classical and neoclassical transport theories which was developed in this book. We are in a position where we can identify some of the common features underlying these theories.

A. The plasma is described by a finite number (typically: two) of distribution functions for the electrons and the various species of ions.

This implies that the mechanisms at work in this transport regime can be entirely described in terms of individual particle interactions, i.e. essentially *binary collisions*. This means that we consider relatively "quiescent" systems. All the *collective aspects* of the plasma are disregarded here. These consist of oscillatory motions, or "waves", involving the synchronized action of a large number of particles. Such strongly organized behaviour is possible because of the long range of the Coulomb interactions. These waves may become unstable under certain circumstances; as their amplitude grows, a number of new types of interactions of these collective entities as such with the particles or among themselves determine the distribution of the energy among all the partners as well as the evolution in time. The *spectral distribution* of the waves becomes an indispensable ingredient in the description of such a "*turbulent plasma*".

B. The distribution functions are, at all times, very close to a local Maxwellian distribution.

This is another consequence of the dominance of the individual particle collisions in the dynamics of the plasma. These collisions maintain the plasma at all times close to the local equilibrium. It must be stressed again that this assumption underlies even the long mean free path regime of the neoclassical theory.

C. The transport phenomena can be evaluated by determining a finite number of Hermitian moments of the distribution functions.

In other words, the infinite set of moment equations which is, in principle, equivalent to the kinetic equations, can be truncated at some finite, usually low, level. This implies that in the classical or neoclassical regimes, the particle and energy fluxes are relatively insensitive to the finer details of the distribution functions. In particular, the "tail" of the distributions, which contains the very few, very energetic particles, does not appreciably influence the macroscopic properties of the plasma. This tail is not explored by a representation in terms of a finite number of moments.

D. The truncated set of moment equations is linearized in some appropriate way.

The truncated set of moment equations is still a set of non-linear differential equations. These are linearized by some procedure, adapted to the problem at hand. The latter specificity makes the treatment non-trivial, as can be seen by comparing the classical and the neoclassical theories.

E. Starting from any initial state, the system "quickly" reaches a quasi-steady state in which the fluxes are related to the thermodynamic forces by a set of linear transport equations.

At this step, the differential equations for the moments are transformed into a set of algebraic equations. This feature is brought about by the disparity existing between the collisional relaxation time and the hydrodynamical time.

F. In a magnetically confined plasma, the magnetic field has the topological structure of a set of nested tori.

This feature is essential to the neoclassical transport theory, to which it provides an underlying reference structure. All the quantities are defined as averages taken over an individual magnetic surface. The radial fluxes of matter and energy traverse these surfaces without seriously perturbing them. Whenever some instability breaks this structure, the transport properties are considerably affected.

It is remarkable to note that within the very constraining limits of these assumptions, there exists such a rich variety of transport phenomena. It was the purpose of this book to explore some of these aspects; even these could not be exhaustively covered and we had to limit ourselves to the most characteristic processes. But we can imagine a whole realm of new phenomena appearing in regimes where the assumptions listed above are no longer appropriate.

We intend to illustrate the possible breakdown of this conceptual framework by considering a simple phenomenon: the *runaway effect*. This study is

of interest in itself, in particular for its impact in fusion physics (and also in astrophysics). But it also provides a very natural transition towards the *anomalous transport phenomena* which will, however, not be treated in this volume.

19.2. The runaway effect

Let us return to the classical theory of transport in an unconfined plasma, as developed in chapter 5. The main assumption there was the linearization of the moment equations, introduced in section **5**.1. In the present section, we shall study one of the simplest problems which clearly exhibits the limitations of the linearization procedure.

We stress the following point. We are not interested here in calculating some small corrections to the linear transport coefficients. This aspect of the problem is rather easily done; in principle, it introduces no conceptual difficulties. Our main concern will be to exhibit the *qualitatively new features introduced by the non-linear character of the moment equations*. These can be most conspicuously demonstrated in a very simplified situation, which we presently analyze.

Consider an unconfined electron–ion plasma, which we assume to be *spatially homogeneous*. An electric field $\boldsymbol{E}$ and a magnetic field $\boldsymbol{B}$, both spatially homogeneous, are present. We shall only consider the fluxes in the direction *parallel* to the magnetic field; hence the latter actually plays no role in the determination of the fluxes. This problem was first solved in a classical paper by Dreicer (1959).

In the situation described here, the only thermodynamic force at work is the *parallel electric field* $E_\parallel$. It drives its conjugate flux, i.e. the *parallel electric current* $j_\parallel$. Note that it also drives a heat flux through the thermoelectric Peltier effect, i.e. through the non-diagonal transport coefficient (see section **5**.5). This aspect, though quite interesting, was not studied in the literature till now and will not be discussed here. We thus concentrate on the determination of the parallel electric current $j_\parallel$ or, equivalently, of the dimensionless moment $h_\parallel^{(1)}$. This quantity obeys eq. (**4**.5.12) which reduces, under our assumptions, to

$$\partial_t h_\parallel^{(1)} = \left(\frac{m_e}{T_e}\right)^{1/2} \frac{e}{m_e} E_\parallel + Q_\parallel^{(1)}\left(h_\parallel^{(1)}\right). \tag{2.1}$$

The main difference between the present problem and the one treated in chapter 5 will be in the treatment of the friction term $Q_\parallel^{(1)}$: we recall its definition by eqs. (**3**.4.9), (**4**.5.7), (**4**.5.17). It is a functional of the distribution

functions $f^{\alpha}(\boldsymbol{v};\ \boldsymbol{x},\ t)$ which, under our assumptions (neglect of the Hermitian moments) reduce to local Maxwellians:

$$f^{\alpha}(\boldsymbol{v};\ \boldsymbol{x},\ t) = n_{\alpha}\left(\frac{m_{\alpha}}{2\pi T_{\alpha}}\right)^{3/2} \exp\left(-\frac{m_{\alpha}}{2T_{\alpha}}\,|\,\boldsymbol{v}-\boldsymbol{u}_{\alpha}\,|^{2}\right). \tag{2.2}$$

When these functions are introduced in the collision term, there appears a non-zero contribution to (3.4.9) from the unlike-particle collisions, because of the different average velocities and temperatures of the electrons and the ions. This implies the existence of a non-vanishing electric current $\boldsymbol{j} = e(Zn_{\mathrm{i}}\boldsymbol{u}_{\mathrm{i}} - n_{\mathrm{e}}\boldsymbol{u}_{\mathrm{e}})$ in this state. The situation is similar to the one encountered in section **11**.1, where we found such a non-zero source term in the linearized collision operator; but here we have an additional current which was not considered in section **11**.1.

Our purpose is to consider the effect of the *exact* friction term, without making any linearization. This implies that we do not make use of the Lorentz process as in section **2**.8, but rather calculate the friction term by using the full electron–ion Landau collision term (**2**.6.24). The calculation can be done exactly when the distribution function has the simple form (2.2), and is further simplified by the assumption

$$T_{\mathrm{e}} \gg T_{\mathrm{i}}. \tag{2.3}$$

Skipping the details of the calculation, which are very similar to those of section **11**.1 (see also Dreicer 1959), we quote the result

$$Q_{\parallel}^{(1)} = -\frac{4\pi Z^{2}e^{4}n_{\mathrm{i}}\ \ln\Lambda}{m_{\mathrm{e}}^{1/2}T_{\mathrm{e}}^{3/2}}\,\mathscr{G}\left(\frac{h_{\parallel}^{(1)}}{\sqrt{2}}\right)\frac{h_{\parallel}^{(1)}}{h^{(1)}}, \tag{2.4}$$

where $\ln\Lambda$ is the familiar Coulomb logarithm (**4**.6.5).

The function $\mathscr{G}(x)$ was first introduced by Chandrasekhar (1943) in his classical studies of the Fokker–Planck equation applied to stellar dynamics [at the same time as the function $\mathscr{H}(x)$ considered in chapter 11]. It is related as follows to the *error function* $\Phi(x)$ (eq. **11**.4.9) and to its derivative $\Phi'(x)$:

$$\mathscr{G}(x) = \frac{1}{2x^{2}}\left[\Phi(x) - x\Phi'(x)\right]. \tag{2.5}$$

[Note that the function $\Psi(x)$ used by Dreicer (1959) is $\Psi(x) = 2\mathscr{G}(x)$].

Equation (2.4) is also conveniently expressed in terms of the electron relaxation time (**4**.6.4),

$$Q_{\parallel}^{(1)} = -3\sqrt{\frac{\pi}{2}}\,\frac{1}{\tau_e}\,\mathcal{G}\left(h_{\parallel}^{(1)}/\sqrt{2}\right)\frac{h_{\parallel}^{(1)}}{h^{(1)}}. \tag{2.6}$$

We now introduce these results into the generalized Ohm law (2.1). We make the additional (unessential) assumption that the electric field is parallel to the magnetic field; the electric current is then necessarily directed along $\boldsymbol{B}$ and we have to consider a single component of the vector equation (2.1). We simplify the notations as follows:

$$E_{\parallel} \to E, \qquad h_{\parallel}^{(1)} \to h,$$

and write (2.1) in the form

$$\partial_t h = \left(\frac{m_e}{T_e}\right)^{1/2}\frac{e}{m_e}E - 3\sqrt{\frac{\pi}{2}}\,\frac{1}{\tau_e}\,\mathcal{G}\left(h/\sqrt{2}\right). \tag{2.7}$$

The equation takes a more transparent form if we define a quantity having the dimension of an electric field, but related to the friction force (hence to the collision term),

$$E_c = 3\sqrt{\frac{\pi}{2}}\,\frac{(m_e T_e)^{1/2}}{e\tau_e}, \tag{2.8}$$

or, alternatively,

$$E_c = \frac{4\pi Z^2 e^3 n_i \ln \Lambda}{T_e}. \tag{2.9}$$

This quantity is called the *critical electric field* or the *Dreicer field* [actually, Dreicer's original definition differs by a factor 2 from eq. (2.9) because of his use of the function $\Psi(x)$ instead of the Chandrasekhar function $\mathcal{G}(x)$; this does not affect the final results; definition (2.9) is used in most of the more recent works]. Equation (2.7) now becomes

$$\partial_t h = \left(\frac{m_e}{T_e}\right)^{1/2}\frac{e}{m_e}\left[E - E_c\,\mathcal{G}\left(h/\sqrt{2}\right)\right]. \tag{2.10}$$

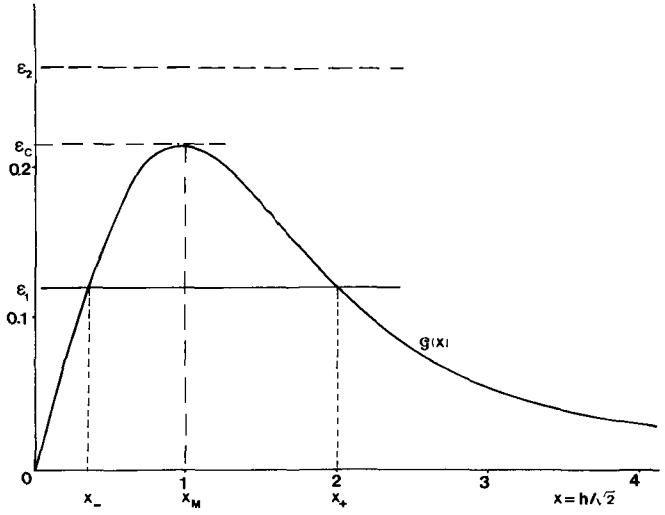

Fig. 2.1. The Chandrasekhar function $\mathcal{G}(x)$ and the graphical determination of the stationary solutions of eq (2.17).

The problem is reduced to the study of a relatively simple, but *non-linear differential equation* for the dimensionless electric current h. *

The crux of the discussion of eq. (2.10) lies in the shape of the Chandrasekhar function $\mathcal{G}(x)$ (see fig. 2.1). The function starts linearly near the origin, where the following expansion is valid:

$$\mathcal{G}(x) \sim \frac{2}{\sqrt{\pi}}\left(\tfrac{1}{3}x - \tfrac{1}{5}x^2 + \tfrac{1}{14}x^5 + \cdots\right), \quad x \ll 1. \tag{2.11}$$

The function, however, does not grow indefinitely: it rather reaches a maximum at $x = 1$, and then starts decreasing towards zero. For large values of x it may be represented as

$$\mathcal{G}(x) \sim \frac{1}{2x^2} - \frac{1}{\sqrt{\pi}}\mathrm{e}^{-x^2}\left(\frac{1}{x} + \frac{1}{2x^3} - \frac{1}{4x^5} + \cdots\right), \quad x \gg 1. \tag{2.12}$$

This peculiar behaviour of the friction force in a plasma (or in a gravitational system) was first pointed out by Chandrasekhar (1943). A charged test particle moving in a plasma suffers a dynamical friction which, for small

* We assume here that the temperature is constant in time, a not altogether consistent assumption, which will be further discussed below.

velocity, is practically proportional to the velocity: this is the "normal" behaviour. But if the velocity of the test particle increases beyond the thermal velocity, the friction due to the medium becomes less and less effective. For very high velocities, the particle moves almost unimpeded through the plasma. Let us now examine how this effect is reflected in the behaviour of the electric current.

We first assume that the electric field is very weak ($E \ll E_c$) and that the electric current it produces is also very weak. We then approximate the Chandrasekhar function by its linear term in (2.11); eq. (2.10) then reduces to the linear equation

$$\tau_e \, \partial_t h = g - h, \tag{2.13}$$

where we reintroduced our usual source term $g \equiv g_{\parallel}^{(1)}$ defined in (5.1.12),

$$g = \tau_e \left(\frac{m_e}{T_e} \right)^{1/2} \frac{e}{m_e} E. \tag{2.14}$$

This equation is precisely identical to the first equation of the set (5.1.7) or (5.2.1) (taken in the parallel direction, and neglecting the Hermitian moments $h_{\parallel}^{e(p)}$, $p \geqslant 3$; we recall that $c_{11}^{e} = 1$). We thus recover, as we should, the linear transport theory. The behaviour of the solutions of (2.13) was studied in a more complicated case in section 5.2. Here the solution is simply

$$h(t) = h(0) \, \mathrm{e}^{-t/\tau} + \frac{1}{\tau} \int_0^t \mathrm{d}\theta \, \mathrm{e}^{-(t-\theta)/\tau} \, g(\theta) \tag{2.15}$$

(where $\tau \equiv \tau_e$). We recover the familiar scenario: starting from some arbitrary initial condition, the current tends towards a unique asymptotic limit, whose value is independent of the initial condition

$$h_{st} = \tilde{\sigma}_{\parallel} \, g, \tag{2.16}$$

with $\tilde{\sigma}_{\parallel} = 1$ ($\equiv c_{11}^{e}$). (The inaccurate value of the parallel conductivity is, of course, due to the neglect of all the Hermitian moments; it might be called the "5M approximation".)

Having recovered the linear result, we now look more globally at the non-linear equation (2.10). We introduce a "scaled" time τ,

$$\tau = \left(\frac{m_e}{T_e} \right)^{1/2} \frac{e}{m_e} E_c \, t, \tag{2.17}$$

and write $\dot{h} = \mathrm{d}h/\mathrm{d}\tau$,

$$\dot{h} = \epsilon - \mathscr{G}\left(h/\sqrt{2}\right) \tag{2.18}$$

with

$$\epsilon \equiv E/E_c. \tag{2.19}$$

Although eq. (2.18) cannot be integrated analytically, we may inquire whether it possesses a *stationary solution*. The answer to this question depends on the value of the "*control parameter*" ϵ, i.e. on the strength of the electric field. It is provided by the solution of the transcendental equation

$$\epsilon - \mathscr{G}\left(h/\sqrt{2}\right) = 0. \tag{2.20}$$

The solutions of this equation can be understood by a graphical method (fig. 2.1). They are obtained as the intersections of the graph of $\mathscr{G}(x)$ with a horizontal line drawn at level ϵ.

For $\epsilon < \mathscr{G}(1)$ there exist *two* solutions to (2.20). The smaller one, $h_{(-)}$, continues the solution of the linear problem (2.16) [from which it deviates slightly because of the non-linearity of the ascending branch of $\mathscr{G}(x)$]. The second solution, $h_{(+)}$, can be found approximately by substituting the asymptotic form (2.12) into (2.20):

$$h_{(-)} \approx \epsilon^{-1/2}. \tag{2.21}$$

This is clearly an unphysical solution: it expresses an "anticonductivity" process in which the current decreases as the field increases; moreover, it leads to an infinite current at zero field. This solution must clearly be discarded.

As ϵ increases, the two solutions move closer together, until they coalesce when ϵ reaches the critical value,

$$\epsilon = \epsilon_c = \mathscr{G}(1) \approx 0.2138. \tag{2.22}$$

This point is, mathematically speaking, a *bifurcation point*. When ϵ increases beyond this value, *a stationary solution no longer exists*. In other words, when the electric field exceeds the value

$$E > 0.2138\, E_c, \tag{2.23}$$

the behaviour of the electric current bears no resemblance to the linear transport theory. It no longer reaches a steady state, in which its value is a

(linear *or* non-linear) functional of the electric field, as predicted by the Ohm law. On the contrary, the current grows indefinitely in time; (almost) nothing counteracts the acceleration of the charged particles by the electric field: for very large values of ϵ, when $\mathscr{G}(x)$ can be altogether neglected, the solution of (2.18) becomes *

$$h(t) \sim \frac{e}{(m_e T_e)^{1/2}} E t, \qquad \epsilon \gg \mathscr{G}(1). \tag{2.24}$$

This behaviour of the electric current for $\epsilon > \mathscr{G}(1)$ is called the *runaway effect.* It was first discovered by Dreicer (1959).

Of course, the present simple model can be refined in various ways. In particular, one may take into account the fact that the temperature is actually not constant. Indeed, the Joule dissipation produces a gradual heating; as a result, even in absence of runaway, the quasi-steady value of the current is actually slowly rising. We do not wish, however, to discuss these aspects here, but rather look at a different aspect of the problem.

19.3. Microscopic aspects of the runaway effect

The result of the previous section can be formulated as follows: Whenever the electric field exceeds a threshold value [such that $\epsilon > \mathscr{G}(1)$], the electric current, i.e. essentially the electron *average* velocity can no longer reach a stationary value. The main reason is the fact that the collision frequencies $\nu^{\alpha\beta}(v)$ in a plasma are decreasing functions of the speed of the particles (see fig. **11**.4.2). This property induces the corresponding decrease of the friction force for large velocities.

Thus, on the *macroscopic level*, there exists a clear-cut separation between a "normal" region, for $\epsilon < \mathscr{G}(1)$, where a steady current exists, and a "runaway" region for $\epsilon > \mathscr{G}(1)$, where no stationary current is possible.

If, instead of the average behaviour, we now consider the *microscopic particle motions*, the perspective radically changes. We fix our attention on a specified test electron, moving with velocity v through the plasma. It is submitted to the accelerating external field $\boldsymbol{E}$ and to the action of the

* Clearly, the acceleration described by (2.24) cannot proceed for arbitrarily long times, lest the average electron velocity exceeds the speed of the light. The consistent long-time treatment of the runaway regime should be relativistic; this was indeed done by Connor and Hastie (1975).

collisions; one of the effects of the latter is to produce a *dynamical friction* (see eqs. **11**.1.9, **11**.1.10) which is easily calculated, as in section 19.2,

$$\boldsymbol{F}(\boldsymbol{v}) = \frac{4\pi Z^2 e^4 n_i \ln \Lambda}{m_e} \mathscr{G}\left(\frac{v}{(2T_e/m_e)^{1/2}}\right) \frac{\boldsymbol{v}}{v}$$

$$\equiv eE_c \mathscr{G}\left(\frac{v}{(2T_e/m_e)^{1/2}}\right) \frac{\boldsymbol{v}}{v}. \tag{3.1}$$

This expression presupposes, of course, the model of section 19.2, viz. a Maxwellian electron distribution and cold ions, $T_e \gg T_i$.

Because of the asymptotic decrease of function $\mathscr{G}(x)$, if the velocity of the test particle is sufficiently high, the dynamic friction can no longer oppose the electric field and the particle will be indefinitely accelerated, i.e. it will run away. *For any given electric field*, this will happen whenever the velocity exceeds a *critical velocity* v_c, at which the external field exactly balances the descending branch of the dynamical friction,

$$eE = eE_c \mathscr{G}\left(\frac{v_c}{(2T_e/m_e)^{1/2}}\right) \approx eE_c \frac{T_e}{m_e} v_c^{-2}, \tag{3.2}$$

where we used approximation (2.12) for high velocities. Thus

$$v_c = \left(\frac{T_e}{m_e} \frac{E_c}{E}\right)^{1/2}. \tag{3.3}$$

We now consider a plasma in some statistical state, described by a distribution function $f^a(\boldsymbol{v}; \boldsymbol{x}, t)$. It is clear that in any such state, and for a given electric field $\boldsymbol{E}$, there *always exists a finite fraction of electrons whose velocity exceeds* v_c *and which will run away.*

We are thus faced with a paradoxical situation. *Under conditions in which the moment equations admit a stationary solution, the full distribution function necessarily contains a runaway population, and is therefore not stationary.*

As a result of this phenomenon, the electrons whose speed exceeds v_c will be systematically accelerated in the direction opposite to the electric field (because $e_e < 0$) and the distribution function develops a tail for $v_\parallel > v_c$, whose importance grows in time and which makes the distribution anisotropic (see fig. 3.1). This process can be described by a specific quantity $\nu_R(E)$, the *rate of runaway*, defined as the number of runaway electrons ($v_\parallel > v_c$) appearing per unit time in the distribution.

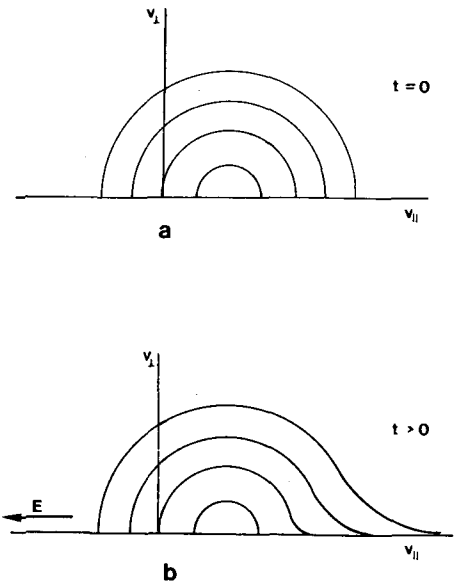

Fig. 3.1. Distortion of the electron distribution function (in the plane $v_{\parallel}$, $v_{\perp}$) in the presence of a d.c. electric field switched on at time $t = 0$ (schematic).

The earliest mention of the runaway electron population seems to appear in connection with the problem of solar flares (Hoyle 1949, Parker 1957). It was later introduced in the context of electric discharges and of fusion physics by Dreicer (1957), Gibson (1957) and by Harrison (1958, 1960), who made a first, oversimplified attempt at a theory. The first non-trivial theoretical formulation of the problem, together with a (not yet very precise) evaluation of the rate of runaway, appears in Dreicer's second classical work (1960). Numerous papers were published later, among which we quote only a few: Gurevich (1961), Gurevich and Zhilyuk (1966), Wiley and Hinton (1980), Fisch and Karney (1985), Karney and Fisch (1986), Fuchs et al. (1986). An excellent and comprehensive review of the problem is due to Knoepfel and Spong (1979). A special mention is due to the very refined analytical works by Kruskal and Bernstein (1964) and by Lebedev (1965). These analytical results were confirmed in the numerical work by Kulsrud et al. (1973).

The problem involves the (asymptotic) solution of the Landau equation in the presence of an electric field. In the simplest model, only electron–ion collisions are considered in the Lorentz approximation ($m_e/m_i \to 0$) for a homogeneous plasma,

$$\partial_t f^e(\boldsymbol{v};\, t) - \frac{e}{m_e}\boldsymbol{E}\cdot\frac{\partial}{\partial \boldsymbol{v}} f^e = A\frac{\partial}{\partial v_r} G_{rs}(\boldsymbol{v})\frac{\partial}{\partial v_s} f^e. \tag{3.4}$$

This is the equation solved by Kruskal and Bernstein (1964) in the *weak field limit* ($E \ll E_c$). The asymptotic solution involves the separation of the velocity space into (at least) three regions:

(a) The main body, in which the distribution function is predominantly determined by the *collision term*, which leads to an almost isotropic function.

(b) The *runaway region* at very large velocities ($v \gg v_c$), where the electric field is the dominant driver.

(c) A *connection region*, in which the electric field and the collisions are comparable and in which a "stretching" method leads to an asymptotic solution (see the treatment of the plateau regime in chapter 16 for an analogous problem).

A proper joining of the three partial solutions leads to a complete asymptotic distribution function, from which the runaway rate can be computed. We will not go into the details of the calculation, but quote directly the result, expressed in terms of the dimensionless parameter ϵ defined in (2.19) *

$$\tilde{\nu}_R \equiv \frac{1}{3}\sqrt{\frac{2}{\pi}}\,\frac{\tau_c}{n_i}\,\nu_R = 0.32\,\epsilon^{-3/8}\exp\left[-\frac{1}{4\epsilon}-\left(\frac{2}{\epsilon}\right)^{1/2}\right]. \tag{3.5}$$

This function is represented in figs. 3.2A, B. The dimensionless runaway rate is very small for small ϵ, and then increases rather sharply, as can be seen in fig. 3.2B. However, its value still remains very small in practice. For a typical tokamak,

$$n_i = 4\times 10^{13}\ \text{cm}^{-3}, \qquad T_e = 1\ \text{keV},$$

$$E = 0.005\ \text{V cm}^{-1}, \qquad \ln\Lambda \simeq 15,$$

we find

$$V_{Te} = 2\times 10^{9}\ \text{cm s}^{-1}, \qquad v_c = 5.7V_{Te},$$

$$\epsilon = 0.03, \qquad \tilde{\nu}_R = 8.9\times 10^{-8}.$$

If we multiply the temperature by 10 ($T_e = 10$ keV), keeping the density and the electric field constant, we find

$$V_{Te} = 6.3\times 10^{9}\ \text{cm s}^{-1}, \qquad v_c = 1.82V_{Te},$$

$$\epsilon = 0.3, \qquad \tilde{\nu}_R = 1.8\times 10^{-2}.$$

* The results published by various authors differ very slightly from each other, the one given here is due to Kulsrud et al (1973), for $Z = 1$.

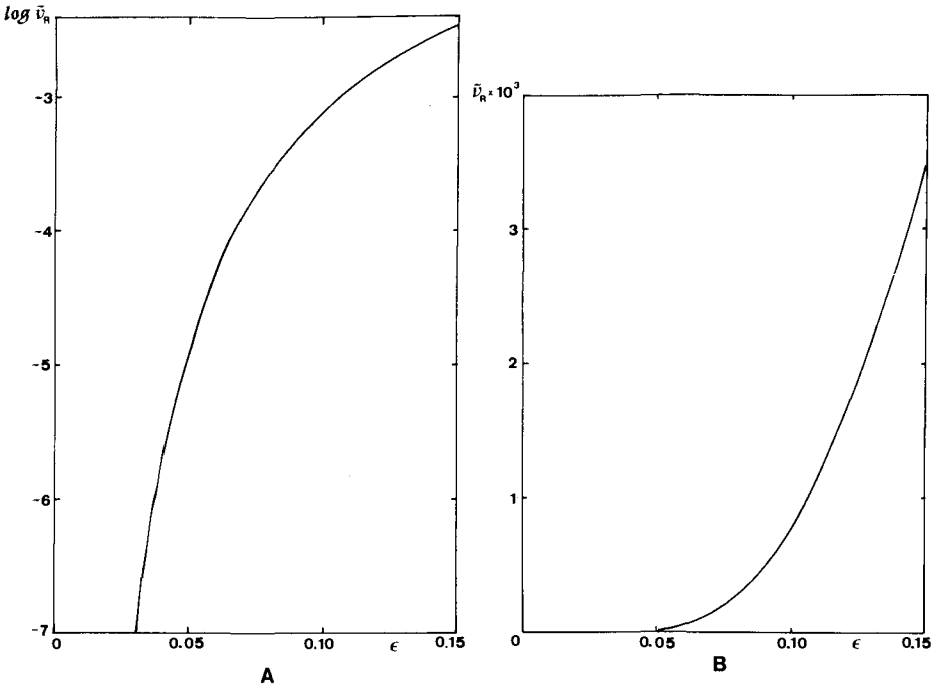

Fig 3.2. Rate of electron runaway (A) logarithmic representation, log $\tilde{\nu}_R(\epsilon)$ vs ϵ; (B) real values of $\tilde{\nu}_R(\epsilon)$, multiplied by 10^3.

If, however, the density also increases by a factor 10, E_c remains the same as in the first case and the first set of values remains applicable.

These results may be summarized as follows.

For very small values of ϵ, the critical velocity is very much larger than the thermal velocity, hence the distorted tail is very far away from the bulk of the Maxwellian. In calculating the average current, the integrand in

$$j_{\parallel} \sim \int_0^{\infty} \mathrm{d}v\, v^2 v f^{\mathrm{e}}(v)$$

is cut down to almost zero by the action of the unperturbed ($\sim$ Maxwellian) part of $f^{\mathrm{e}}(v)$ well before reaching the distorted tail beyond the critical velocity v_c. Thus, the macroscopic current "does not feel" the presence of the distortion. Moreover, the non-stationarity of the distribution is exceedingly weak, due to the smallness of the runaway rate (compared to the collision frequency). Thus, the electric current can be calculated with a (quasi-)stationary distribution function and *all the results of the classical theory are recovered*. The paradox mentioned at the beginning of this section is resolved.

As ϵ *increases*, the distorted, anisotropic part of the distribution moves closer to the bulk and the rate of runaway increases. After a certain point, the distortion starts "biting" more and more deeply into the bulk of the distribution and the runaway production becomes intensive. When the distortion reaches a point (which is still beyond the thermal velocity, $v_c/V_{Te} > 1$) where a non-negligible fraction of the electrons run away, the non-stationarity of the distribution shows up at the macroscopic level, and a steady current is no longer observed. As we know from the previous section, this happens when $\epsilon = \mathscr{G}(1) \approx 0.21$. Beyond that value of the electric field, the distribution no longer bears any resemblance to the unperturbed Maxwellian.

Before closing this section, we mention that the theory of runaway electrons has been generalized to the case of a confined plasma. Here, one must consider, of course, separately the action of the electric field on the trapped and on the passing particles (see chapters 9 and 14). The expression of the runaway rate found by Gurevich & Dimant (1978) is not very different from (3.5),

$$\tilde{\nu}_R = 0.32 \langle \epsilon_\parallel \rangle^{-3/8} \exp\left[-\frac{1}{4\langle \epsilon_\parallel \rangle} - \left(\left\langle \frac{1}{B} \right\rangle \frac{2\langle B \rangle}{\langle \epsilon_\parallel \rangle} \right)^{1/2} \right], \tag{3.6}$$

where

$$\langle \epsilon_\parallel \rangle = \frac{\langle E_\parallel \rangle}{E_c}.$$

The behaviour of runaway electrons in tokamaks and in other confinement devices has been studied quite intensively, both theoretically and experimentally. The review article by Knoepfel and Spong (1979) gives an excellent account of these studies.

19.4. The emergence of anomalous transport

We now have a pretty clear picture of the runaway phenomenon, which is at the root of a limitation of the classical or neoclassical transport theories. Three important features appeared in this discussion. In the presence of a constant electric field that is a sizeable fraction of the critical runaway field E_c,
- the distribution function is no longer quasi-stationary;
- its shape is strongly distorted, compared to a local Maxwellian;
- the electric current is no longer controlled by a transport coefficient. *

* A similar statement actually holds for the electron heat flux as well: the latter depends on the electric field through the thermoelectric coefficient in the classical theory. When the distribution function is distorted by a large electric field, the heat flux will also run away.

This picture is, however, still incomplete, as we presently show by a qualitative discussion. After switching on the (relatively strong) electric field, the runaway electrons (i.e. the tail of the distribution function) build up a *beam* of non-thermal, high-energy particles moving through the (thermal) plasma. Such a situation is potentially *unstable*: this instability is actually a "fortunate" event, because it prevents the plasma from indefinitely running away in time. One possible scenario can be described as follows.

A plasma is able to support (i.e. allow the propagation of) some types of *waves*, which are defined as eigenmodes of the collisionless Vlasov equation. Among these, the most characteristic are the *plasma oscillations* or *Langmuir oscillations*, whose frequency is close to the well-known electron plasma frequency ω_{Pe} (see eq. **2**.6.10). The possibility exists for a *resonance* between the collective waves of frequency ω ($\approx \omega_{Pe}$) and wave vector $k_{\|}$ on one hand, and the high-velocity particles on the other hand, in such a way that $\omega - k_{\|} v_{\|} = 0$. When this condition is satisfied, the runaway particles will systematically generate a spectrum of Langmuir waves. The situation is quite analogous to the Cherenkov effect: a rapid particle moving through a dielectric generates electromagnetic waves in its wake. This is the crux of the instability; the collective plasma waves grow systematically at the expense of the runaway energy. In a later stage, the fast electrons are scattered off the wave spectrum, and the distribution tends to become again isotropic. This "feedback mechanism" tends to *saturate* the initial instability by bringing the system into a new (possibly quasi-stationary) non-equilibrium state that is quite "far" from a Maxwellian [see Knoepfel and Spong (1979) and for a more complete treatment Parail and Pogutse (1978)]. In this non-equilibrium state, the energy is distributed among the individual particles and the collective waves. Because of this feature, the new state is called a *turbulent state*. Clearly, the transport processes occurring in such a situation will be radically different from the neighborhood of the local equilibrium.

This qualitative discussion is grossly oversimplified. It is only meant to open a small window towards a realm of phenomena that are generically described as *anomalous transport*. Indeed, a plasma "does not like" to be forced out of equilibrium or to be confined in a magnetic bottle. It will therefore react to an unstable situation; this reaction may become quite violent, thus leading to a *tearing* of the magnetic surfaces, through which the particles and the energy will leak at a high rate. The plasma then excites a spectrum of collective waves, and this redistribution of energy tends to "calm down" the initial instability. (For a recent review of this type of phenomena, see Kadomtsev 1987.) Fortunately for us, such chaotic phenomena are often localized in space and time. The task of the fusion physicist is to arrive at a detailed understanding of these processes, which allows him to control the phenomena, and in particular, the leaks.

We may also note that anomalous transport is the paradigm of all the complex transport phenomena occurring in interplanetary or interstellar space (for instance, in our own magnetosphere). In these media the dilution of the particles is such that individual collisions (which control the classical and neoclassical transport) are exceedingly rare and the transport processes are exclusively driven by collective effects, in conjunction with the external electric and magnetic fields.

These phenomena will be the subject of the future volume on anomalous transport processes in plasmas.

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Appendix

G2

Curvilinear coordinate systems

G2.1. Non-orthogonal coordinate systems

In the study of magnetic confinement systems one is necessarily faced with the study of non-trivial, "unusual" geometric surfaces. In order to obtain maximum simplicity in the expressions, it is necessary to use a *curvilinear coordinate system* adapted to the shape of the surface. We collect here a number of formulae allowing the reader to deal easily with quantities expressed in such coordinates.

We first define an "*absolute*" *Cartesian rectangular reference frame*, fixed once for all (fig. 1.1). Each point P in space is characterized by its three Cartesian coordinates x, y, z. The triad of unit orthogonal vectors along the three coordinate axes will be denoted by $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$.

Next, we suppose that through each point of the relevant region of space passes a magnetic surface. We introduce a unit vector $\boldsymbol{e}_1$ which is normal to the surface at the given point P. Next, we choose two other unit vectors $\boldsymbol{e}_2$, $\boldsymbol{e}_3$, which are tangent to the surface. Apart from these requirements, there is still much freedom left in the final choice of these vectors, which will be called the *physical basis vectors*. Clearly, from their definition, the orientation of these vectors will be different in each point. By continuity, we may construct a set of three curves passing through P, which are everywhere tangent to $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$, respectively: these are the coordinate lines q^1, q^2, q^3. The latter two curves lie entirely on the magnetic surface passing through P.

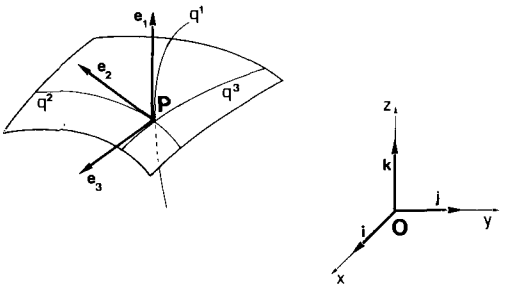

Fig. 1.1 General curvilinear coordinate system adapted to the magnetic surface $q^1 = \text{const.}$

We have constructed in this way a coordinate system, enabling us to characterize each point by the values of q^1, q^2, q^3. These are related to the absolute Cartesian coordinates by a set of invertible functional relations:

$$x = x(q^1, q^2, q^3), \qquad y = y(q^1, q^2, q^3), \qquad z = z(q^1, q^2, q^3), \quad (1.1)$$

$$q^i = q^i(x, y, z), \qquad i = 1, 2, 3. \quad (1.2)$$

In the main text, q^1 is called a *radial* coordinate and is usually denoted by ρ; q^2 corresponds to the *poloidal* coordinate θ and q^3 to the *toroidal* coordinate ζ.

Clear and concise treatments of general, non-orthogonal coordinate systems can be found in classical textbooks, such as the one by Margenau and Murphy (1943) or by Korn and Korn (1968) (see also Fock 1964). Shorter presentations, applied to plasma physics, are found in the article by Soloviev and Shafranov (1970) and in the book by Bateman (1978).

A fundamental formula in the study of any coordinate system is the one expressing in the length ds between two adjacent points:

$$ds^2 = dx^2 + dy^2 + dz^2$$

$$= \sum_{r=1}^{3} \sum_{s=1}^{3} g_{rs}(q^1, q^2, q^3)\, dq^r\, dq^s. \quad (1.3)$$

From (1.1) we find the *(covariant) components of the metric tensor* g_{rs},

$$g_{rs} = \frac{\partial x}{\partial q^r}\frac{\partial x}{\partial q^s} + \frac{\partial y}{\partial q^r}\frac{\partial y}{\partial q^s} + \frac{\partial z}{\partial q^r}\frac{\partial z}{\partial q^s} = g_{sr}. \quad (1.4)$$

We also define the *contravariant components* of the metric tensor g^{rs} as

$$g^{rs} = \frac{\partial q^r}{\partial x}\frac{\partial q^s}{\partial x} + \frac{\partial q^r}{\partial y}\frac{\partial q^s}{\partial y} + \frac{\partial q^r}{\partial z}\frac{\partial q^s}{\partial z} = g^{sr}. \quad (1.5)$$

We then easily check the relation

$$\sum_{n=1}^{3} g_{rn} g^{ns} = \delta_r^s, \quad (1.6)$$

where δ_r^s are the components of the (mixed) unit tensor.

In this book we limit ourselves to coordinate systems characterized by the property that the *determinant g of the metric tensor is positive*:

$$\| g_{rs} \| \equiv g > 0. \quad (1.7)$$

We then clearly have

$$\| g^{rs} \| = g^{-1} > 0. \tag{1.8}$$

We now define two sets of three vectors as

$$\hat{\boldsymbol{e}}_r = \frac{\partial x}{\partial q^r}\boldsymbol{i} + \frac{\partial y}{\partial q^r}\boldsymbol{j} + \frac{\partial z}{\partial q^r}\boldsymbol{k}, \quad r = 1, 2, 3, \tag{1.9}$$

$$\hat{\boldsymbol{e}}^r = \frac{\partial q^r}{\partial x}\boldsymbol{i} + \frac{\partial q^r}{\partial y}\boldsymbol{j} + \frac{\partial q^r}{\partial z}\boldsymbol{k}, \quad r = 1, 2, 3. \tag{1.10}$$

The set of vectors $\hat{\boldsymbol{e}}_r$ are called the *covariant basis vectors* and $\hat{\boldsymbol{e}}^r$ are the *contravariant basis vectors*. From (1.4), (1.5) we immediately derive

$$\hat{\boldsymbol{e}}_r \cdot \hat{\boldsymbol{e}}_s = g_{rs}, \tag{1.11}$$

$$\hat{\boldsymbol{e}}^r \cdot \hat{\boldsymbol{e}}^s = g^{rs}. \tag{1.12}$$

Thus, the basis vectors are *not* unit vectors, and they are *not* mutually orthogonal, except when $g^{rs} = 0$ for $r \neq s$. In that case the coordinate system is called *orthogonal*. Orthogonal coordinates will be treated in section G2.2. We continue here the treatment of the general case.

We note that the equations of the coordinate line q^1 are

$$x = x(q^1;\, q^2,\, q^3), \qquad y = y(q^1;\, q^2,\, q^3), \qquad z = z(q^1;\, q^2,\, q^3), \tag{1.13}$$

where q^2, q^3 are held fixed (similar equations hold for the two other coordinate lines). It follows that the covariant basis vector $\hat{\boldsymbol{e}}_r$ is tangent to the coordinate line q^r. It is therefore parallel to the physical basis vector $\boldsymbol{e}_r$ defined above. But it is *not* a unit vector like $\boldsymbol{e}_r$; rather

$$\boldsymbol{e}_r = \frac{1}{(\hat{\boldsymbol{e}}_r \cdot \hat{\boldsymbol{e}}_r)^{1/2}}\hat{\boldsymbol{e}}_r = \frac{1}{\sqrt{g_{rr}}}\hat{\boldsymbol{e}}_r. \tag{1.14}$$

The contravariant basis vector $\hat{\boldsymbol{e}}^r$ (eq. 1.10) are equivalently defined in a compact form as

$$\hat{\boldsymbol{e}}^r = \nabla q^r. \tag{1.15}$$

They are, in general, not parallel to the convariant basis vectors. The following relations can be checked:

$$\hat{\boldsymbol{e}}^1 = \frac{1}{J}(\hat{\boldsymbol{e}}_2 \wedge \hat{\boldsymbol{e}}_3), \qquad \hat{\boldsymbol{e}}_1 = \frac{1}{J'}(\hat{\boldsymbol{e}}^2 \wedge \hat{\boldsymbol{e}}^3) \tag{1.16}$$

(together with the relations obtained by circular permutations). The factor J is related in a simple way to the Jacobian of the transformation $(x, y, z) \rightarrow (q^1, q^2, q^3)$, and also to the determinant of the metric tensor (1.7):

$$J = \hat{\boldsymbol{e}}_1 \cdot (\hat{\boldsymbol{e}}_2 \wedge \hat{\boldsymbol{e}}_3) = \frac{\partial(x, y, z)}{\partial(q^1, q^2, q^3)} = \sqrt{g}\,. \tag{1.17}$$

The triad $(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3)$ is called *right-handed* when $J > 0$. All coordinate systems used in this book are supposed to be right-handed. Similarly,

$$J' = \hat{\boldsymbol{e}}^1 \cdot (\hat{\boldsymbol{e}}^2 \wedge \hat{\boldsymbol{e}}^3) = \frac{\partial(q^1, q^2, q^3)}{\partial(x, y, z)} = \frac{1}{\sqrt{g}}\,. \tag{1.18}$$

Finally, from (1.16) we find the fundamental relation

$$\hat{\boldsymbol{e}}^r \cdot \hat{\boldsymbol{e}}_s = \delta^r_s. \tag{1.19}$$

This relation, rather than (1.11) and (1.12), generalizes properly the usual orthonormality relation.

For any vector $\boldsymbol{V}$, one defines *covariant components* $\hat{V}_r$ and *contravariant components* $\hat{V}^s$ as

$$\boldsymbol{V} = \hat{V}_r \hat{\boldsymbol{e}}^r = \hat{V}^s \hat{\boldsymbol{e}}_s. \tag{1.20}$$

(From here on, in the remainder of this section, we use the convention of summation over repeated subscripts and superscripts.) Clearly,

$$\hat{V}_j = \boldsymbol{V} \cdot \hat{\boldsymbol{e}}_j, \qquad \hat{V}^j = \boldsymbol{V} \cdot \hat{\boldsymbol{e}}^j, \tag{1.21}$$

and we also have the well-known rule for raising and lowering the indices (see eqs. 1.11, 1.12),

$$\hat{V}_j = g_{jk}\hat{V}^k, \qquad \hat{V}^j = g^{jk}\hat{V}_k. \tag{1.22}$$

The *scalar product* of two vectors must be defined as a bilinear form involving co- and contravariant components,

$$\boldsymbol{V}\cdot\boldsymbol{W}=\hat{V}^j\hat{W}_j=\hat{V}_j\hat{W}^j. \tag{1.23}$$

The *vector product* is defined consistently with (1.16) as

$$(\boldsymbol{V}\wedge\boldsymbol{W})^1=\frac{1}{\sqrt{g}}\left(\hat{V}_2\hat{W}_3-\hat{V}_3\hat{W}_2\right), \tag{1.24}$$

$$(\boldsymbol{V}\wedge\boldsymbol{W})_1=\sqrt{g}\,(\hat{V}^2\hat{W}^3-\hat{V}^3\hat{W}^2). \tag{1.25}$$

These formulae can be written compactly by using the well-known *Levi–Civita symbols*. But the latter must be handled with great care in non-orthogonal coordinate systems. We define contravariant components $\hat{\varepsilon}^{ijk}$ and covariant components $\hat{\varepsilon}_{ijk}$ of these symbols in the familiar way:

$$\begin{aligned}
&\hat{\varepsilon}^{ijk}=\hat{\varepsilon}_{ijk}=0 && \text{when two indices have the same value,}\\
&\hat{\varepsilon}^{ijk}=\hat{\varepsilon}_{ijk}=+1 && \text{when the sequence } (ijk) \text{ is an even permutation of } (123),\\
&\hat{\varepsilon}^{ijk}=\hat{\varepsilon}_{ijk}=-1 && \text{when the sequence } (ijk) \text{ is an odd permutation of } (123).
\end{aligned} \tag{1.26}$$

The subtle point is that *the Levi–Civita symbols do not transform like tensors* under general coordinate transformations. It can be shown, on the other hand, that the two quantities

$$\hat{E}_{rmn}=\sqrt{g}\,\hat{\varepsilon}_{rmn}, \qquad \hat{E}^{rmn}=\frac{1}{\sqrt{g}}\,\hat{\varepsilon}^{rmn} \tag{1.27}$$

are, respectively, covariant and contravariant components of a tensor, and are therefore related by

$$\hat{E}^{rmn}=g^{rk}g^{ml}g^{np}\hat{E}_{klp}. \tag{1.28}$$

It then follows that the consistent recipe for raising and lowering the indices of the Levi–Civita symbols is

$$\hat{\varepsilon}^{ijk} = g\, g^{il} g^{jm} g^{kn} \hat{\varepsilon}_{lmn},$$

$$\hat{\varepsilon}_{ijk} = \frac{1}{g} g_{il} g_{jm} g_{kn} \hat{\varepsilon}^{lmn}. \tag{1.29}$$

We may then rewrite eqs. (1.24) and (1.25) as

$$(\boldsymbol{V} \wedge \boldsymbol{W})^i = \frac{1}{\sqrt{g}} \hat{\varepsilon}^{ijk} \hat{V}_j \hat{W}_k, \tag{1.30}$$

$$(\boldsymbol{V} \wedge \boldsymbol{W})_i = \sqrt{g}\, \hat{\varepsilon}_{ijk} \hat{V}^k \hat{W}^k. \tag{1.31}$$

it is easily checked that, due to (1.29), we have

$$(\boldsymbol{V} \wedge \boldsymbol{W})_i = g_{ij} (\boldsymbol{V} \wedge \boldsymbol{W})^j, \qquad (\boldsymbol{V} \wedge \boldsymbol{W})^i = g^{ij} (\boldsymbol{V} \wedge \boldsymbol{W})_j$$

and the whole formalism is internally consistent.

Besides covariant and contravariant components, one can also define the *physical components* of the vector $\boldsymbol{V}$, by resolving the latter along the physical basis,

$$\boldsymbol{V} = \sum_j V_j \boldsymbol{e}_j. \tag{1.32}$$

From the previous results we find

$$V_j = \sqrt{g_{jj}}\, \hat{V}^j = \sum_k \sqrt{g_{jj}}\, g^{jk} \hat{V}_k \tag{1.33}$$

(no summation over j!). The scalar product does not have a simple form in terms of the physical components:

$$\boldsymbol{V} \cdot \boldsymbol{W} = \sum_j \sum_k \frac{g_{jk}}{\sqrt{g_{jj} g_{kk}}} V_j W_k. \tag{1.34}$$

This is to be contrasted with the simple form (1.23). For this reason, the use of the *dual basis* (covariant and contravariant components) is to be preferred in general, non-orthogonal coordinates. We will see in the next section that these expressions are considerably simpler in orthogonal coordinates, in which case the use of physical coordinates leads to the simplest and clearest formulae.

We now introduce the main quantities used in integral and differential vector calculus. The *differential length vector* $\mathrm{d}s_j$ directed along the coordinate line q^j is

$$\mathrm{d}s_j = \mathrm{d}q^j\, \hat{\boldsymbol{e}}_j = l_j\, \mathrm{d}q^j\, \boldsymbol{e}_j \tag{1.35}$$

(no summation over j!), where we introduced the *scale factor* l_j,

$$l_j(q^1, q^2, q^3) = \sqrt{g_{jj}}\,. \tag{1.36}$$

The infinitesimal *surface element* $\mathrm{d}\boldsymbol{S}^1$ perpendicular to the coordinate line q^1 is

$$\mathrm{d}\boldsymbol{S}^1 = \mathrm{d}s_2 \wedge \mathrm{d}s_3 = (\boldsymbol{e}_2 \wedge \boldsymbol{e}_3)\, \mathrm{d}q^2\, \mathrm{d}q^3. \tag{1.37}$$

The length of this vector is easily found as

$$|\mathrm{d}\boldsymbol{S}^1| = \sqrt{g_{22}g_{33} - g_{23}^2}\, \mathrm{d}q^2\, \mathrm{d}q^3. \tag{1.38}$$

Finally, the *volume element* is obtained from the important relation (1.17) by

$$\mathrm{d}^3\boldsymbol{x} = \sqrt{g}\, \mathrm{d}q^1\, \mathrm{d}q^2\, \mathrm{d}q^3. \tag{1.39}$$

The *gradient* of a scalar field is easily obtained by

$$\nabla A = \frac{\partial A}{\partial q^i}\, \nabla q^i = \frac{\partial A}{\partial q^i}\, \hat{\boldsymbol{e}}^i.$$

Hence, we obtain the covariant components of the gradient by

$$\hat{\nabla}_i A = \frac{\partial A}{\partial q^i}\,. \tag{1.40}$$

The *divergence of a vector* is defined as

$$\nabla \cdot \boldsymbol{V} = \frac{1}{\sqrt{g}}\, \frac{\partial}{\partial q^j}(\sqrt{g}\, \hat{V}^j). \tag{1.41}$$

The *curl of a vector* is defined by three relations of the form

$$(\nabla \wedge \boldsymbol{V})^1 = \frac{1}{\sqrt{g}}\left(\frac{\partial}{\partial q^2}\hat{V}_3 - \frac{\partial}{\partial q^3}\hat{V}_2\right). \tag{1.42}$$

Finally, the *Laplacian* of a scalar is obtained from (1.40), (1.41) by

$$\nabla^2 A = \frac{1}{\sqrt{g}}\frac{\partial}{\partial q^i}\left(\sqrt{g}\, g^{ik}\frac{\partial}{\partial q^k}A\right). \tag{1.43}$$

G2.2. *Orthogonal coordinate systems*

We now consider the important special case of *orthogonal (curvilinear) coordinate systems* *. These are defined by the diagonal character of the metric tensor (1.4),

$$\mathrm{d}s^2 = \sum_{j=1}^{3} g_{jj}(q^1, q^2, q^3)\, \mathrm{d}q^j\, \mathrm{d}q^j. \tag{2.1}$$

The metric tensor has only three non-vanishing components which are directly related to the *scale factors* (1.36),

$$\mathrm{d}s^2 = \sum_{j=1}^{3} l_j^2 (\mathrm{d}q^j)^2. \tag{2.2}$$

It then follows that the elements of the contravariant metric tensor (1.5), which is the reciprocal of the tensor g_{ij}, are simply

$$g^{JJ} = \frac{1}{g_{jj}} = \frac{1}{l_j^2}. \tag{2.3}$$

It follows from (1.19), (1.12) and (1.14) that the three sets $(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3)$, $(\hat{\boldsymbol{e}}^1, \hat{\boldsymbol{e}}^2, \hat{\boldsymbol{e}}^3)$ and $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ all form triorthogonal triads having the same

* One of the most thorough treatments of orthogonal coordinate systems is found in the book by Morse and Feshbach (1953).

orientation. In other words, for every j, $\hat{\boldsymbol{e}}^j$, $\hat{\boldsymbol{e}}_j$ and $\boldsymbol{e}_j$ are parallel vectors, differing only in their lengths:

$$\hat{\boldsymbol{e}}_j = \sqrt{g_{jj}}\,\boldsymbol{e}_j = l_j\,\boldsymbol{e}_j,$$

$$\hat{\boldsymbol{e}}^j = \sqrt{g^{jj}}\,\hat{\boldsymbol{e}}_j = \frac{1}{l_j}\boldsymbol{e}_j, \tag{2.4}$$

and we clearly have

$$\sum_j \hat{\boldsymbol{e}}^j \cdot \hat{\boldsymbol{e}}_j = \sum_j \boldsymbol{e}_j \cdot \boldsymbol{e}_j = 1, \tag{2.5}$$

and

$$\boldsymbol{e}_1 \wedge \boldsymbol{e}_2 = \boldsymbol{e}_3, \tag{2.6}$$

i.e. the familiar relation defining a right-handed triorthogonal triad.

Furthermore, the covariant, contravariant and physical components of any vector are simply proportional to each other (see eqs. 1.33),

$$\hat{V}_j = l_j\,V_j, \qquad \hat{V}^j = \frac{1}{l_j}V_j. \tag{2.7}$$

In orthogonal coordinates, (1.34) reduces to a simple expression for the *scalar product* in terms of physical components,

$$\boldsymbol{V} \cdot \boldsymbol{W} = \sum_j V_j\,W_j. \tag{2.8}$$

The physical components of the *vector product* also reduce to the familiar form

$$(\boldsymbol{V} \wedge \boldsymbol{W})_1 = V_2\,W_3 - V_3\,W_2. \tag{2.9}$$

There is no point in continuing to distinguish between contravariant, covariant and physical components of a vector. We shall make, once for all, the following choice. *When using orthogonal coordinates, we shall resolve all vectors along the physical basis*

$$\boldsymbol{V} = \sum_j V_j\,\boldsymbol{e}_j. \tag{2.10}$$

In other words, when orthogonal coordinates are used, a subscript always denotes a physical component (*not* a covariant component) of a vector. This choice has many advantages in clarity. Among these, let us mention the following ones.

– *The physical basis vectors* $\boldsymbol{e}_j$ *are unit vectors*, unlike ($\hat{\boldsymbol{e}}_j$, $\hat{\boldsymbol{e}}^j$). Hence the length of a vector is expressed in terms of physical components by the simple formula

$$|\boldsymbol{V}|^2 = \sum_j V_j^2. \tag{2.11}$$

– *The physical basis vectors* $\boldsymbol{e}_j$ *are dimensionless*. It follows that the *components* V_j *have all the same dimension*, the latter being the dimension of the physical quantity $\boldsymbol{V}$. This is not so for the covariant and contravariant components. (Think, for instance, of eq. (1.15) applied to the case when q^1 is the pressure, q^2 the poloidal angle and q^3 the toroidal angle!)

The *differential length vector* $\mathrm{d}\boldsymbol{s}_j$ has the same form as (1.35),

$$\mathrm{d}\boldsymbol{s}_j = l_j \, \mathrm{d}q^j \, \boldsymbol{e}_j. \tag{2.12}$$

Taking account of eqs. (2.4)–(2.6), eq. (1.37) for the *surface element* perpendicular to q^1 becomes

$$\mathrm{d}\boldsymbol{S}_1 = l_2 l_3 \, \mathrm{d}q^2 \, \mathrm{d}q^3 \, \boldsymbol{e}_1, \tag{2.13}$$

and the *volume element* reduces to

$$\mathrm{d}^3\boldsymbol{x} = l_1 l_2 l_3 \, \mathrm{d}q^1 \, \mathrm{d}q^2 \, \mathrm{d}q^3. \tag{2.14}$$

We now list the following useful expressions involving the nabla operator (Proofs can be found, e.g., in the book by Morse and Feshbach 1953). For vectors, we only write out one component, the others being obtained by circular permutation of the indices.

Gradient of a scalar:

$$\nabla_1 A = \frac{1}{l_1} \frac{\partial A}{\partial q^1}. \tag{2.15}$$

Divergence of a vector:

$$\nabla \cdot \boldsymbol{V} = \frac{1}{l_1 l_2 l_3} \left(\frac{\partial}{\partial q^1}(l_2 l_3 V_1) + \frac{\partial}{\partial q^2}(l_3 l_1 V_2) + \frac{\partial}{\partial q^3}(l_1 l_2 V_3) \right). \tag{2.16}$$

Curl of a vector:

$$(\nabla \wedge V)_1 = \frac{1}{l_2 l_3}\left(\frac{\partial}{\partial q^2}(l_3 V_3) - \frac{\partial}{\partial q^3}(l_2 V_2)\right). \tag{2.17}$$

Laplacian of a scalar:

$$\nabla^2 A = \frac{1}{l_1 l_2 l_3}\left(\frac{\partial}{\partial q^1}\frac{l_2 l_3}{l_1}\frac{\partial}{\partial q^1} + \frac{\partial}{\partial q^2}\frac{l_3 l_1}{l_2}\frac{\partial}{\partial q^2} + \frac{\partial}{\partial q^3}\frac{l_1 l_2}{l_3}\frac{\partial}{\partial q^3}\right)A. \tag{2.18}$$

Gradient of the divergence of a vector:

$$[\nabla(\nabla \cdot V)]_1 = \frac{1}{l_1}\frac{\partial}{\partial q^1}\frac{1}{l_1 l_2 l_3} \times\left(\frac{\partial}{\partial q^1}(l_2 l_3 V_1) + \frac{\partial}{\partial q^2}(l_3 l_1 V_2) + \frac{\partial}{\partial q^3}(l_1 l_2 V_3)\right). \tag{2.19}$$

Double curl of a vector:

$$[\nabla \wedge (\nabla \wedge V)]_1 = \frac{1}{l_2 l_3}\left(\frac{\partial}{\partial q^2}\frac{l_3}{l_1 l_2}\frac{\partial}{\partial q^1} l_2 V_2 + \frac{\partial}{\partial q^3}\frac{l_2}{l_1 l_3}\frac{\partial}{\partial q^1} l_3 V_3 - \frac{\partial}{\partial q^2}\frac{l_3}{l_1 l_2}\frac{\partial}{\partial q^2} l_1 V_1 - \frac{\partial}{\partial q^3}\frac{l_2}{l_1 l_3}\frac{\partial}{\partial q^3} l_1 V_1\right). \tag{2.20}$$

Laplacian of a vector:

$$\begin{aligned}(\nabla^2 V)_1 = \frac{1}{l_1 l_2 l_3}\Bigg(& l_2 l_3 \frac{\partial}{\partial q^1}\frac{1}{l_1 l_2 l_3}\frac{\partial}{\partial q^1} l_2 l_3 V_1 \\ &+ l_1 \frac{\partial}{\partial q^2}\frac{l_3}{l_1 l_2}\frac{\partial}{\partial q^2} l_1 V_1 + l_1 \frac{\partial}{\partial q^3}\frac{l_2}{l_3 l_1}\frac{\partial}{\partial q^3} l_1 V_1 \\ &+ l_2 l_3 \frac{\partial}{\partial q^1}\frac{1}{l_1 l_2 l_3}\frac{\partial}{\partial q^2} l_3 l_1 V_2 - l_1 \frac{\partial}{\partial q^2}\frac{l_3}{l_1 l_2}\frac{\partial}{\partial q^1} l_2 V_2 \\ &+ l_2 l_3 \frac{\partial}{\partial q^1}\frac{1}{l_1 l_2 l_3}\frac{\partial}{\partial q^3} l_1 l_2 V_3 - l_1 \frac{\partial}{\partial q^3}\frac{l_2}{l_1 l_3}\frac{\partial}{\partial q^1} l_3 V_3\Bigg).\end{aligned} \tag{2.21}$$

Directional derivative of a vector:

$$[(\boldsymbol{W}\cdot\nabla)\boldsymbol{V}]_1 = \left(\frac{W_1}{l_1}\frac{\partial}{\partial q^1} + \frac{W_2}{l_2}\frac{\partial}{\partial q^2} + \frac{W_3}{l_3}\frac{\partial}{\partial q^3}\right)V_1$$
$$+ \frac{V_2}{l_1 l_2}\left(W_1\frac{\partial}{\partial q^2}l_1 - W_2\frac{\partial}{\partial q^1}l_2\right)$$
$$+ \frac{V_3}{l_1 l_3}\left(W_1\frac{\partial}{\partial q^3}l_1 - W_3\frac{\partial}{\partial q^1}l_3\right). \tag{2.22}$$

Divergence of a second-rank tensor:

$$(\nabla\cdot\boldsymbol{A})_1 = \frac{1}{l_1}\left[\frac{\partial}{\partial q^1}A_{11} + \frac{l_1}{l_2}\frac{\partial}{\partial q^2}A_{21} + \frac{l_1}{l_3}\frac{\partial}{\partial q^3}A_{31}\right.$$
$$+ \frac{1}{l_2}\left(\frac{\partial l_1}{\partial q^2}\right)A_{12} + \frac{1}{l_3}\left(\frac{\partial l_1}{\partial q^3}\right)A_{13}$$
$$- \frac{1}{l_2}\left(\frac{\partial l_2}{\partial q^1}\right)A_{22} - \frac{1}{l_3}\left(\frac{\partial l_3}{\partial q^1}\right)A_{33}$$
$$+ \left(\frac{1}{l_2}\frac{\partial l_1}{\partial q^2} + \frac{l_1}{l_2 l_3}\frac{\partial l_3}{\partial q^2}\right)A_{21} + \left(\frac{1}{l_3}\frac{\partial l_1}{\partial q^3} + \frac{l_1}{l_2 l_3}\frac{\partial l_2}{\partial q^3}\right)A_{31}$$
$$\left. + \left(\frac{1}{l_2}\frac{\partial l_2}{\partial q^1} + \frac{1}{l_3}\frac{\partial l_3}{\partial q^1}\right)A_{11}\right]. \tag{2.23}$$

G2.3. Concentric circular toroidal coordinates

We now specialize some of the previous results to a particular coordinate system which is very often used in simple calculations concerning axisymmetric toroidal confinement systems. It is, in particular, the basis of the *standard model* defined in section **8**.9.

The system is most simply described as a grid of poloidal and toroidal angular coordinates lying on each one of a set of concentric circular tori. Consider a rectangular Cartesian reference frame whose z-axis coincides with

the symmetry axis of the tori and whose x–y plane coincides with the symmetry plane of the tori. Let R_0 be their major radius. Any point in the confinement region is defined by the following coordinates (fig. 8.9.2):

r: the minor radius of the torus passing through the point;
θ: the poloidal angle measured counterclockwise from the outer edge;
ζ: the toroidal angle measured clockwise from the y-axis.

The relation between Cartesian and toroidal coordinates is

$$x = (R_0 + r\cos\theta)\sin\zeta,$$

$$y = (R_0 + r\cos\theta)\cos\zeta,$$

$$z = r\sin\zeta. \tag{3.1}$$

The inverse relations are

$$r = \left\{\left[R_0 - (x^2+y^2)^{1/2}\right]^2 + z^2\right\}^{1/2},$$

$$\theta = \sin^{-1}\left(z\left\{\left[R_0 - (x^2+y^2)^{1/2}\right]^2 + z^2\right\}^{-1/2}\right),$$

$$\zeta = \tan^{-1}(x/y). \tag{3.2}$$

The relevant confinement region is defined by

$$r \leqslant R_0. \tag{3.3}$$

The elements of the metric tensor are easily calculated by using (1.4); one thus finds that the coordinate system is orthogonal, and obtains the following values for the scale factors (1.36):

$$l_r = 1, \qquad l_\theta = r,$$

$$l_\zeta = R_0 + r\cos\theta. \tag{3.4}$$

In each point a local triad of physical basis vectors is defined, with

$$\boldsymbol{e}_r \wedge \boldsymbol{e}_\theta = \boldsymbol{e}_\zeta. \tag{3.5}$$

These unit vectors are related to the Cartesian (absolute) triad $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ as

$$\boldsymbol{e}_r = \cos\theta \sin\zeta\, \boldsymbol{i} + \cos\theta \cos\zeta\, \boldsymbol{j} + \sin\theta\, \boldsymbol{k},$$

$$\boldsymbol{e}_\theta = -\sin\theta \sin\zeta\, \boldsymbol{i} - \sin\theta \cos\zeta\, \boldsymbol{j} + \cos\theta\, \boldsymbol{k},$$

$$\boldsymbol{e}_\zeta = \cos\zeta\, \boldsymbol{i} - \sin\zeta\, \boldsymbol{j}, \tag{3.6}$$

and inversely

$$\boldsymbol{i} = \cos\theta \sin\zeta\, \boldsymbol{e}_r - \sin\theta \sin\zeta\, \boldsymbol{e}_\theta + \cos\zeta\, \boldsymbol{e}_\zeta,$$

$$\boldsymbol{j} = \cos\theta \cos\zeta\, \boldsymbol{e}_r - \sin\theta \cos\zeta\, \boldsymbol{e}_\theta - \sin\zeta\, \boldsymbol{e}_\zeta,$$

$$\boldsymbol{k} = \sin\theta\, \boldsymbol{e}_r + \cos\theta\, \boldsymbol{e}_\theta. \tag{3.7}$$

The three *surface elements* are (eq. 2.13)

$$\mathrm{d}\boldsymbol{S}_r = r(R_0 + r\cos\theta)\, \mathrm{d}\theta\, \mathrm{d}\zeta\, \boldsymbol{e}_r,$$

$$\mathrm{d}\boldsymbol{S}_\theta = (R_0 + r\cos\theta)\, \mathrm{d}\zeta\, \mathrm{d}r\, \boldsymbol{e}_\theta,$$

$$\mathrm{d}\boldsymbol{S}_\zeta = r\, \mathrm{d}r\, \mathrm{d}\theta\, \boldsymbol{e}_\zeta. \tag{3.8}$$

The *volume element* is (eq. 2.14)

$$\mathrm{d}^3\boldsymbol{x} = r(R_0 + r\cos\theta)\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\zeta. \tag{3.9}$$

The *gradient of a scalar* is (eq. 2.15)

$$\nabla A = \frac{\partial A}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial A}{\partial \theta}\boldsymbol{e}_\theta + \frac{1}{R_0 + r\cos\theta}\frac{\partial A}{\partial \zeta}\boldsymbol{e}_\zeta. \tag{3.10}$$

The *divergence of a vector* is (eq. 2.16)

$$\nabla \cdot \boldsymbol{V} = \frac{1}{r(R_0 + r\cos\theta)}\left(\frac{\partial}{\partial r}\left[r(R_0 + r\cos\theta)V_r\right] + \frac{\partial}{\partial \theta}\left[(R_0 + r\cos\theta)V_\theta\right] + \frac{\partial}{\partial \zeta}(rV_\zeta)\right). \tag{3.11}$$

The *curl of a vector* is (eq. 2.17)

$$\nabla \wedge \boldsymbol{V} = \frac{1}{r(R_0 + r\cos\theta)}\left(\frac{\partial}{\partial\theta}(R_0 + r\cos\theta)V_\zeta - \frac{\partial}{\partial\zeta}rV_\theta\right)\boldsymbol{e}_r$$

$$+ \frac{1}{R_0 + r\cos\theta}\left(\frac{\partial}{\partial\zeta}V_r - \frac{\partial}{\partial r}(R_0 + r\cos\theta)V_\zeta\right)\boldsymbol{e}_\theta$$

$$+ \frac{1}{r}\left(\frac{\partial}{\partial r}rV_\theta - \frac{\partial}{\partial\theta}V_r\right)\boldsymbol{e}_\zeta. \tag{3.12}$$

The other, more complicated expressions of section G2.2 can be readily made explicit by using the scale factors (3.4).

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